PML and WAML

Model theory and Craig Interpolation

刘佶鑫

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目录

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鸽笼原理：每对夫妻中至少有一人满足 A, 至少有一人满足 B, 且至少有一人满足 C，故每对夫妻中至少有一人满足 A 且 B, A 且 C，B 且 C 中的一个性质。

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  - $\mathsf{K}_n : \Box p_0 \wedge \cdots \wedge \Box p_n \rightarrow \Box \bigvee_{(0 \leq i < j \leq n)} (p_i \wedge p_j)$

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- 用 K_n 在 $\mathbb{K}$ 中替换 C 得到弱聚合模态逻辑（*Weakly Aggregative Modal Logics*（WAML）[SJ80]）。

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- C 同样不是有效的：
  你知道如何喝醉，也知道如何证明某个数学定理，但你很可能并不知道如何在喝醉的时候证明这个定理。（在 "knowing how" 认知逻辑中 [Wan17, FHLW17]。）

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- 对 $\Box\varphi$ 的解释恰好符合一种特殊情况下的 $\nabla(\varphi_1, \ldots, \varphi_n)$ 语义：$\varphi_1 = \cdots = \varphi_n$。

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- 称 $\Box$ 为 N 元对角线算子。[2]

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$$\mathcal{M}, w \models \nabla(\varphi_1, \ldots, \varphi_n) \quad \text{iff} \quad \forall v_1, \ldots v_n \in W(R_\nabla w v_1 \ldots, v_n \Rightarrow \mathcal{M}, v_i \models \varphi_i \text{ for some } i \leq n).$$

$$\mathcal{M}, w \models \Delta(\varphi_1, \ldots, \varphi_n) \quad \text{iff} \quad \exists v_1, \ldots v_n \in W(R_\nabla w v_1 \ldots, v_n \& \mathcal{M}, v_i \models \varphi_i \text{ for all } i \leq n).$$

Normal polyadic modal logic

- A modal logic Λ is *normal* if it contains the axiom K^i_∇ and is closed under N^i_∇ for each $i \in [1, n]$.[BDRV02]

$$\begin{array}{ll} \mathsf{K}^i_\nabla & \nabla(r_1, \dots r_{i-1}, p \to q, r_{i+1} \dots, r_n) \to \\ & \quad (\nabla(r_1, \dots r_{i-1}, p, r_{i+1} \dots, r_n) \to \nabla(r_1, \dots r_{i-1}, q, r_{i+1} \dots, r_n)) \\ \mathsf{N}^i_\nabla & \quad \text{from } \vdash_\Lambda \varphi \text{ infer } \vdash_\Lambda \nabla(\psi_1 \dots, \psi_{i-1}, \varphi, \psi_{i+1}, \dots, \psi_n) \end{array}$$

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- We call the resulting minimal normal modal logic $\mathbb{K}_n$.

Alternative Axiomatization

In [Joh78], the author used the following axiom $\mathtt{G}^i_\nabla$ instead of $\mathtt{K}^i_\nabla$,[3] and besides $\mathtt{N}^i_\nabla$, a monotonicity rule $\mathtt{RM}^i_\nabla$ is also used.

$$\begin{array}{ll} \mathtt{G}^i_\nabla & \nabla(r_1,\dots r_{i-1},p,r_{i+1}\dots,r_n) \to (\nabla(r_1,\dots r_{i-1},q,r_{i+1}\dots,r_n) \\ & \qquad \to \nabla(r_1,\dots r_{i-1},p\wedge q,r_{i+1}\dots,r_n)) \\ \mathtt{RM}^i_\nabla & \text{from } \vdash_\wedge \varphi\to\psi \text{ infer} \\ & \vdash_\wedge \nabla(\psi_1\dots,\psi_{i-1},\varphi,\psi_{i+1},\dots,\psi_n) \to (\psi_1\dots,\psi_{i-1},\psi,\psi_{i+1},\dots,\psi_n) \end{array}$$

[3] The name $\mathtt{G}^i_\nabla$ is in recognition of the contribution of Goldblatt.

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- Define a new semantics $\Vdash$:
- $w \Vdash \nabla(\varphi_1, \ldots, \varphi_n)$ iff one of the followings hold:
  1. w is a dead end, i.e. there is no $v_1, \ldots, v_n$ s.t. $Rwv_1, \ldots, v_n$.
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  $Rwv_1, \ldots, v_n \wedge \exists k \in [1, n] \forall w_1, \ldots, w_n (Rww_1, \ldots, w_n \rightarrow$
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- (There is a unique argument which is true at the corresponding position of every sequence of successors.)

Wrong Axiomatizations

- Also note that the following axiom mentioned in the definition of normal polyadic modal logics from [GPV03] is not valid:[4]

$$\nabla(p_1 \to q_1, \ldots, p_n \to q_n) \to (\nabla(p_1, \ldots, p_k) \to \nabla(q_1, \ldots, q_n))$$

[4] In [GPV03], Δ is used as the polyadic box.

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- The *ultrafilter extension* of an n-model $\mathcal{M} = (\mathcal{F}, V)$ is the model $ue\mathcal{M} = (ue\mathcal{F}, V^{ue})$ where $V^{ue}(p_i) = \{u$ is an ultrafilter on $W \mid V(p_i) \in u\}$.[BDRV02]

An Important Lemma

Lemma

Suppose u is an ultrafilter on W^n. Let $\prod_i : W^n \to W$ be the i-th coordinate projection and $b_i = \{\prod_i(x) \mid x \in b\}$ be the projection of u. Then $u_i = \{b_i \mid b \in u\}$ is an ultrafilter on W.

Proof

- Define a function $'$: $W \to W^n$:
  $a' = \{(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) \mid x \in a$ and $a_j \in W$ for each $j \leq n\} = \{x \in W^n \mid \prod_i(x) \in a\}$. Obviously, $a \subseteq b$ only if $a' \subseteq b'$ and one can check that $(a')_i = a$.

Proof

- Define a function $'$: $W \to W^n$: $a' = \{(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) \mid x \in a$ and $a_j \in W$ for each $j \le n\} = \{x \in W^n \mid \prod_i(x) \in a\}$. Obviously, $a \subseteq b$ only if $a' \subseteq b'$ and one can check that $(a')_i = a$.
- If $a \supseteq b \in u_i$, then $\exists c \in u$ s.t. $b = c_i$. To get $a \in u_i$, we only need to show $a' \supseteq c$. If $x \in c$, then $\prod_i(x) \in b$ and hence $\prod_i(x) \in a$, which means $x \in a'$.

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- Define a function $' : W \to W^n$:
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- If $a \supseteq b \in u_i$, then $\exists c \in u$ s.t. $b = c_i$. To get $a \in u_i$, we only need to show $a' \supseteq c$. If $x \in c$, then $\prod_i(x) \in b$ and hence $\prod_i(x) \in a$, which means $x \in a'$.
- If $a \notin u_i$, then $a' \notin u$, which means $W^n - a' \in u$. It follows that $(W^n - a')_i \in u_i$, but $(W^n - a')_i = \{\prod_i(x) \mid x \in W^n - a'\}$. Assume that $y \in \{\prod_i(x) \mid x \in W^n - a^{'}\}$, then $y = \prod_i(x)$ for some $x \in W^n - a'$. If $y \in a$, then $\prod_i(x) \in a$ which means $x \in a'$, a contradiction. So $(W^n - a')_i \subseteq W - a$. By the above result, $W - a \in u_i$.

Saturation Models

- Let $\mathcal{M} = (W, R_\nabla, V)$ be an n-model. $\mathcal{M}$ is called $\mathfrak{m}$-saturated if for every state $w \in W$ and every sequence $\Sigma_1, \ldots, \Sigma_n$ of sets of PML formulas we have:

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- If for every sequence of finite subsets $\Delta_1 \subseteq \Sigma_1, \ldots, \Delta_n \subseteq \Sigma_n$ there are states $v_1, \ldots, v_n$ s.t. $R_\nabla wv_1, \ldots, v_n$ and for each i $v_i \models \Delta_i$. then there are $w_1, \ldots, w_n$ s.t. $R_\nabla ww_1, \ldots, w_n$ and for each i $w_i \models \Sigma_i$.

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- The name '$\mathfrak{m}$-saturation' stems from [Vis94], but actually the notion is older: its first occurrence is in [Fin75]. In those original papers, the notion is monadic, while the polyadic case is a direct generalization.

Saturation Theorem of UE

Theorem

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- Proof.

Let $A_i = \{W_1 \times \cdots \times W_{i-1} \times V(\varphi) \times W_{i+1} \cdots \times W_n \mid \varphi \in \Delta_i$ and $W_j = W$ for all $j\}$.

$A = \cup_{1 \leq i \leq n} A_i$.

$B = \{\cup_{1 \leq i \leq n}(W_1 \times \cdots \times W_{i-1} \times Y_i \times W_{i+1} \cdots \times W_n) \mid m^{\delta}_{\Delta}(Y_1, \ldots, Y_n) \in w$ and $W_j = W$ for all $j\}$.

Let $\Delta = A \cup B$. Check that Δ has the finite intersection property.

Use the above lemma.

□

Theorem ([Gol00])

Let $\mathcal{M}$ be an n-model. Then, for any formula φ and any ultrafilter u over W, $V(\varphi) \in u$ iff $ue\mathcal{M}, u \models \varphi$.

- Hence, for each state w in $\mathcal{M}$ we have $w \leftrightsquigarrow \prod_w$, where $\prod_w$ is the principal ultrafilter generated by $\{w\}$.

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- **Proof.**

  Use the lemma again. $\square$

Ultraproduct

- Let $C = \prod_{i \in I} W_i$ be the Cartesian product of $\{W\}_{i \in I}$ and u be an ultrafilter on the index set I. For two functions $f, g \in C$ we say that f and g are u-equivalent $(f \sim_u g$ if $\{i \in I \mid f(i) = g(i)\} \in u$. One can easily check this is indeed an equivalence relation.

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- Let $f_u = \{g \in C \mid g \sim_u f\}$. The ultraproduct of $\{W\}_{i \in I}$ modulo u is define as follows:

$$\prod_u W_i = \{f_u \mid f \in \prod_{i \in I} W_i\}$$

Ultraproduct

- Definition (ultraproduct)

  Let $\mathcal{M}_i = (W_i, R_{\Delta i}, V_i)(i \in I)$ be n-models. The ultraproduct $\prod_u \mathcal{M}$ modulo u is described as follows.

- (i) The universe W_u is the set $\prod_u W_i = \{f_u \mid f \in \prod_{i \in I} W_i\}$.
- (ii) The valuation V_u is defined by

$$f_u \in V_u(p) \text{ iff } \{i \in I \mid f(i) \in V_i(p)\} \in u.$$

- (iii) The n-ary relation $R_{\Delta u}$ is given by

$$f_u^0 R_{\Delta u} f_u^1 ... f_u^n \text{ iff } \{i \in I \mid f^0(i) R_{\Delta i} f^1(i) ... f^n(i)\} \in u.$$

  - If all the $\mathcal{M}_i$ are the same model $\mathcal{M}$, we say $\prod_u \mathcal{M}$ the ultrapower of $\mathcal{M}$ modulo u.

Theorem (AC)

Let $\prod_u \mathcal{M}$ *be an ultrapower of* $\mathcal{M}$*. Then for all PML formulas* φ*, we have* $\mathcal{M}, w \models \varphi$ *iff* $\prod_u \mathcal{M}, ((f_w)_u \models \varphi$*, where* f_w *is the constant function s.t.* $f_w(i) = w$ *for all* $i \in I$*.*

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- Doets and Van Benthem [VBD83] gave an intuitive explanation of the ultraproduct construction.

Craig Interpolation Theorem

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- We do have a algebraic proof for CIT of PML in [Nem85].
- In [H+01], there is a deep connection between the amalgamation on algebras and the interpolation on logic.

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- We choose to give a proof for PML using the method in [ANvB98].
- Rosen's method is very important in proving CIT when the logic is just weakly complete.

Craig Interpolation Theorem

Theorem

Each normal polyadic modal logic $\mathbb{K}_n$ *has the Craig Interpolation Theorem. More precisely, let* $\varphi \vdash_{\mathbb{K}_n} \psi$*, then there is a formula* α *s.t.* $\varphi \vdash_{\mathbb{K}_n} \alpha \vdash_{\mathbb{K}_n} \psi$ *and* $atom(\alpha) \subseteq atom(\varphi) \cap atom(\psi)$*.*

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- **Proof.**

  First we fix an n and just use $\vdash$ without a subscript. Since we already know that $\mathbb{K}_n$ is strongly complete w.r.t to all n-frames, we could freely switch between $\models$ and $\vdash$. For convenience, let $P = atom(\varphi)$, $Q = atom(\psi)$ and $R = atom(\alpha)$. We show that the set $cons_R(\varphi)$ of all consequences of φ in R language satisfies the following claim:

  $$cons_R(\varphi) \models \psi.$$

  By a standard compactness argument, we can find the interpolant.

  □

Proof of the Claim

Let $(\mathcal{M}, a) = (W, R_\Delta, V)$ be any pointed n-model s.t. $(\mathcal{M}, a) \models cons_R(\varphi)$. We show that $(\mathcal{M}, a) \models \psi$. By a routine argument, the R-theory $Th_R(\mathcal{M}, a)$ is consistent with $\{\varphi\}$, and by compactness again, there is a P-model $(\mathcal{N}, b) \models \varphi$ s.t. $(\mathcal{M}, a) \equiv_R (\mathcal{N}, b)$. Suppose that $(\mathcal{N}, b) = (W', R'_\Delta, V')$. We have already shown that there are m-saturated models which can preserve modal truth in this paper before, so without loss of generality we assume that both $(\mathcal{M}, a)$ and $(\mathcal{N}, b)$ are m-saturated. It follows that the $\equiv_R$ is indeed an R-bisimulation. Next we construct a product model $\mathcal{MN}, (a, b)$ s.t. $(\mathcal{M}, a) \leftrightarroweq_Q \mathcal{MN}, (a, b)$ and $(\mathcal{N}, b) \leftrightarroweq_P \mathcal{MN}, (a, b)$, which is sufficient for our proof:

$$(\mathcal{N}, b) \models \varphi \Rightarrow \mathcal{MN}, (a, b) \models \varphi \Rightarrow \mathcal{MN}, (a, b) \models \psi \Rightarrow (\mathcal{M}, a) \models \psi$$

Proof of the Claim

Let $Z = \{(x, y) \in W \times W' \mid x \leftrightarrows_R y\}$, and define $\mathcal{MN} = (Z, R^*_\Delta, V^*)$ as follows:

$$(x, y)R^*_\Delta(x_1, y_1), ..., (x_n, y_n) \text{ iff } xR_\Delta x_1, ..., x_n \text{ and } yR'_\Delta y_1, ..., y_n$$

For each $(x, y) \in Z$,

$$(x, y) \in V^*(p) \iff \begin{cases} x \in V(p) & \text{if} \quad p \in Q \\ y \in V'(p) & \text{if} \quad p \in P \\ \text{never} & \text{if} \quad \text{otherwise} \end{cases}$$

Notice that V^* is well-defined since every $(x, y) \in Z$ satisfies $x \leftrightarrows_R y$. Now it is sufficient to check that our construction satisfies the requirement.

Let $B_1 = \{(x, (z_1, z_2)) \mid x \leftrightarrows_Q z_1 \text{ and } z_2 \in W'\}$ be a relation on $W \times Z$ and $B_2 = \{(y, (z_1, z_2)) \mid y \leftrightarrows_P z_2 \text{ and } z_1 \in W\}$ be a relation on $W' \times Z$.

Weakly Aggregative Modal Logic

Language and Semantics

- $$\varphi := p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi$$

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- Definition (n-Semantics)

  R_∇ is an $n+1$-ary relation over W. The semantics for $\Box\varphi$ (and $\Diamond\varphi$) is defined by:

$$\begin{array}{lll} \mathcal{M}, w \models \Box\varphi & \text{iff} & \forall v_1, \ldots v_n \in W(R_\nabla w v_1 \ldots, v_n \to \mathcal{M}, v_i \models \varphi \text{ for some } i \leq n). \\ \mathcal{M}, w \models \Diamond\varphi & \text{iff} & \exists v_1, \ldots v_n \in W(R_\nabla w v_1 \ldots, v_n \& \mathcal{M}, v_i \models \varphi \text{ for all } i \leq n). \end{array}$$

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  - It is not hard to see that the aggregation axiom $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$ is not valid on n-frames for any $n \geq 2$.

Systems

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- The logic $\mathbb{K}_n^w$ is a modal logic including propositional tautologies, the axiom K_n and closed under the rules $\mathtt{N}$ and $\mathtt{RM}$:

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\begin{array}{ll}
\mathsf{K}_n & \Box p_0 \wedge \cdots \wedge \Box p_n \to \Box \bigvee_{(0 \le i < j \le n)} (p_i \wedge p_j) \\
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- K_1 is just the aggregation axiom C and thus $\mathbb{K}_1^w$ is just the normal monadic modal logic $\mathbb{K}$.

Neighborhood Semantics

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- A pair $\mathfrak{F} = \langle W, \nu \rangle$ is a called a *neighborhood frame*, if W is a non-empty set and ν is a neighborhood function from W to $\mathcal{P}(\mathcal{P}(W))$. $\mathcal{M} = \langle \mathfrak{F}, V \rangle$ is a model if $V : \Phi \to 2^W$ is a valuation function.

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- The semantics is defined as follows:

$$\mathcal{M}, w \models \Box\varphi \iff [\![\varphi]\!]^{\mathcal{M}} \in \nu(w) \mid \mathcal{M}, w \models \Diamond\varphi \iff W - [\![\varphi]\!]^{\mathcal{M}} \notin \nu(w)$$

where $[\![\varphi]\!]^{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$, i.e. the truth set of φ.

Bisimulation

Let $\mathcal{M} = (W, R_\nabla, V)$ and $\mathcal{M}' = (W', R'_\nabla, V')$ be two n-models. A non-empty binary relation $Z \subseteq W \times W'$ is called a *wa^n-bisimulation* between $\mathcal{M}$ and $\mathcal{M}'$ if the following conditions are satisfied:

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- **forth** If wZw' and $R_\nabla wv_1, \ldots, v_n$ then there are $v'_1, \ldots, v'_n$ in W' s.t. $R'_\nabla w'v'_1, \ldots, v'_n$ and for each v'_j there is a v_i such that $v_iZv'_j$ where $1 \leq i, j \leq n$.

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- **back** If wZw' and $R'_\nabla w'v'_1, \ldots, v'_n$ then there are $v_1, \ldots, v_n$ in W s.t. $R_\nabla wv_1, \ldots, v_n$ and for each v_i there is a v'_j such that $v_iZv'_j$ where $1 \leq i, j \leq n$.

Bisimulation

- Consider the following two 2-models where $\{\langle w, w_1, w_2\rangle, \langle w, w_2, w_3\rangle\}$ is the ternary relation in the left model, and $\{\langle v, v_1, v_2\rangle\}$ is the ternary relation in the right model.

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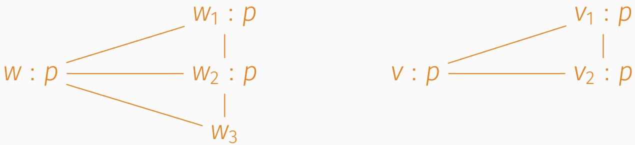

$Z = \{\langle w, v\rangle, \langle w_1, v_1\rangle, \langle w_2, v_2\rangle, \langle w_2, v_1\rangle\}$ is a wa^2-bisimulation. A polyadic modal formula $\neg\nabla\neg(p, \neg p)$, not expressible in WAML^2, can distinguish w and v.

Definition (Standard translation)

ST : WAML$^n \to$ FOL:

$$
\begin{array}{lcl}
ST_x(p) & = & Px \\
ST_x(\neg\varphi) & = & \neg ST_x(\varphi) \\
ST_x(\varphi \wedge \psi) & = & ST_x(\varphi) \wedge ST_x(\psi) \\
ST_x(\Box\varphi) & = & \forall y_1 \forall y_2 \ldots \forall y_n (Rxy_1 y_2 \ldots y_n \to ST_{y_1}(\varphi) \vee \cdots \vee ST_{y_n}(\varphi))
\end{array}
$$

n-Tree Unraveling [dR93]

- We use an example of a graph with ternary relations to illustrate the intuitive idea behind the general n-ary unraveling.

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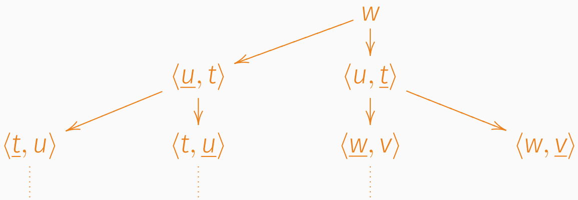
- Example

Given the 2-model with ternary relations $\langle \{w, v, u, t\}, \{\langle w, u, t\rangle, \langle u, t, u\rangle, \langle t, w, v\rangle\}, V\rangle$. It is quite intuitive to first unravel it into a binary tree with pairs of states as nodes, illustrated below:



Definition

Given an n-model $\mathcal{M} = \langle W, R, V \rangle$ and $w \in W$, we first define the binary unraveling $\mathcal{M}_w^b$ of $\mathcal{M}$ around w as $\langle W_w, R^b, V' \rangle$ where:

W_w is the set of sequences $\langle \langle \vec{v}_0, i_0 \rangle, \langle \vec{v}_1, i_1 \rangle, \ldots, \langle \vec{v}_m, i_m \rangle \rangle$ where:

$m \in \mathbb{N}$; for each $j \in [0, m]$, $\vec{v}_j \in W^n$ and $i_j \in [1, n]$ such that $R(\vec{v}_j[i_j])\vec{v}_{j+1}$; $\vec{v}_0$ is the constant n-sequence $w \ldots w$ and $i_0 = 1$;

$R^b ss'$ iff s' extends s with some $\langle \vec{v}, i \rangle$

$V'(s) = V(r(s))$, where $r(s) = \vec{v}_m[i_m]$ if $s = \langle \ldots, \langle \vec{v}_m, i_m \rangle \rangle$.

The unraveling $\mathcal{M}_w = \langle W_w, R', V' \rangle$ is based on $\mathcal{M}_w^b$ by defining $R' s_0 s_1 \ldots s_n$ iff $Rr(s_0)r(s_1) \ldots r(s_n)$ and $R^b s_0 s_i$ for all $i \in [1, n]$.

Rosen van Benthem Characterization Theorem

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  - Following the general strategy in [Ott04], the only non-trivial part is to show that the FOL formula $\alpha(x)$ that is invariant under wa^n-bisimulation has some locality property w.r.t. its bounded unraveling $\mathcal{M}_w|_l$ for some l.

Rosen van Benthem Characterization Theorem

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  A first-order formula $\alpha(x)$ *is invariant under* $\underline{\leftrightarrow}^n$ *(over finite models) iff* $\alpha(x)$ *is equivalent to a WAML*^{n} *formula (over finite models).*

  - Following the general strategy in [Ott04], the only non-trivial part is to show that the FOL formula $\alpha(x)$ that is invariant under wa^n-bisimulation has some locality property w.r.t. its bounded unraveling $\mathcal{M}_w|_l$ for some l.

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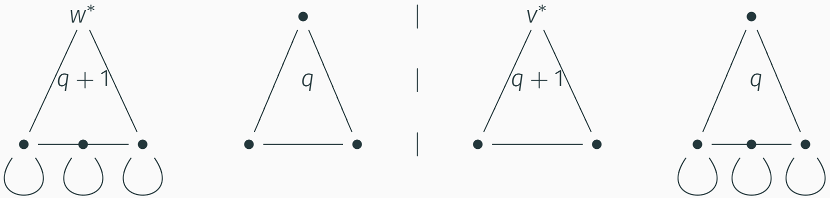
- **Lemma (locality)**

  FOL formula $\alpha(x)$ is invariant under $\leftrightarroweq$ (over finite models) implies that for some $l \in \mathbb{N}$, for any n-model $\mathcal{M}, w$: $\mathcal{M}, w \Vdash \alpha(x)[w]$ iff $\mathcal{M}_w|_l \Vdash \alpha(x)[(\langle \vec{w}, 1)\rangle]$.

Rosen van Benthem Characterization Theorem

$\mathcal{M}^* \quad \equiv_{\mathsf{FOL}_q} \quad \mathcal{N}^*$



Interpolation

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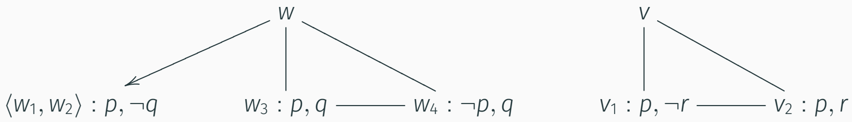
- The counterexamples are actually from what we lack in the bisimulation relation–which is not a bijection.

Counterexample 1

Example

Consider the following two 2-models where $\{\langle w, w_1, w_2\rangle, \langle w, w_3, w_4\rangle\}$ is the ternary relation in the left model $\mathcal{M}_2$, and $\{\langle v, v_1, v_2\rangle\}$ is the ternary relation in the right model $\mathcal{N}_2$.



- $Z = \{\langle w, v\rangle, \langle w_1, v_1\rangle, \langle w_2, v_2\rangle, \langle w_3, v_1\rangle, \langle w_3, v_2\rangle\}$ is a $\{p\}$-wa^2-bisimulation.

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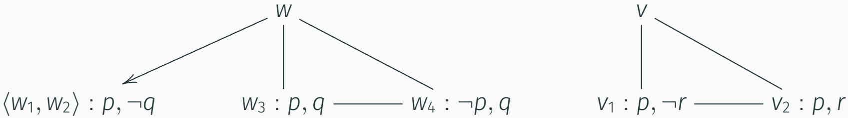

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- $\mathcal{M}_2, w \models \varphi_2 = \Diamond(p \wedge \neg q) \wedge \Box p \wedge \Diamond q \wedge \Box(\neg p \vee \neg q)$; $\mathcal{N}_2, v \models \psi_2 = \Box(r \wedge p) \wedge \Box(\neg r \wedge p) \wedge \Box(\neg p)$.

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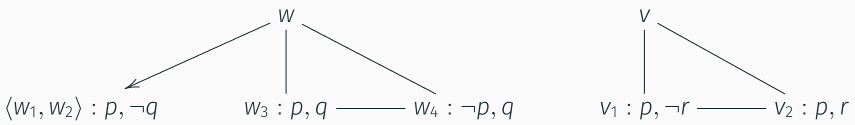

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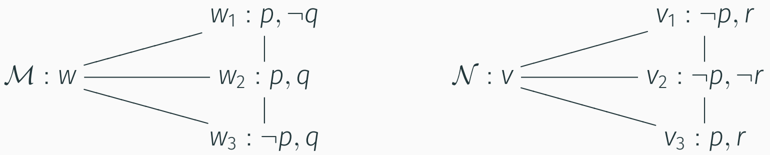
Counterexample 2(Due to Yifeng)

Example

Consider the following two 3-models where $\{\langle w, w_1, w_2, w_3\rangle\}$ is the relation in $\mathcal{M}_3$, and $\{\langle v, v_1, v_2, v_3\rangle\}$ is the relation in $\mathcal{N}_3$.



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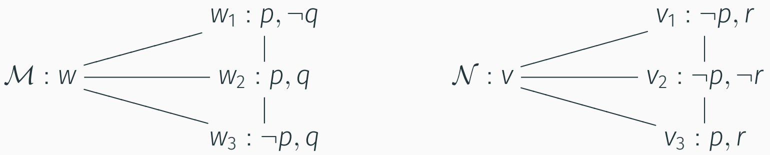

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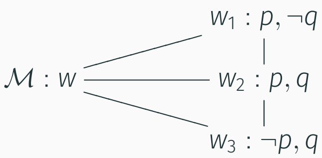

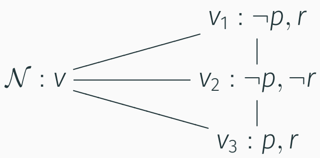

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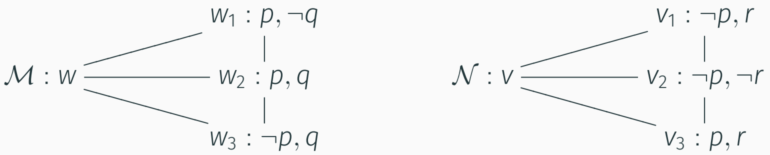

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- $\vdash_3 \varphi_3 \rightarrow \neg\psi_3$.

Proposition

K_2 and K_3 lack the Craig Interpolation Theorem.

Proof.

We will see that the above two are counterexamples for CIP: $\varphi_i \to \neg\psi_i$ do not have any interpolant in K_i. Since the only common proposition letter is p, we assume for contradiction that $\theta_i(p)$ is a interpolant for $\varphi_i \to \neg\psi_i$. Then We have $\vdash_i \varphi_i \to \theta_i(p)$ and $\vdash_i \theta_i(p) \to \neg\psi_3$. By the above two examples we know that $\mathcal{M}_i, w \models \theta_i(p)$, which also means $\mathcal{N}_i, v \models \theta_i(p)$ by p-bisimulation. But it follows that $\mathcal{N}_i, v \models \neg\psi_i$, a contradiction.

$\square$

Repair CIT

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- $\exists\langle a_1, ..., a_n\rangle(wRa_1...a_n \wedge$
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- $\forall A, B((A \cup B \in \nu(w) \rightarrow A \in \nu(w) \vee B \in \nu(w))$.

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- **Proposition (Prop 2.8 in [Pac17])**

  Every first degree modal logic is complete with respect to the class of finite neighborhood frames which validate it.

Let $\mathcal{M} = \langle W, N, V \rangle$ be a neighborhood model, we say $\mathcal{M}^{\bar{Q}}$ is a $\bar{Q}$-variant of $\mathcal{M}$ if they only differ in $\bar{Q}$-valuation. More precisely, $\mathcal{M}^{\bar{Q}} = \langle W, N, V' \rangle$, where $V(p) = V'(p)$ for all $p \notin \bar{Q}$.

Definition

Let $\varphi(\bar{P}, \bar{Q})$ be a modal formula s.t. $\bar{P} \cap \bar{Q} = \emptyset$. Then $\mathcal{M}, w \models \exists \bar{Q} \varphi(\bar{P}, \bar{Q})$ iff there is some $\bar{Q}$-variant $\mathcal{M}^{\bar{Q}}, w \models \varphi(\bar{P}, \bar{Q})$. The formula $\forall \bar{Q} \varphi(\bar{P}, \bar{Q})$ is defined similarly.

$\mathbb{K}_2^c$ has CIT

Proposition

$\mathbb{K}_2^c$ has the Craig interpolation. More precisely, let $\varphi \vdash_{\mathbb{K}_2^c} \psi$, then there is a formula α s.t. $\varphi \vdash_{\mathbb{K}_2^c} \alpha \vdash_{\mathbb{K}_2^c} \psi$ and $atom(\alpha) \subseteq atom(\varphi) \cap atom(\psi)$. Furthermore, $deg(\alpha) \leq max(deg(\varphi), deg(\psi))$.

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- Let $k = max(deg(\varphi), deg(\psi))$. There are only finitely many $\sim_k^P$-equivalence classes.

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Proof.

- Let $k = max(deg(\varphi), deg(\psi))$. There are only finitely many $\sim_k^P$-equivalence classes.
- Claim: For any $\sim_k^P$-equivalence classes C, if there is some $\mathcal{M}, w \in C$ s.t. $\mathcal{M}, w \models \exists \bar{Q} \varphi(\bar{P}, \bar{Q})$, then for all $\mathcal{N}, v \in C$, $\mathcal{N}, v \models \forall \bar{R} \psi(\bar{P}, \bar{R})$.

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- Claim: For any $\sim_k^P$-equivalence classes C, if there is some $\mathcal{M}, w \in C$ s.t. $\mathcal{M}, w \models \exists \bar{Q}\varphi(\bar{P}, \bar{Q})$, then for all $\mathcal{N}, v \in C$, $\mathcal{N}, v \models \forall \bar{R}\psi(\bar{P}, \bar{R})$.
- For each class contain such an $\mathcal{M}, w$, let β_i be the formula defines that class. One can check that $deg(\beta_i) \leq k$ and the disjunction of all such β_i, say $\beta = \bigvee \beta_i$, will be an interpolant.

□

Proof of the Claim

- Suppose for contradiction that there are $\mathcal{M}, w \sim_k^P \mathcal{N}, v$ s.t. $\mathcal{M}, w \models \exists \bar{Q} \varphi(\bar{P}, \bar{Q})$ and $\mathcal{N}, v \models \exists \bar{R} \neg \psi(\bar{P}, \bar{R})$. We need to find another model $\mathcal{D}, u \models \exists \bar{Q} \varphi(\bar{P}, \bar{Q}) \wedge \exists \bar{R} \neg \psi(\bar{P}, \bar{R})$, which is a contradiction.

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- Every finite models of $\mathbb{K}_2^c$ is bisimular with some determined 2-tree model:
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- Every finite models of $\mathbb{K}_2^c$ is bisimular with some determined 2-tree model:
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- Let $\mathcal{M}^* = \langle W_1, R_1, V_1 \rangle$ and $\mathcal{N}^* = \langle W_2, R_2, V_2 \rangle$.
  If $k = 0$, then Let $\mathcal{D} = \langle W, T, V^* \rangle$, where $W = \{(w, v)\}, T = \emptyset$, and V^* is defined as follows:

$$(x, y) \in V^*(p) \iff \begin{cases} x \in V_1(p) & \text{if} \quad p \in Q \\ y \in V_2(p) & \text{if} \quad p \in R \\ \text{never} & \text{if} \quad \text{otherwise} \end{cases}$$

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- Since $w \leftrightarroweq_0^P v$, the definition is well-defined, and it's easy to check the two required bisimulation relation.

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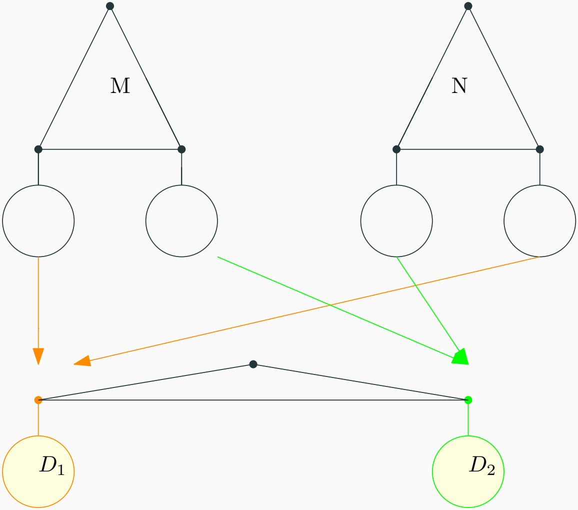

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- Define $\mathcal{D} = \langle W, T, V^* \rangle$ as follows:
  $W = (w^*, v^*) \cup Dom(D_1) \cup Dom(D_2)$;
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  For each $x \in Dom(D_i)$, $x \in V^*(p)$ iff $x \in V_{\mathcal{D}_i}(p)$.
- It's a routine argument to check that $\mathcal{D}, (w^*, v^*)$ satisfies the two bisimulation relations.

Conclusion and Further Work

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- The logic need to have ”tree-like” model property.

Further Work

- Relativized interpolation.

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- Strong interpolation properties: Uniform Interpolation.
- Applications, especially of neighborhood frame: n-filter.

相关论文

- 1. Model Theoretical Aspects of Normal Polyadic Modal Logic: An Exposition. **逻辑学研究已收录**;
- 2. Weakly Aggregative Modal Logic: completeness, characterization and interpolation：**修改中**，**和丁一峰**，**王彦晶老师合作**。

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