Call Me by Your Name: An Introduction to Epistemic Logic with Assignments

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Outline

1 Introduction

2 Over Constant and Varying Domain Models

3 When Names Fail to Designate

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Introduction

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For the most part, epistemic logic focuses on propositional knowledge. (*Stanford Encyclopedia of Philosophy*)

First-order modal logics, as traditionally formulated, are not expressive enough. It is implicitly assumed that the names of agents are rigid designators in standard epistemic logic, and thus that it's common knowledge to whom they refer.

Why Constants should not be constant. (Fitting and Mendelsohn, 1998)

Introduction

Further complexities arise with higher-order knowledge.

Suppose there are two robotic agents, A and B, and A has just broken down. He sends a cry for help over a public broadcast system. B, who is the agent responsible for dealing with such matters, may or may not have heard. So A's subsequent action depends on whether he can deduce that "I know that B knows that I need help" (if this is true, he can just wait, but otherwise he should try something else).Grove (1995)

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The standard formulation of A's knowledge: $K_a K_b H(a)$.
Without the assumption that constants are constant, a formula like this can be read many ways:

- the robot named '*b*' knows that the robot named '*a*' needs help.
- the robot named '*b*' knows that it, i.e. the broken robot needs help.
- the maintenance robot knows that the robot named '*a*' needs help.
- the maintenance robot knows that it, i.e. the broken robot, needs help.

Introduction

A modal sentence $\Box F(c)$ containing the non-rigid constant c has a syntactic ambiguity that can engender a semantic ambiguity.

- c designates an object in the actual world and that object is said, in every possible world, to be F.
- in every possible world, the object designated by c in that world is said to be F.

Two operations are involved: one is the world-shift operation, $\Box$. The other is the designation of an object/agent by a constant symbol. These two operations do not commute.

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Informally, classical first-order formulas represent predicates. $\phi(x)$. In a modal setting things are quite different.

- "necessary P" predicate applied to c
- P predicate applied to c is necessary.

Formulas can no longer be thought of as representing predicates, pure and simple. Rather, a representation of a predicate can be abstracted from a formula. (Fitting and Mendelsohn, 1998)

This is the purpose of the device of predicate abstraction.

Make a distinction between **a formula** and **the predicate abstracted from it**.

$$\phi(x)\textbf{VS.}\langle\lambda x.\phi(x)\rangle$$

By analogy with the lambda-calculus, in which a distinction is made between an expression like $x+3$, and the function abstracted from it, $\langle\lambda x.x+3\rangle$.

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$$\langle\lambda x.\Box P(x)\rangle(c)\ VS.\ \Box\langle\lambda x.P(x)\rangle(c)$$

All the basic ideas of predicate abstraction were introduced into modal logic by Stalnaker and Thomason (1968), further modal applications of predicate abstraction appear in (Fitting, 1972; Fitting, 1973). Handbook of modal logic (Blackburn et al., 2006)

$$\langle\lambda x, y.K_c M(x, y)\rangle(b, a)\ VS.\ K_d(M(b, a))$$

In multi-modal logics for knowledge representation, logicians use a finite set of modalities, indexed by the first n natural numbers, usually denoted either $[1], [2], \ldots, [n]$ or $K_1, K_2 \ldots, K_n$. Each number is here naming some agent.

The problem, roughly, is that agents are (denoted by) a finite set of indexes that, so to say, live outside of the logic. (Orlandelli and Corsi, 2017)

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The idea of **term modal logic** is to treat the agents' names as terms from first-order modal logic. Specifically, it extends first order logic with modalities of the form K_t where t is a term. (Fitting et al., 2001)

This is quite natural in a first-order setting, and one can subsequently also quantify over the epistemic modalities associated with the agents.

We could develop an alternative system in which the epistemic modalities are treated as “relative” to persons. In this system we should have to deal with expressions like “known to somebody”, “unknown to everybody”, etc. (Wright, 1951)

$K_{f(a)}\neg\forall x K_x\varphi$ says that “*a*’s father knows that not everyone knows φ”

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The above readings can be expressed as $K_b H(a)$, $\langle \lambda x.\ K_b H(x)\rangle(a)$, $\langle \lambda y.\ K_y H(a)\rangle(b)$, $\langle \lambda x, y.\ K_y H(x)\rangle(a, b)$ respectively.

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Based on term-modal logic, Kooi (2007) proposes dynamic term modal logic approch, borrowing dynamic assignment modalities from first-order dynamic logic so as to adjust the designation of names.

$[x := b]K_a Px$ says that "*a* knows *de re* of *b* that it is *P*.

Now a minimalistic approach: a small fragment of dynamic term modal logic. (Wang and Seligman, 2018)

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Definition (ELAS)

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$$t ::= x \mid a$$

$$\phi ::= \top \mid t \approx t \mid P(\mathbf{t}) \mid \neg\phi \mid \phi \wedge \phi \mid K_t\phi \mid [x := t]\phi$$

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Before the formal details:
The formula $[x := t]\phi$ holds after assigning the current value of t to x.
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Therefore: $K_b H(a)$, $[x := a]K_b H(x)$, $[y := b]K_y H(a)$, $[x := a][y := b]K_y H(x)$.

Moreover, since names are non-rigid, we can express a knowing who b is by $[x := b]K_a(x \approx b)$ ($K_a b$). a identifies the right person with name b on all relevant possible worlds.
For the case Bob murdered Adam,
$[x := b][y := a](K_c M(x, y)) \wedge \neg K_c(a \approx x \wedge b \approx y)$ says Charles knows who killed whom that night but does not know the names of the murderer and the victim.
$K_d M(b, a) \wedge \neg K_d a \wedge \neg K_d b$ says Dave knows that a person named Bob murdered a person named Adam without knowing who they are.

The notion of free variables $Fv(\phi)$ is standard, except that $Fv(K_t\phi) = Var(t) \cup Fv(\phi)$, and $Fv([x := t]\phi) = (Fv(\phi) \setminus \{x\}) \cup Var(t)$.

Definition

A constant domain Kripke model is $\mathfrak{M} = \langle W, I, R, \rho, \eta \rangle$, where:

- W is a non-empty set of possible worlds.
- I is a non-empty set of agents.
- $R \subseteq (W \times I \times W)$, where $(w, i, v) \in R$ is denoted by wR_iv.
- $\rho : \mathbf{P} \times W \to \bigcup_{n \in \omega} 2^{I^n}$ assigns an n-ary relation $\rho(P, w)$ between agents to each n-ary predicate P at each world w.
- $\eta : \mathbf{N} \times W \to I$ assigns an agent $\eta(n, w)$ to each name n at each world w.

We call $\mathfrak{M}$ an epistemic model if R_i is an equivalence relation for each $i \in I$.

The semantics of term modal logic differs from the standard semantics of first-order modal logics by the treatment of the accessibility relation on worlds. The agent set is not fixed in the language, but specified along with the structure.

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Definition

A varying domain Kripke model is $\mathfrak{M} = \langle W, I, R, D, \rho, \eta \rangle$, where:

- W is a non-empty set of possible worlds.
- I is a non-empty set of agents.
- $R \subseteq (W \times I \times W)$, where $(w, i, v) \in R$ is denoted by wR_iv.
- $D : W \to 2^D$ assigns to each $w \in W$ a subset of I. The set $D(w)$ is denoted by I_w, and called the domain of w.
- $\rho : \mathbf{P} \times W \to \bigcup_{n \in \omega} 2^{I^n}$ assigns an n-ary relation $\rho(P, w)$ between agents to each n-ary predicate P at each world w.
- $\eta : \mathbf{N} \times W \to I$ assigns an agent $\eta(n, w)$ to each name n at each world w.

I is called the domain of the model.

In the semantics by Hintikka (1962), the alternatives to w is the worlds the knower in w considers possible, that is, the knower can't distinguish these worlds and w based on what information she possesses.
By Padmanabha and Ramanujam (2019), the demands of $(w, i, v) \in R$ is that only an agent alive at w can consider v accessible.
Also, (Grove, 1995) does not allow meaningless assertions about an agent's knowledge at a world where the agent does not exist.

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Definition

$\mathfrak{M}$ is an epistemic-like model over D if for any $i \in D,\ w, v, u \in W,\ R_i \subseteq W \times W$:

1. $i \in I_w \Longrightarrow wR_iw$.
2. wR_iv *and* $vR_iu \Longrightarrow wR_iu$.
3. $wR_iv \Longrightarrow vR_iw$.

Remark

(1) $i \in I_w \iff wR_iw$, *for any* $i \in D,\ w, v \in W,\ R_i \subseteq W \times W$. *It is straightforward by the first condition in Definition.*

(2) For any $w, v \in W$, wR_iv *implies* $i \in I_w$ *and* $i \in I_v$. *Suppose* wR_iv *(therefore* $i \in I_w$*), then* vR_iw *by condition 3, which implies* $i \in I_v$. *Combined with (1), we have* $I_w = \{i \mid wR_iw\} = \{i \mid \text{ there is a } v \in W \text{ s.t. } wR_iv\} = \{i \mid \text{ there is a } v \in W \text{ s.t. } vR_iw\}$.

(3) R_i *is an equivalence relation on* $W^* \times W^*$ *where* $W^* = \{w \mid i \in I_w\}$. *That is why the model is said to be epistemic-like. First* R_i *is reflexive on* $W^* \times W^*$ *by (1). Second,* R_i *is transitive and symmetric by condition 2 and 3 respectively.*

To interpret free variables, we need a variable assignment $\sigma : \mathbf{X} \to D$. Formulas are interpreted on pointed models $\mathfrak{M}, w$ with variable assignments σ. Given an assignment σ and a world $w \in W$, let $\sigma_w(a) = \eta(a, w)$ and $\sigma_w(x) = \sigma(x)$.

Definition

- $\mathfrak{M}, w, \sigma \vDash t \approx t' \Leftrightarrow \sigma_w(t) = \sigma_w(t')$
- $\mathfrak{M}, w, \sigma \vDash P(t_1 \ldots t_n) \Leftrightarrow (\sigma_w(t_1), \ldots, \sigma_w(t_n)) \in \rho(P, w)$
- $\mathfrak{M}, w, \sigma \vDash \neg\phi \Leftrightarrow \mathfrak{M}, w, \sigma \nvDash \phi$
- $\mathfrak{M}, w, \sigma \vDash (\phi \wedge \psi) \Leftrightarrow \mathfrak{M}, w, \sigma \vDash \phi$ *and* $\mathfrak{M}, w, \sigma \vDash \psi$
- $\mathfrak{M}, w, \sigma \vDash K_t\phi \Leftrightarrow \mathfrak{M}, v, \sigma \vDash \phi$ for all v s.t. $wR_{\sigma_w(t)}v$
- $\mathfrak{M}, w, \sigma \vDash [x := t]\phi \Leftrightarrow \mathfrak{M}, v, \sigma[x \mapsto \sigma_w(t)] \vDash \phi$
  where $\sigma[x \mapsto \sigma_w(t)]$ denotes an interpretation that is the same as σ except for mapping x to $\sigma_w(t)$.

An **ELAS** formula is valid if it holds on all the epistemic (-like) models with assignments $\mathfrak{M}, w, \sigma$.

Note that validity is defined not as in the increasing domain case. In the constant and varying domain cases all assignments are considered whereas in the increasing domain case the only assignments considered are assignments where every variable is assigned an existent. The reason why we make use of different definitions of validity is that straightforward and simple axiom systems are available with these choices. (Blackburn et al., 2006)

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In varying domain case:

Proposition

(i) *The **ELAS** formula $K_t\bot$ is true over an epistemic-like model with assignment $\mathfrak{M}, w, \sigma$ iff. $\sigma_w(t) \notin I_w$.*

(ii) *The **ELAS** formula $\hat{K}_t\top$ is true over an epistemic-like model with assignment $\mathfrak{M}, w, \sigma$ iff. $\sigma_w(t) \in I_w$.*

So the formula $\hat{K}_t\top$ is intended to assert that the agent $\sigma_w(t)$ exists at w.
The key reason we introduce $K_t\bot$ and $\hat{K}_t\top$ is that it allows us to transfer a meta-level notion "existence" into the object language. Interestingly, the formula $\hat{K}_t\top$ looks very much like the slogan of Descartes: **Cogito, ergo sum** (I think, therefore I am).

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First-order intensional logic can be translated into three-sorted first-order logic with equality. There is one sort for **worlds**, one sort for **objects**, and one sort for **intensions**. We translate ELAS, considered a language for talking about constant or varying domain models, into a (2-sorted) first-order language with a ternary relation symbol R for the accessibility relation, a function symbol f^a for each name a, and an n+1-ary relation symbol Q^P for each predicate symbol P.

$$Tr_w(x) = x \qquad Tr_w(a) = f^a(w)$$

$$Tr_w(t \approx t') = Tr_w(t) \approx Tr_w(t') \qquad Tr_w(Pt) = Q^P(w, Tr_w(t))$$

$$Tr_w(\neg\psi) = \neg Tr_w(\psi) \qquad Tr_w(\phi \wedge \psi) = Tr_w(\phi) \wedge Tr_w(\psi)$$

$$Tr_w(K_t\psi) = \forall v(R(w, v, Tr_w(t)) \to Tr_v(\psi))$$

$$Tr_w([x := t]\psi) = \begin{cases} \exists x(x \approx Tr_w(t) \wedge Tr_w(\psi)) & \text{if} t \neq x \\ Tr_w(\psi) & \text{if} t = x \end{cases}$$

valid $x \approx y \to K_t x \approx y, \quad x \not\approx y \to K_t x \not\approx y.$

invalid $x \approx a \to K_t x \approx a, \quad x \not\approx a \to K_t x \not\approx a, \quad a \approx b \to K_t a \approx b$

valid $K_x \phi \to K_x K_x \phi, \quad \neg K_x \phi \to K_x \neg K_x \phi, \quad (\hat{K}_t \to) K_t \phi \to \phi.$

invalid $K_t \phi \to K_t K_t \phi, \quad \neg K_t \phi \to K_t \neg K_t \phi$

valid $[x := y]\phi \to \phi[y/x]$($\phi[y/x]$is admissible)

invalid $[x := a]\phi \to \phi[a/x]$

valid $x \approx a \to (K_x \phi \to K_a \phi)$

invalid $x \approx a \to (K_b Px \to K_b Pa)$

valid $[x := y]K_a \phi \to K_a[x := y]\phi$

invalid $[x := b]K_a \phi \to K_a[x := b]\phi$

System SELAS (SELAS')

Axioms

TAUT	Propositional tautologies
DISTK	$K_t(\phi \to \psi) \to (K_t\phi \to K_t\psi)$
Tx	$\hat{K}_x\top \to (K_x\phi \to \phi)$
4x	$K_x\phi \to K_xK_x\phi$
5x	$\neg K_x\phi \to K_x\neg K_x\phi$
ID	$t \approx t$
ISUBP	$\mathbf{t} \approx \mathbf{t'} \to (P\mathbf{t} \leftrightarrow P\mathbf{t'})$ (P can be $\approx$)
SUBK	$t \approx t' \to (K_t\phi \leftrightarrow K_{t'}\phi)$
SUBAS	$t \approx t' \to ([x := t]\phi \leftrightarrow [x := t']\phi)$
RIGIDP	$x \approx y \to K_t x \approx y$
RIGIDN	$x \not\approx y \to K_t x \not\approx y$
KAS	$[x := t](\phi \to \psi) \to ([x := t]\phi \to [x := t]\psi)$
DETAS	$\langle x := t\rangle\phi \to [x := t]\phi$
DAS	$\langle x := t\rangle\top$
EFAS	$[x := t]x \approx t$
SUB2AS	$\phi[y/x] \to [x := y]\phi$ ($\phi[y/x]$is admissible)

Rules

MP $\dfrac{\phi, \phi \to \psi}{\psi}$ NECK $\dfrac{\vdash \phi}{\vdash K_t\phi}$ *NECAS* $\dfrac{\vdash \phi \to \psi}{\vdash \phi \to [x := t]\psi}(x \notin Fv(\phi))$

where $\mathbf{t} \approx \mathbf{t'}$ means point-wise equivalence for sequences of terms $\mathbf{t}$ and $\mathbf{t'}$ such that $|\mathbf{t}| = |\mathbf{t'}|$.

Theorem

Soundness *SELAS is sound over constant domian epistemic models with assignments. SELAS' is sound over varing domian epistemic-like models with assignments.*

Completeness

First note that every point w in every model $\mathfrak{M}$ for a logic is associated with a set of formulas, namely $\{\phi \mid \mathfrak{M}, w \vDash \phi\}$. This set of formulas is actually a MCS. That is: if ϕ is true in some model, then it belongs to a MCS. Second, if w is related to w' in some model $\mathfrak{M}$, then it is clear that the information embodied in the MCSs associated with w and w' is 'coherently related'. Thus our second observation is: models give rise to collections of coherently related MCSs.

The idea behind the canonical model construction is to try and turn these observations round: that is, to work backwards from collections of coherently related MCSs to the desired model.
(Blackburn et al., 2005)

Completeness

We first extend the language **ELAS** with countably infinitely many new variables, and call the new language $\mathbf{ELAS}^+$ with the variable set $\mathbf{X}^+$. We say a language L is an infinitely proper sublanguage of another language L' if:

- L and L' only differ in their sets of variables (*proper*),
- $L \subset L'$ (*sublanguage*),
- there are infinitely many new variables in L' that are not in L (*infinitely*).

We use maximal consistent sets w.r.t. different infinitely proper sublanguages of $\mathbf{ELAS}^+$ that are extensions of **ELAS** to build a pseudo canonical frame.

Definition (Pseudo canonical frame)

$\mathfrak{F}^c = \langle W, R \rangle$ is defined as follows:

- W is the set of *MCS* Δ *w.r.t.* some infinitely proper sublanguages L_Δ of $\mathbf{ELAS}^+$ such that for each $\Delta \in W$:
  $\mathbf{ELAS} \subset L_\Delta$,
  For each $a \in N$ there is a variable x in L_Δ ($x \in Var(\Delta)$) such that $x \approx a \in \Delta$ (call it $\exists-$property)
- For each $x \in \mathbf{X}^+$, $\Delta R_x \Theta$ iff.:
  (i) $x \in Var(\Delta)$ $(\hat{K}_x \top \in \Delta)$,
  (ii) $\{\phi \mid K_x \phi \in \Delta\} \subset \Theta$,
  (iii) if $y \in Var(\Theta) \setminus Var(\Delta)$ then $y \not\approx z \in \Theta$ for all $z \in Var(\Theta)$ such that $z \neq y$

Observation For each Δ from the pseudo frame, it is easy to see that it $t \in L_\Delta$ then there is $x \in Var(\Delta)$ such that $x \approx t \in \Delta$ by $\exists-$property and ID.

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Proposition

If $\Delta R_x \Theta$ in $\mathfrak{F}^c$, then:

- *L_Δ is a sublanguage of L_Θ*
- *for any $y \neq z \in Var(\mathbf{ELAS}^+)$: $y \approx z \in \Delta$ iff. $y \approx z \in \Theta$.*

This proposition makes sure that we do not have confliting equalities in diffferent states which are accessible from one to another.

Lemma (Existence Lemma)

If $\Delta \in W$ and $\hat{K}_t\phi \in \Delta$ then there is a $\Theta \in W$ and an $x \in Var(L_\Delta)$ such that $\phi \in \Theta,\ x \approx t \in \Delta$ and $\Delta R_x \Theta$

It states that there are enough coherently related MCSs to ensure the success of the canonical model construction.

To prove the completeness of SELAS over constant domian epistemic Kripke models with assignments, Wang and Seligman (2018) uses the following proof strategy to consruct a canonical model:

- Extend the language with countably many new variables.
- Build a pseudo canonical frame using maximal consistent sets for various sublanguages of the extended language, with witnesses for names.
- Given a maximal consistent set (MCS), cut out its generated subframe from the pseudo frames, and build a constant-domian canonical, by taking certain equivalence classes of variables as the domain.
- Show that the truth lemma holds for the canonical model.
- Take the reflexive symmetric transitive closure of the relations in the pseudo model and show that the truth of the formulas in the original language are preserved..

Given a state Γ in $\mathfrak{F}^c$, we can define an equivalence relation $\sim_\Gamma$: $x \sim_\Gamma y$ iff. $x \approx y \in \Gamma$ or $x = y$. $\sim_\Gamma$ is indeed an equivalence relation.
Let $|x|_\Gamma = \{y \mid x \sim_\Gamma y\}$, it is easy to show that $\Delta R_z \Theta$ implies $|x|_\Delta = |x|_\Theta$. $|x|_\Gamma$ is abbreviated to $|x|$ in the following when Γ is fixed.

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Definition (Canonical model for constant domian)

Given a Γ in $\mathfrak{F}^c$ we define the canonical model $\mathfrak{M}_\Gamma = \langle W_\Gamma, R^c, I^c, \rho^c, \eta^c \rangle$ over based on the psuedo canonical frame $\langle W, R \rangle$.

- W_Γ is the subset of W generated from Γ w.r.t. the relations R_x.
- $I^c = \{|x| \mid x \in Var(W_\Gamma)\}$ where $Var(W_\Gamma)$ is the set of all the variables appearing in W_Γ.
- $\Delta R^c_{|x|} \Theta$ iff $\Delta R_x \Theta$, for any $\Delta, \Theta \in W_\Gamma$.
- $\eta^c(a, \Delta) = |x|$ iff $a \approx x \in \Delta$.
- $\rho^c(P, \Delta) = \{|\mathbf{x}| \mid P\mathbf{x} \in \Delta\}$.

Proposition

The canonical model is well-defined.

Proof.

- For $R^c_{|x|}$: the choice of the representative in $|x|$ does not change the definition.
- For $\eta^c(a, \Delta)$: first, the choice of the representative in $|x|$ does not change the definition. Then, $\eta^c(a, \Delta)$ is unique.

Proposition

$R_{|x|}$ is transitive.

Remark

- $R_{|x|}$ is not reflexive, some x may not be in the language of some state. **(condition 1)**
- $R_{|x|}$ is not symmetric. Suppose $\Delta R_{|x|} \Theta$, for any $y \in Var(\Theta) \setminus Var(\Delta)$, $K_x y \approx y \in \Theta$ but $y \approx y \notin \Delta$ since Δ is a MCS w.r.t. L_Δ. **(condition 2)**

We will turn this model into an epistemic one later on. Before that:

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We will turn this model into an epistemic one later on. Before that:

Proposition (Truth lemma)

*For any $\phi \in$ **ELAS**$^+$ and any $\Delta \in W$, if $\phi \in L_\Delta$ then:*

$$\mathfrak{M}_\Gamma, \Delta, \sigma^* \vDash \phi \Leftrightarrow \phi \in \Delta$$

where σ^* is the canonical assignment such that $\sigma^*(x) = |x|$ for all $x \in VAr(W_\Gamma)$

Proof.

For the case of $[x := t]\psi \in L_\Delta$:

- $\Longrightarrow$) Suppose $\mathfrak{M}_\Gamma, \Delta, \sigma^* \vDash [x := t]\psi$.
  If $t \in N$, by $\exists$-property, we have $y \approx t \in \Delta$ for some $y \in Var(\Delta)$. By induction hypothesis, $\mathfrak{M}_\Gamma, \Delta, \sigma^* \vDash y \approx t$. Therefore $\sigma^*(\Delta, t) = |y|$ thus $\mathfrak{M}_\Gamma, \Delta, \sigma^*[x \mapsto |y|] \vDash \psi$. Now if $\psi[y/x]$ is admissible then we have $\mathfrak{M}_\Gamma, \Delta, \sigma^* \vDash \psi[y/x]$. By IH, $\psi[y/x] \in \Delta$. Thus $[x := y]\psi \in \Delta$ by SUB2AS. Since $t \approx y \in \Delta$, thus $[x := t]\psi \in \Delta$ by SUBAS. Note that if $\psi[y/x]$ is not admissible, then we can reletter ψ to have an equivalent formula $\psi' \in L(\Delta)$ such that $\psi'[y/x]$ is admissible. Then the above proof still works to show that $[x := t]\psi' \in \Delta$. Since relettering can be done in the proof system by Proposition 3.3, we have $[x := t]\psi \in \Delta$.
  If t is a variable y, then $\mathfrak{M}_\Gamma, \Delta, \sigma^*[x \mapsto |y|] \vDash \psi$. From here a similar (but easier) proof like the above suffices.
- $\Longleftarrow$) Supposing $[x := t]\psi \in \Delta$, by the $\exists$-property of Δ, we have some $y \in Var(\Delta)$ such that $t \approx y \in \Delta$. Like the proof above we can assume w.l.o.g. that $\psi[y/x]$ is admissible, for otherwise we can reletter ψ first. Thus $[x := y]\psi \in \Delta$ by SUBAS. Then by SUBASEQ, $\psi[y/x] \in \Delta$. By IH, $\mathfrak{M}_\Gamma, \Delta.\sigma^* \vDash \psi[y/x] \wedge t \approx y$. By the semantics and the assumption that $\psi[y/x]$ admissible, $\mathfrak{M}_\Gamma, \Delta, \sigma^* \vDash [x := t]\psi$.

Now we transform the canonical model into an epistemic model by taking reflexive, symmetric and transitive closure of each $R_{|x|}$ in $\mathfrak{M}_\Gamma$.
Although $\mathfrak{M}_\Gamma$ is a transitive model, the symmetric closure will break the transitivity. If $\Delta R_{|x|}\Theta,\ \Gamma R_{|x|}\Theta$ then we have $\Delta R'_{|x|}\Theta,\ \Theta R'_{|x|}\Gamma$ by taking symmetric closure of $R_{|x|}$.

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Taking the reflexive transitive closure via undirected paths. let $\mathfrak{N}_\Gamma$ be the model like $\mathfrak{M}_\Gamma$ but with the revised relation $R^*_{|x|}$ for each $x \in D^c$, defined as:

$$\begin{aligned}\Delta R^*_{|x|}\Theta \quad \Leftrightarrow \quad & \text{either } \Delta = \Theta \text{ or there are some } \Delta_1 \cdots \Delta_n \text{ for some } n \geq 0 \\ & \text{such that } \Delta_k R_{|x|}\Delta_{k+1} \text{ or } \Delta_{k+1} R_{|x|}\Delta_k \\ & \text{for each } 0 \leq k \leq n \text{ where } \Delta_0 = \Delta \text{ and } \Delta_{n+1} = \Theta.\end{aligned}$$

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Lemma

$\mathfrak{N}^\Gamma$ *is an epistemic model.*

We will show that it preserves the truth value of **ELAS** formulas.

Lemma (Preservation lemma)

*For all $\phi \in$ **ELAS**:*

$$\mathfrak{N}_\Gamma, \Delta, \sigma^* \vDash \phi \Leftrightarrow \phi \in \Delta$$

Proof. Since we only altered the relations, we just need to check $K_t\psi \in$ **ELAS**. Note that then $K_t\psi$ is in all the local language L_Δ.

- $\Longrightarrow$) Since the closure only adds relations then we know $\mathfrak{M}^\Gamma, \Delta, \sigma^* \vDash K_t\psi$ and therefore $K_t\psi \in \Delta$ by induction hypothesis and Truth Lemma.
- $\Longleftarrow$) Suppose $K_t\psi \in \Delta$. By $\exists$-property, there is some $x \in Var(\Delta)$ s.t. $x \approx t \in \Delta$, $K_x\psi \in \Delta$.
  Consider an arbitrary $R^*_{|x|}$-successor Θ in $\mathfrak{N}^\Gamma$. If $\Delta = \Theta$ then by T it is trivial to show that $\psi \in \Delta$. Suppose there are some $\Delta_1 \cdots \Delta_n$ such that $\Delta_k R_{|x|} \Delta_{k+1}$ or $\Delta_{k+1} R_{|x|} \Delta_k$ for each $0 \leq k \leq n$ where $\Delta_0 = \Delta$ and $\Delta_{n+1} = \Theta$. Now we do induction on n to show that $K_x\psi \in \Delta_k$ for all those $k \leq n+1$. Note that if the claim is correct then by T we have $\psi \in \Delta_{K+1}$ thus by IH we have $\mathfrak{N}^\Gamma, \Delta, \sigma^* \vDash K_t\psi$.

To consruct a varying domain epistemic-like canonical model for SELAS', by contrast, we need to construct domains for every possible worlds. Since $\hat{K}_x\top$ asserts that the agent $\sigma_\Delta(x)$ exists at w, we have to collect the x satisfying $\hat{K}_x\top$ belongs to Δ to form the domain of Δ.

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Given a Γ in $\mathfrak{F}^c$ we define the canonical model $\mathfrak{M}'_\Gamma = \langle W_\Gamma, R^c, I^c, D^c, \rho^c, \eta^c\rangle$ over based on the psuedo canonical frame $\langle W, R\rangle$.

- W_Γ is the subset of W generated from Γ w.r.t. the relations R_x.
- $I^c = \{|x| \mid x \in Var(W_\Gamma)\}$
- $\Delta R^c_{|x|}\Theta$ iff $\Delta R_x \Theta$, for any $\Delta, \Theta \in W_\Gamma$.
- $D^c_\Delta = \{|x| \mid \hat{K}_x\top \in \Delta\}$
- $\eta^c(a, \Delta) = |x|$ iff $a \approx x \in \Delta$.
- $\rho^c(P, \Delta) = \{|\mathbf{x}| \mid P\mathbf{x} \in \Delta\}$.

Definition

For any $\Delta, \Theta, \Lambda \in W_\Gamma$, any $x \in Var(W_\Gamma)$:

(i) $\Delta R_{|x|} \Delta \iff |x| \in I^c_\Delta$

(ii) $\Delta R_{|x|} \Theta$ *and* $\Theta R_{|x|} \Lambda \implies \Delta R_{|x|} \Lambda$

Proof.

(i) We just need to show $\Delta R_{|x|} \Delta$ iff. $\hat{K}_x \top \in \Delta$ by definition.

$\implies$) Suppose $\Delta R_{|x|} \Delta$ then in $\mathfrak{F}^c$ $\Delta R_x \Delta$. Thus $\hat{K}_x \top \in \Delta$ by definition.

$\impliedby$) Suppose $\hat{K}_x \top \in \Delta$. We have $K_x \phi \to \phi \in \Delta$ by T'x and the property of MCS, so $\{\phi \mid K_x \phi \in \Delta\} \subseteq \Delta$ by using the property of MCS again. Thus $\Delta R_x \Delta$ by definition, which means $\Delta R_{|x|} \Delta$.

(ii) Suppose $\Delta R_{|x|} \Theta$ and $\Theta R_{|x|} \Lambda$ then in $\mathfrak{F}^c$ $\Delta R_x \Theta$ and $\Theta R_x \Lambda$. We have to show the three conditions for $\Delta R_x \Lambda$. □

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*For any $\phi \in$ **ELAS**$^+$ and any $\Delta \in W$, if $\phi \in L_\Delta$ then:*

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Proof

For the case of $K_t\psi \in L_\Delta$:

- $\Longrightarrow$) Suppose $K_t\psi \notin \Delta$, then $\hat{K}_t\neg\psi \in \Delta$ which implies $\hat{K}_t\top \in \Delta$ by EXIS. Analogously, by **Existence Lemma** there is some variable $x \in Var(\Delta)$ and $\Theta \in W_\Gamma$ such that $\Delta R_x \Theta$ (which means $\Delta R_{|x|}\Theta$), $x \approx t \in \Delta$ and $\neg\psi \in \Theta$.
- $\Longleftarrow$) Suppose $K_t\psi \in \Delta$, then by $\exists$-property, there ia an $x \in Var(\Delta)$ s.t. $x \approx t \in \Delta$ and $K_x\psi \in \Delta$. Therefore by IH, $\mathfrak{M}'_\Gamma, \Delta, \sigma^* \vDash x \approx t$.
  For the case of $K_x\bot \in \Delta$, i.e. $\hat{K}_x\top \notin \Delta$ we have $|x| \notin I^c_\Delta$ by definition of I^c_Δ, therefore $\mathfrak{M}'_\Gamma, \Delta, \sigma^* \vDash K_x\bot$.
  Otherwise consider any $R_{|x|}$- successor Θ of Δ, we have $\psi \in \Theta$. By IH again, $\mathfrak{M}_\Gamma, \Theta, \sigma^* \vDash \psi$. Thus $\mathfrak{M}'_\Gamma, \Delta, \sigma^* \vDash K_x\psi$ which implies $\mathfrak{M}'_\Gamma, \Delta, \sigma^* \vDash K_t\psi$.

Now we transform the canonical model into an epistemic-like model by taking **symmetric and transitive closure** of each $R_{|x|}$ in $\mathfrak{M}'_\Gamma$. Although $\mathfrak{M}'_\Gamma$ is a transitive model, the symmetric closure will break the transitivity. If $\Delta R_{|x|}\Theta$, $\Gamma R_{|x|}\Theta$ then we have $\Delta R'_{|x|}\Theta$, $\Theta R'_{|x|}\Gamma$ by taking symmetric closure of $R_{|x|}$. However we do not have $\Delta R'_{|x|}\Gamma$. let $\mathfrak{N}_\Gamma$ be the model like $\mathfrak{M}_\Gamma$ but with the revised relation $R^*_{|x|}$ for each $x \in I^c$, defined as:

$$\Delta R^*_{|x|}\Theta \quad \Leftrightarrow \quad \begin{array}{l} \text{there are some } \Delta_1 \cdots \Delta_n \text{ for some } n \geq 0 \\ \text{such that } \Delta_k R_{|x|} \Delta_{k+1} \text{ or } \Delta_{k+1} R_{|x|} \Delta_k \\ \text{for each } 0 \leq k \leq n \text{ where } \Delta_0 = \Delta \text{ and } \Delta_{n+1} = \Theta. \end{array}$$

We need to check the four conditions for an epistemic-like model $\mathfrak{N}'_\Gamma$:

Lemma

$\mathfrak{N}^\Gamma$ *is an epistemic-like mode with assignment, satisfies:*

(i) $\Delta R^*_{|x|}\Delta \Longrightarrow |x| \in I^c_\Delta$

(ii) $\Delta R^*_{|x|}\Theta$ *and* $\Theta R^*_{|x|}\Lambda \Longrightarrow \Delta R^*_{|x|}\Lambda$

(iii) $\Delta R^*_{|x|}\Theta \Longrightarrow \Theta R^*_{|x|}\Delta$

Proof.
Firstly, suppose $\Delta R^*_{|x|}\Theta$ then there are some $\Delta_1 \cdots \Delta_n$ such that $\Delta_k R_{|x|}\Delta_{k+1}$ or $\Delta_{k+1} R_{|x|}\Delta_k$ for each $0 \leq k \leq n$ where $\Delta_0 = \Delta$ and $\Delta_{n+1} = \Theta$. **(If $\Delta R_{|x|}\Theta$ then $|x| \in I^c_\Delta$, $|x| \in I^c_\Theta$.$|x| \in I^c_\Delta$ is trivial. By 5x $\hat{K}_x\top \to K_x\hat{K}_x\top$, thus $K_x\hat{K}_x\top \in \Delta$ which implies $\hat{K}_x\top \in \Theta$. Therefore $|x| \in I^c_\Theta$.)** Thus $|x| \in I^c_{\Delta_k}$ for all those $k \leq n+1$.

□

Outline

1 Introduction

2 Over Constant and Varying Domain Models

3 When Names Fail to Designate

We will show that it preserves the truth value of **ELAS** formulas.

Lemma (Preservation lemma)

For all $\phi \in$ ***ELAS****:*

$$\mathfrak{M}'_{\Gamma}, \Delta, \sigma^* \vDash \phi \Leftrightarrow \phi \in \Delta$$

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- $\Longrightarrow$) If $\mathfrak{N}'_{\Gamma}, \Delta, \sigma^* \vDash K_t\psi$ then since the closure only adds relations then we know $\mathfrak{M}'^{\Gamma}, \Delta, \sigma^* \vDash K_t\psi$.
- $\Longleftarrow$) Suppose $K_t\psi \in \Delta$. Since Δ has $\exists$-property, there is some $x \in Var(\Delta)$ such that $x \approx t \in \Delta$ thus $K_x\psi \in \Delta$.
  For the case of $K_x\bot \in \Delta$ i.e. $\hat{K}_x\top \notin \Delta$ then $|x| \notin I^c_\Delta$. Then Δ is not in $R|x|$ relation with itself or others. Then no $R_{|x|}$ relations are added w.r.t Δ, which means $\mathfrak{N}'_{\Gamma}, \Delta, \sigma^* \vDash K_t\bot$.
  Otherwise consider an arbitrary $R^*_{|x|}$-successor Θ in $\mathfrak{N}'_{\Gamma}$.

Note that $I \neq \bigcup_{w \in W} I_w$ actually. Just consider a model for a set of sentences like $\{P(a), K_{t_0}\bot, K_{t_1}\bot, \ldots, K_{t_n}\bot, \ldots\}$ where $t_0, \ldots, t_n, \ldots$ is a enumeration of all terms in **ELAS**. The set of possible worlds can be a singleton $\{w\}$ while the domain $I_w = \emptyset$.

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A term may designate, at a world, an agent not in the domain of that world at the same time not in the domain of all possible worlds. However we have adopted the priciple that terms always do designate.
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While according to (Fitting and Mendelsohn, 1998) definite descriptions are the classic examples of agents that may fail to designate, we do not introduce them formally for simplicity.
The key points about terms failing to designate are:

- Variables do designate.
- Names are allows to be undefined at some possible worlds.
- Dynamic assignment operators do not perform assignment actions when contain names without denotation in themselves.

We now modify the notation of a varying domain Kripke model to allow for undefined names.

Definition

Let $\mathfrak{M} = \langle W, I, R, \rho, \eta \rangle$ be a varying domain Kripke model over D, its definition is exactly as in Definition in varying domain case above, except that:

- $\eta : \mathbf{N} \times W \to D$ partially assigns an agent $\eta(n, w)$ to each name n at some (possibly no) worlds $w \in W$.

Given an assignment σ and a world $w \in W$, still $\sigma_w(x) = \sigma(x)$. A name a designates at w provided $\eta(a, w)$ is defined, and if it is, $\sigma_w(a) = \eta(a, w)$. If a does not designate at w, $\sigma_w(a)$ is undefined.

Definition

- $\mathfrak{M}, w, \sigma \Vdash t \approx t' \Leftrightarrow$ t, t' *designate at* w *in* $\mathfrak{M}$ *w.r.t* σ, $\sigma_w(t) = \sigma_w(t')$
- $\mathfrak{M}, w, \sigma \Vdash P(t_1 \ldots t_n) \Leftrightarrow$
  $t_1, \ldots, t_n$ *designate at* w *in* $\mathfrak{M}$ *w. r. t.* σ, $(\sigma_w(t_1), \ldots, \sigma_w(t_n)) \in \rho(P, w)$
- $\mathfrak{M}, w, \sigma \Vdash \neg\phi \Leftrightarrow \mathfrak{M}, w, \sigma \nVdash \phi$
- $\mathfrak{M}, w, \sigma \Vdash (\phi \wedge \psi) \Leftrightarrow \mathfrak{M}, w, \sigma \Vdash \phi$ *and* $\mathfrak{M}, w, \sigma \Vdash \psi$
- $\mathfrak{M}, w, \sigma \Vdash K_t\phi \Leftrightarrow$
  t *fails to designate at* w *in* $\mathfrak{M}$ *w. r. t.* σ, *or* t *designates at* w *in* $\mathfrak{M}$ *w. r. t.* σ, $\mathfrak{M}, v, \sigma \Vdash \phi$ for all v s.t. $wR_{\sigma_w(t)}v$
- $\mathfrak{M}, w, \sigma \Vdash [x := t]\phi \Leftrightarrow$
  t *fails to designate at* w *in* $\mathfrak{M}$ *w. r. t.* σ, *or* t *designates at* w *in* $\mathfrak{M}$ *w. r. t.* σ, $\mathfrak{M}, v, \sigma[x \mapsto \sigma_w(t)] \Vdash \phi$.

An **ELAS** formula is valid (over epistemic-like model) if it holds on all the (epistemic-like) models with assignments $\mathfrak{M}, s, \sigma$

we have several observations as a consequence of the Definition.

Proposition

(i) *The **ELAS** formula* $[x := t]\bot$ *is true over an epistemic-like model with assignment* $\mathfrak{M}, w, \sigma$ *iff* t *fails to designate at* w *in* $\mathfrak{M}$.

(ii) *The **ELAS** formula* $\langle x := t\rangle\top$ *is true over an epistemic-like model with assignment* $\mathfrak{M}, w, \sigma$ *iff* t *designates at* w *in* $\mathfrak{M}$.

(iii) *The **ELAS** formula* $a \approx a$ *is true over an epistemic-like model with assignment* $\mathfrak{M}, w, \sigma$ *iff* a *designates at* w *in* $\mathfrak{M}$.

The two formulas $\langle x := t\rangle\top$ and $a \approx a$ assert the same things.They allow us to move a meta-level notion designation into the object language.

System SELAS*

Axioms

TAUT	Propositional tautologies	*INTER*	$\hat{K}_a\top \to \langle x := a\rangle\top$
DISTK	$K_t(\phi \to \psi) \to (K_t\phi \to K_t\psi)$	SUBAS	$t \approx t' \to$ $([x := t]\phi \leftrightarrow [x := t']\phi)$
T'x	$\hat{K}_x\top \to K_x\phi \to \phi$		
4x	$K_x\phi \to K_xK_x\phi$	RIGIDP	$x \approx y \to K_t x \approx y$
5x	$\neg K_x\phi \to K_x\neg K_x\phi$	RIGIDN	$x \not\approx y \to K_t x \not\approx y$
IDx	$x \approx x$	KAS	$[x := t](\phi \to \psi) \to$ $([x := t]\phi \to [x := t]\psi)$
IDa	$\langle x := a\rangle\top \to a \approx a$		
ID∗	$a \approx t \to \langle x := a\rangle\top$	DETAS	$\langle x := t\rangle\phi \to [x := t]\phi$
SUBP	$\mathbf{t} \approx \mathbf{t'} \to (P\mathbf{t} \leftrightarrow P\mathbf{t'})$	*DAS*∗	$\langle x := y\rangle\top$
	(P can be $\approx$)	EFAS	$[x := t]x \approx t$
SUBK	$t \approx t' \to (K_t\phi \leftrightarrow K_{t'}\phi)$	SUB2AS	$\phi[y/x] \to [x := y]\phi$
PREDID	$P\mathbf{t} \to \langle \mathbf{x} := \mathbf{t}\rangle\top$		($\phi[y/x]$is admissible)

Rules

MP $\dfrac{\phi, \phi \to \psi}{\psi}$ NECK $\dfrac{\vdash \phi}{\vdash K_t\phi}$ NECAS $\dfrac{\vdash \phi \to \psi}{\vdash \phi \to [x := t]\psi}(x \notin Fv(\phi))$

SYM	$t \approx t' \to t' \approx t$	*TRANS*	$t \approx t' \wedge t' \approx t'' \to t$
DBASEQ*	$\langle x := t\rangle\top \to [x := t]\phi \to \langle x := t\rangle\phi$	SUBASEQ	$\phi[y/x] \leftrightarrow [x := y]\phi$
EAS*	$\langle x := a\rangle\top \to [x := t]\phi \leftrightarrow \phi \ (x \notin Fv(\phi))$	T'	$\hat{K}_t\top \to (K_t\phi \to \phi)$
CNECAS*	$\dfrac{\vdash \phi \to \psi}{\vdash \langle x := t\rangle\top \to [x := t]\phi \to \psi} \ (x \notin Fv(\psi))$	NECAS'	$\dfrac{\vdash \phi}{\vdash [x := t]\phi}$
EX	$[x := x]\phi \leftrightarrow \phi$	EXIS	$\hat{K}_t\phi \to \hat{K}_t\top$
DESI	$\langle x := t\rangle\phi \to \langle x := t\rangle\top$	**INTERt**	$\hat{K}_t\top \to \langle x := t\rangle\top$

we adopt the same strategy to prove the completeness of SELAS*. First we extend the language with countably many new variables. Then we build a pseudo canonical frame using maximal consistent sets for various sublanguages of the extended language, with witnesses for certain formulas.

Definition (Pseudo canonical frame)

$\mathfrak{F}^c = \langle W, R\rangle$ is defined as follows:

- W is the set of *MCS* Δ *w.r.t.* some infinitely proper sublanguages L_Δ of **ELAS**$^+$ such that for each $\Delta \in W$:
  ELAS $\subseteq L_\Delta$,
  For each $\langle x := a\rangle\top$ **in** L_Δ, if $\langle x := a\rangle\top \in \Delta$ then there is a variable $y \in Var(\Delta)$ such that $y \approx a \in \Delta$ (call it E-property)
- For each $x \in \mathbf{X}^+$, $\Delta R_x \Theta$ iff.:
  (i) $\hat{K}_x\top \in \Delta$,
  (ii) $\{\phi \mid K_x\phi \in \Delta\} \subseteq \Theta$,
  (iii) if $y \in Var(\Theta) \setminus Var(\Delta)$ then $y \not\approx z \in \Theta$ for all $z \in Var(\Theta)$ such that $z \neq y$

Observation. For each Δ from the pseudo frame, it is easy to see that in each case we have $\langle x := a\rangle\top \in \Delta$ whenever $a \approx t \in \Delta$, $P(\mathbf{t}) \in \Delta$ **and some** t **in the sequence t is** a**,** $\hat{K}_a\phi \in \Delta$**, or** $\langle x := a\rangle\phi \in \Delta$ by ID*, PREDID, EXIS and INTER, DESI respectively.

More than that, if $t \approx t' \in \Delta$, then there is $y, z \in Var(\Delta)$ s.t. $t \approx y \in \Delta,\ t' \approx z \in \Delta$. If $P(\mathbf{t}) \in \Delta$, then there exists $\mathbf{y} \in Var(\Delta)$ such that $\mathbf{y} \approx \mathbf{t} \in \Delta$. If $\hat{K}_t\phi \in \Delta$, or $\langle x := t\rangle\phi \in \Delta$ then there is $y \in Var(\Delta)$ s.t. $y \approx t \in \Delta$. These results are all by E-property and IDx.

Observation. For each Δ from the pseudo frame, it is easy to see that in each case we have $\langle x := a\rangle\top \in \Delta$ whenever $a \approx t \in \Delta$, $P(\mathbf{t}) \in \Delta$ **and some** t **in the sequence t is** a**,** $\hat{K}_a\phi \in \Delta$**, or** $\langle x := a\rangle\phi \in \Delta$ by ID*, PREDID, EXIS and INTER, DESI respectively.

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Also, if $\Delta R_x \Theta$ in $\mathfrak{F}^c$, then L_Δ is a sublanguage of L_Θ, and for any $y \neq z \in Var(\mathbf{ELAS}^+)$: $y \approx z \in \Delta$ iff. $y \approx z \in \Theta$, which makes sure that we do not have conflicting equalities in Δ and Θ.

The proof of Existence Lemma w.r.t. SELAS* is different.

Lemma (Existence Lemma)

If $\Delta \in W$ *and* $\hat{K}_t\phi \in \Delta$ *then there is a* $\Theta \in W$ *and an* $x \in Var(L_\Delta)$ *such that* $\phi \in \Theta,\ x \approx t \in \Delta,\ \hat{K}_x\top \in \Delta$ *and* $\Delta R_x \Theta$

The proof of Existence Lemma w.r.t. SELAS* is different.

Lemma (Existence Lemma)

If $\Delta \in W$ and $\hat{K}_t\phi \in \Delta$ then there is a $\Theta \in W$ and an $x \in Var(L_\Delta)$ such that $\phi \in \Theta,\ x \approx t \in \Delta,\ \hat{K}_x\top \in \Delta$ and $\Delta R_x \Theta$

Proof. If $\hat{K}_t\phi \in \Delta$, then we have $x \approx t \in \Delta$ and $\hat{K}_x\top \in \Delta$ for some x.
Let$\Theta^{--} = \{\phi\} \cup \{\psi | K_x\psi \in \Delta\}$, Θ will be constructed from Θ^{--}. Θ^{--} is consistent by DISTK and NECK (routine). Next we want to show that it can be extended to a state in W. Select an infinitely proper sublanguage L of $\mathbf{ELAS}^+$ such that L_Δ is an infinitely proper sublanguage of L. List:

- the new variables in L but not in L_Δ by $y_0,\ y_1,\ y_2, \ldots$,
- the sentences as $\langle x := a_0\rangle\top,\ \langle x := a_1\rangle\top,\ \langle x := a_2\rangle\top, \ldots$,

Define the following chain of sets of sentences of L.
Suppose Θ_k has already defined and new variables occuring in Θ_k are $y_0, , \ldots, y_n$.
Case(1). $\Theta_k \cup \{\langle x := a_k\rangle\top\}$ is L-consistent.
Case(1.1). For some variable $x_i \in Var(\Delta) \cup \{y_0, \ldots, y_n\}$, $\Theta_k \cup \{x_i \approx a_k\}$ is L-consistent. Define $\Theta_{k+1} = \Theta_k \cup \{x_i \approx a_k\}$.

Case(1.2). For all $x \in Var(\Delta) \cup \{y_0, \ldots, y_n\}$, $\Theta_k \cup \{x_i \approx a_k\}$ is not L-consistent. Take a y in L not occuring in $Var(\Delta) \cup \{y_0, \ldots, y_k\}$ and define
$\Theta_{k+1} = \Theta_k \cup \{y \approx a_k\} \cup \{y \not\approx z \mid z \in Var(\Delta) \cup \{y_0, \ldots, y_n\}\}$.

Case(2). $\Theta_k \cup \{\langle x := a_k\rangle\top\}$ is not L-consistent. Define $\Theta_{k+1} = \Theta_k$.

Definition (Canonical model)

Given a Γ in $\mathfrak{F}^c$ we define the canonical model $\mathfrak{M}_\Gamma = \langle W_\Gamma, R^c, I^c, D^c, \rho^c, \eta^c \rangle$, based on the psuedo canonical frame $\langle W, R \rangle$.

- W_Γ is the subset of W generated from Γ w.r.t. the relations R_x.
- $\Delta R^c_{|x|} \Theta$ iff $\Delta R_x \Theta$, for any $\Delta, \Theta \in W_\Gamma$. I^c where $D^c = \{|x| \mid x \in Var(W_\Gamma)\}$
- $D^c_\Delta = \{|x| \mid \hat{K}_x \top \in \Delta\}$
- $\eta^c(a, \Delta) = |x|$ iff $a \approx x \in \Delta$.
- $\rho^c(P, \Delta) = \{|\mathbf{x}| \mid P\mathbf{x} \in \Delta\}$.

$\eta^c(a, \Delta)$ is undefined iff $a \approx x \notin \Delta$ for any $x \in Var(W_\Gamma)$. The canonical model is well-defined.

Now we are ready to prove the truth lemma.

Lemma (Truth Lemma)

For any $\phi \in \textbf{\textit{ELAS}}^+$ and any $\Delta \in W$, if $\phi \in L_\Delta$ then:

$$\mathfrak{M}_\Gamma, \Delta, \sigma^* \Vdash \phi \Leftrightarrow \phi \in \Delta$$

where σ^* is the canonical assignment such that $\sigma^*(x) = |x|$ for all $x \in VAr(W_\Gamma)$

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