What we know and don't know about Craig Interpolation and Propositional Dynamic Logic.

Malvin Gattinger
ILLC, Amsterdam

2016-12-06
Department of Philosophy
Peking University

Based on many discussions with Yde Venema.

Disclaimer

I don't know whether PDL has Craig Interpolation.

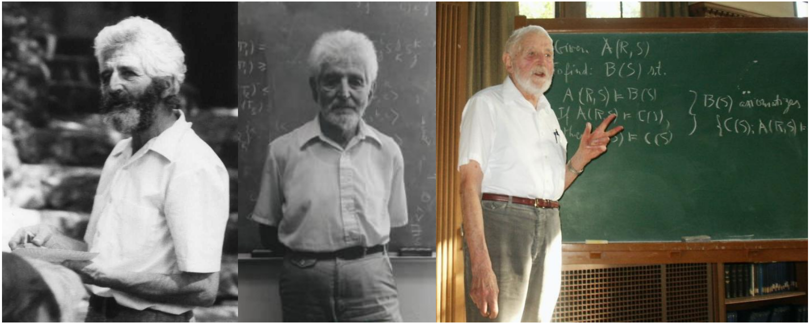
(I don't know whether)^2 PDL has Craig Interpolation.

I don't know whether I know whether PDL has Craig Interpolation.

Contents

Craig-Interpolation

William Craig (November 13, 1918 – January 13, 2016)



- born in Nürnberg
- first proved Interpolation for first-order logic in 1957

Interpolation in Natural Language I

If god exists, then the world will never end and all humans and cats will live forever.

$\Rightarrow$ If god exists and I am a cat, then I will live forever.

Interpolation in Natural Language II

If god exists, then the world will never end and all humans and cats will live forever.

$\Rightarrow$ If god exists, then all cats will live forever.

$\Rightarrow$ If god exists and I am a cat, then I will live forever.

Craig-Interpolation: Definition

Given:

- Λ – logic as set of formulas, given by semantics or proof system
- $L(\varphi)$ – language of a formula
  - e.g. proposition letters: $L(p \to ((r \vee p) \wedge q)) = \{p, q, r\}$

Definition

Λ has *Craig Interpolation* iff for any $\varphi \to \psi \in \Lambda$, there is a μ s.t.:

- $L(\mu) \subseteq L(\varphi) \cap L(\psi)$,
- $\varphi \to \mu \in \Lambda$
- and $\mu \to \psi \in \Lambda$.

Then call μ an *interpolant* for $\varphi \to \psi$.

Example: Propositional Interpolation

Consider: $(q \vee (r \wedge s)) \;\rightarrow\; (\neg q \rightarrow (t \vee s))$

$L(q \vee (r \wedge s)) = \{q, r, s\}$

$L(\neg q \rightarrow (t \vee s)) = \{t, s, q\}$

$L(q \vee (r \wedge s)) \cap L(\neg q \rightarrow (t \vee s)) = \{q, s\}$

Find a φ such that

- $L(\varphi) = \{q, s\}$
- $(q \vee (r \wedge s)) \rightarrow \; \varphi$
- $\varphi \;\rightarrow\; (\neg q \rightarrow (t \vee s))$

Solution: $\varphi = q \vee s$

Craig-Interpolation: We can always find such interpolants.

An Easy Interpolation Proof

Consider Propositional Logic:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi$$

Given $\vDash \varphi \to \psi$, define an interpolant by:

- If $L(\varphi) \subseteq L(\psi)$, then use φ.
- Given $p \in L(\varphi) \setminus L(\psi)$,
  1. Let $\varphi' := [\top/p]\varphi \vee [\bot/p]\varphi$
  2. Find an interpolant for $\varphi' \to \psi$.

An Easy Interpolation ~~Proof~~ Program

```
interpolate :: (Form,Form) -> Form
interpolate (phi, psi)
  | not (isValid (phi --> psi)) = error "Not valid!"
  | atomsIn phi `subseteq` atomsIn psi = phi
  | otherwise = interpolate (phi', psi) where
      p    = head (atomsIn phi \\ atomsIn psi)
      phi' = Disj [ substitute top p phi
                  , substitute bot p phi ]
```

Output:

$\lambda>$ `interpolate (Disj[q,Conj[r,s]],Neg q-->Disj[t,s])`
$((q \vee (\top \wedge s)) \vee (q \vee (\bot \wedge s)))$

(which is equivalent to our guess $q \vee s$ above.)

Proof Methods: Different Roads to Craig Interpolation

- What we just saw: Purely Syntactic
  - constructive
  - no proof system needed
- Not today: Algebraic
  - not constructive
  - amalgamation $\approx$ interpolation
- What we will see: Proof Theoretic
  - sometimes constructive
  - start with a proof of $\varphi \to \psi$
  - construct interpolants for each step of the proof
  - usually done with sequent or tableaux systems

First-Order Logic Example

$$(Eg \rightarrow Lw \wedge \forall x : (Hx \vee Cx \rightarrow Ix)) \;\rightarrow\; (Eg \rightarrow Cm \rightarrow Im)$$

Interpolant: $Eg \rightarrow \forall x : (Cx \rightarrow Ix)$

Interpolation with Proofs

Consider proof trees where nodes are sets of formulas.

Given a proof of $\varphi \to \psi$, define interpolants at each node.

Additionally, use $L(\cdot)$ and $R(\cdot)$ for bookmarking.

Propositional Example:

$$\frac{S \cup \{L(X)\} \overset{\text{int}}{\to} A \qquad S \cup \{L(Y)\} \overset{\text{int}}{\to} B}{S \cup \{L(X \vee Y)\} \overset{\text{int}}{\to} A \vee B} \vee$$

Proper FOL Example: $\forall$ rules look like this:

$$\frac{S \cup \{L(\gamma(c))\} \overset{\text{int}}{\to} A}{S \cup \{L(\forall x \gamma(x))\} \overset{\text{int}}{\to} \forall x[c/x]A} \forall_3$$

(Melvin Fitting: *First-Order Logic and Automated Theorem Proving* [5])

Logics that have Craig Interpolation

- Propositional Logic
- First-Order Logic
- Intuitionistic Logic
- Basic and Multi-modal logic [11]
- μ-Calculus [1]

- What about Propositional Dynamic Logic (PDL)?
  Yde Venema in the Amsterdam course on Model Theory:
  "By the way, for PDL this is an open question . . . "

Propositional Dynamic Logic

Propositional Dynamic Logic (PDL)

Michael J. Fischer and Richard E. Ladner

"fundamental propositional logical system based on modal logic for describing correctness, termination and equivalence of programs." [4]

Related Topics: regular expressions, automata theory, multi-agent knowledge, programming language semantics, . . .

PDL: Basic Definitions

Syntax

Formulas and Programs:

$$\begin{array}{lcl} \varphi & ::= & p \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi \rightarrow \varphi \mid \langle\alpha\rangle\varphi \\ \alpha & ::= & a \mid \alpha;\alpha \mid \alpha \cup \alpha \mid \alpha^* \end{array}$$

Models

$\mathcal{M} = (W, \mathcal{R}, V)$ where

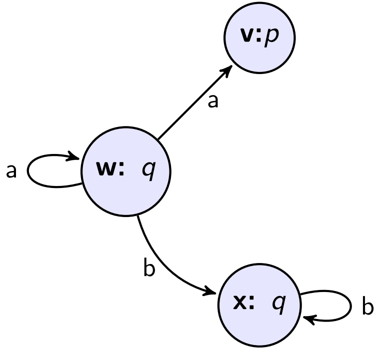
- W: set of worlds/states
- $\mathcal{R} = (R_\xi)_\xi$: family of binary relations on W such that
  - $R_{\chi;\xi} = R_\chi; R_\xi$ (consecution)
  - $R_{\chi\cup\xi} = R_\chi \cup R_\xi$ (union)
  - $R_{\chi^*} = (R_\chi)^*$ (reflexive-transitive closure)
  - $R_{\varphi?} = \{w \in W \mid w \vDash \varphi\}$ (where $\vDash$ is on the next slide)
- $V : \Phi \rightarrow \mathcal{P}(W)$: valuation function

PDL: Semantics

- $\mathcal{M}, w \models p$ iff $w \in V(p)$
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$
- $\mathcal{M}, w \models \varphi \vee \psi$ iff $\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \varphi \wedge \psi$ iff $\mathcal{M}, w \models \varphi$ and $\mathcal{M}, w \models \psi$
- $\mathcal{M}, w \models \varphi \to \psi$ iff $\mathcal{M}, w \not\models \varphi$ or $\mathcal{M}, w \models \psi$

- $\mathcal{M}, w \models [\alpha]\varphi$ iff for all $w' \in W : \; wR_\alpha w' \; \Rightarrow \; \mathcal{M}, w' \models \varphi$.

PDL: Example



$\mathcal{M}, w \models \langle a; b \rangle q$
$\mathcal{M}, w \models \neg[a \cup b]p$
$\mathcal{M}, w \models [b]q$
$\mathcal{M}, w \models [b^*]q$
$\mathcal{M}, w \models [b; a]\bot$

$\mathcal{M}, w \models \langle a \rangle(\langle a \rangle \neg q \wedge \langle b \rangle [b^*]q)$

PDL Axioms

Axioms:

- all propositional tautologies
- $[\alpha](\varphi \wedge \psi) \leftrightarrow ([\alpha]\varphi \wedge [\alpha]\psi)$
- $[\alpha; \beta]\varphi \leftrightarrow [\alpha][\beta]\varphi$
- $[\alpha \cup \beta]\varphi \leftrightarrow ([\alpha]\varphi \wedge [\beta]\varphi)$
- $[\alpha^*]\varphi \leftrightarrow (\varphi \wedge [\alpha][\alpha^*]\varphi)$
- $[\psi?]\varphi \leftrightarrow (\psi \rightarrow \varphi)$

Rules:

- Modus Ponens: $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$ imply $\vdash \psi$.
- Distribution: $\vdash \varphi \rightarrow \psi$ implies $\vdash [\alpha]\varphi \rightarrow [\alpha]\psi$
- Induction: $\vdash \varphi \rightarrow [\alpha]\varphi$ implies $\vdash \varphi \rightarrow [\alpha^*]\varphi$.

This Hilbert-style proof system is not useful to find interpolants ☹

PDL: Language of a formula

$$\begin{aligned}
L(p) &:= \{p\} \\
L(\varphi \wedge \psi) &:= L(\varphi) \cup L(\psi) \\
L(\varphi \vee \psi) &:= L(\varphi) \cup L(\psi) \\
L(\varphi \rightarrow \psi) &:= L(\varphi) \cup L(\psi) \\
L(\langle \tau \rangle \varphi) &= L(\tau) \cup L(\varphi)
\end{aligned}
\qquad
\begin{aligned}
L(a) &:= \{a\} \\
L(\sigma; \tau) &:= L(\sigma) \cup L(\tau) \\
L(\sigma \cup \tau) &:= L(\sigma) \cup L(\tau) \\
L(\sigma^*) &:= L(\sigma)
\end{aligned}$$

Example: $L([a; b]p \rightarrow \langle c \rangle q) = \{a, b, c, p, q\}$

Craig Interpolation for PDL

Why is this difficult?

Short answer: The star.

Without star PDL is multi-modal logic and we would be done. [11]

- But how do we systematically remove programs under the star to get interpolants in which those must not occur?

And more:

- Note that with tests we have a double recursion:
  - formulas in programs
  - programs in formulas
- The proof system contains an infinitary induction rule!

Interpolation via Translation?

Wait, but we can translate PDL to the μ-Calculus, right?

Yes, but this does not give us interpolants:

0. Let $t : \mathcal{L}_{PDL} \to \mathcal{L}_\mu$ be the translation.
1. Suppose $\vDash_{PDL} \varphi \to \psi$.
2. Then we have $\vDash_\mu t(\varphi) \to t(\psi)$.
3. μ-Calculus has C-I, there is an interpolant $\gamma_\mu \in \mathcal{L}_\mu$:
   - $\vDash_\mu t(\varphi) \to \gamma_m u$
   - $\vDash_\mu \gamma_m u \to t(\psi)$
   - $L(\gamma_\mu) = L(t(\varphi)) \cap L(t(\psi))$
4. But now we still need $\gamma \in \mathcal{L}_{PDL}$ such that $t(\gamma) = \gamma_\mu$!?!?

☹

History

Whether PDL has Craig-Interpolation seems to be an open question.

But there are at least three proof(attempt)s!

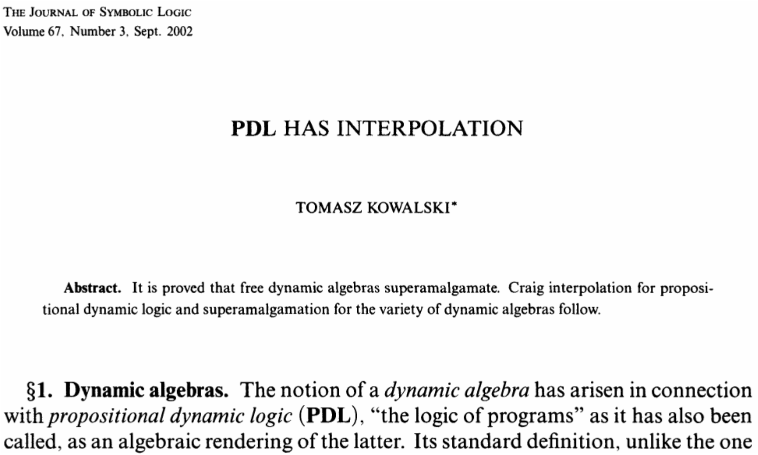
- Daniel Leivant: *Proof theoretic methodology for propositional dynamic logic*. Conference paper in LNCS, 1981.
- Manfred Borzechowski: *Tableau–Kalkül für PDL und Interpolation*. Diploma thesis, FU Berlin, 1988. Unpublished.
- Tomasz Kowalski: *PDL has interpolation*. Journal of Symbolic Logic, 2002. Revoked in 2004.

History

Other notable references:

- Marcus Kracht: Chapter *The open question* in *Tools and Techniques in Modal Logic*, 1999.
- D'Agostino & Hollenberg: *Logical questions concerning the μ-Calculus: Interpolation, Lyndon and Łoś-Tarski*. JSL, 2000.
- Johan van Benthem: *The many faces of Interpolation*. Synthese, 2008.

Kowalski 2002


THE JOURNAL OF SYMBOLIC LOGIC
Volume 67, Number 3, Sept. 2002

PDL HAS INTERPOLATION

TOMASZ KOWALSKI*

Abstract. It is proved that free dynamic algebras superamalgamate. Craig interpolation for propositional dynamic logic and superamalgamation for the variety of dynamic algebras follow.

§1. Dynamic algebras. The notion of a *dynamic algebra* has arisen in connection with *propositional dynamic logic* (**PDL**), "the logic of programs" as it has also been called, as an algebraic rendering of the latter. Its standard definition, unlike the one

Kowalski 2004

THE JOURNAL OF SYMBOLIC LOGIC
Volume 69, Number 3, Sept. 2004

RETRACTION NOTE FOR
"**PDL** HAS INTERPOLATION"

TOMASZ KOWALSKI

In this journal I published a paper [1] entitled "**PDL** has interpolation" purporting to prove what the title announced. It has been pointed out to me by Yde Venema that my argument contains a serious error. As I have not been able to correct it, the problem of interpolation for Propositional Dynamic Logic is still open.

Leivant 1981

Proof theoretic methodology for Propositional Dynamic Logic

Daniel Leivant
Department of Computer Science
Cornell University
Ithaca, NY 14853, USA

Abstract. We relate by syntactic techniques finitary and infinitary axiomatizations for the iterator-construct * of Propositional Dynamic Logic PDL. This is applied to derive the Interpolation Theorem for PDL, and to provide a new proof of the semantic completeness of Segerberg's axiomatic system for PDL.

Contrary to semantic techniques used to date, our proof of completeness is relatively insensitive to changes in the language and axioms used, provided some minimum syntactic closure properties hold. For instance, the presence of the test-operator adds no difficulty, and the proof also establishes the Interpolation Theorem and the closure under iteration of a constructive variant of PDL.

Simplifying the question

Completeness of the original axioms by Segerberg is also shown by Leivant, but not our interest here.

In ~~2014~~ 2016 we also know:

- PDL does not have uniform interpolation. [1]
- Test-free PDL has interpolation iff PDL has. [8]

Hence we can

- really stop looking at the μ-calculus for help
- reduce the syntax a bit

Proof Outline

- Define a sound and complete sequent calculus for PDL.
- Use Maehara's method to show Partition-Interpolation.
  - Show that the calculus has the "step-by-step property".
  - For the $*$ case, find a repetitive scheme in long enough proofs.
  - Use linear transformations of programs to imply a $*$ formula.
- Check that Partition-Interpolation implies Craig Interpolation.

Sequent Calculus for PDL

Notation

X, Y, Z: formulas

f, g: sets of formulas

α, β: programs

Sequent example: $f, X \vdash Y$

Proof example

$$\dfrac{\dfrac{\dfrac{\dfrac{[a]p \ \vdash \ [a]p}{[a]p, [b]p \ \vdash \ [a]p}\,\text{WEAK} \qquad \dfrac{[b]p \ \vdash \ [b]p}{[a]p, [b]p \ \vdash \ [b]p}\,\text{WEAK}}{[a]p, [b]p \ \vdash \ [a \cup b]p}\,(\cup\text{R})}{[a]p \ \vdash \ [b]p \to [a \cup b]p}\,(\to \text{R})}{\vdash \ [a]p \to ([b]p \to [a \cup b]p)}\,(\to \text{R})$$

Let CD be the following proof system where g is $\emptyset$ or a singleton.

$$(\neg R)\ \frac{f, X \vdash}{f, \vdash \neg X} \qquad (\neg L)\ \frac{f \vdash X}{f, \neg X \vdash}$$

$$(\to R)\ \frac{f, X \vdash Y}{f \vdash X \to Y} \qquad (\to L)\ \frac{f \vdash X \qquad f, Y \vdash g}{f, X \to Y \vdash g}$$

$$(;R)\ \frac{f \vdash [\alpha][\beta]X}{f \vdash [\alpha;\beta]X} \qquad (;L)\ \frac{f, [\alpha][\beta]X \vdash g}{f, [\alpha;\beta]X \vdash g}$$

$$(\cup R)\ \frac{f \vdash [\alpha]X \qquad f \vdash [\beta]X}{f \vdash [\alpha \cup \beta]X} \qquad (\cup L)\ \frac{f, [\alpha]X, [\beta]X \vdash g}{f, [\alpha \cup \beta]X \vdash g}$$

$$(*L)\ \frac{f, X, [\alpha][\alpha^*]X \vdash g}{f, [\alpha^*]X \vdash g} \qquad (*R)\ \frac{f \vdash \varphi \qquad f \vdash [\alpha]\varphi \qquad \cdots \qquad f \vdash [\alpha]^k\varphi}{f \vdash [\alpha^*]\varphi}$$

where $k = 2^{|f|+|\varphi|}$

$$(\mathrm{GEN})\ \frac{f \vdash X}{[\alpha]f \vdash [\alpha]X} \qquad (\mathrm{WEAK})\ \frac{f \vdash g}{f' \vdash g'}$$

where $f \subseteq f'$ and $g \subseteq g'$.

Completeness

Theorem (Leivant 1981)
CD is a intuitionistic/constructive variant of D which is a sound an complete system for PDL, i.e. we have:

$$\vDash X \text{ iff } \vdash_{\mathrm{D}} X \text{ iff } \vdash_{\mathrm{CD}} X^0$$

where X^0 is the result of inserting $\neg\neg$ in front of everything in X.

NB: CD is *not* sound and complete for intuitionistic/constructive PDL.

Remaining goal: Show that CD has interpolation.

Maehara's Method for Partition-Interpolation

Idea

Find interpolants by going along the proof tree.
Given the previous interpolants, we define the next one.

Example

Suppose the last step is $\cup$R:

$$\dfrac{\dfrac{\vdots}{f \ \vdash \ [\alpha]X} \qquad \dfrac{\vdots}{f \ \vdash \ [\beta]X}}{f \vdash [\alpha \cup \beta]X} \, (\cup\mathrm{R})$$

Given any two interpolants Z_1 and Z_2 for $f \vdash [\alpha]X$ and $f \vdash [\beta]X$, let $Z := Z_1 \wedge Z_2 = \neg(Z_1 \to \neg Z_2)$. This interpolates $f \vdash [\alpha \cup \beta]X$.

(See [13] for a detailed explanation.)

Partition-Interpolation

Definition

Given a sequent $f \vdash X$ and a partition of f into $f^-; f^+$, we say that K is an interpolant for $f^-; f^+ \vdash X$ iff

$$L(K) \subseteq L(f^-) \cap L(f^+, X) \;\text{ and }\; f^- \vdash K \;\text{ and }\; f^+, K \vdash X$$

Lemma 5.3.1 (Leivant 1981)

Let $f^-; f^+$ be any partition of f and q not occur in f.

1. If $f \vdash_{\mathrm{CD}} X$, then there is an interpolant for $f^-; f^+ \vdash X$.
2. Suppose P is a proof of $f \vdash [\alpha]q$ from $\{f_i \vdash q\}_{i<k}$ and let $f_i^-; f_i^+$ be the partitions of f_i induced by $f^-; f^+$ for all $i < k$. If K_i is an interpolant for $f_i^-; f_i^+ \vdash X$ for all $i < k$, then there is an interpolant of the form $\bigwedge_i [\beta_i] K_i$ for $f^-; f^+ \vdash [\alpha]X$.

Proof. By tree-induction on P, simultaneously for (i) and (ii).

Partition-Interpolation: Easy Warm-Up Case

Suppose the last step is $\to$ L:

$$\frac{f \ \vdash \ X \qquad f, Y \ \vdash \ Z}{f, X \to Y \ \vdash \ Z} \, (\to \mathsf{L})$$

Case a) partition $f^-, X \to Y; f^+$. By induction hypothesis:

- $f^+; f^- \vdash X$ (Note: flipped!) yields K_1 such that

  $$L(K_1) \subseteq L(f^+) \cap L(f^-, X) \text{ and } f^+ \vdash K_1 \text{ and } f^-, K_1 \vdash X$$

- $f^-, Y; f^+ \vdash Z$ yields K_2 such that

  $$L(K_2) \subseteq L(f^-, Y) \cap L(f^+, Z) \text{ and } f^-, Y \vdash K_2 \text{ and } f^+, K_2 \vdash Z$$

Let $K := K_1 \to K_2$. This is interpolates $f^-, X \to Y; f^+ \vdash Z$.

Case b) partition $f^-; X \to Y, f^+$. Then $K := K_1 \wedge K_2$ works.

Partition-Interpolation: The evil $*$ case

Suppose the last step of P is ($*$R). For each $h = 1 \leq M$ let P_h be the proof of $f \vdash [\alpha]^h X$ occurring in P above this premise:

$$\dfrac{\dfrac{P_0}{f \ \vdash \ X} \qquad \dfrac{P_1}{f \ \vdash \ [\alpha]X} \qquad \cdots \qquad \dfrac{P_M}{f \ \vdash \ [\alpha]^M X}}{f \ \vdash \ [\alpha^*]X} \, (*\mathrm{R})$$

Note: all active formulas on the right. Hence, only consider the given partition f^-, f^+ without further manipulation.

Given: M many interpolants. Goal: find a formula K such that

$$L(K) \subseteq L(f^-) \cap L(f^+, [\alpha^*]X) \text{ and } f^- \vdash K \text{ and } f^+, K \vdash [\alpha^*]X$$

How?!

Down the rabbit hole . . .

Nice Properties of Long Proofs: Positive Closure

Definition

The *positive closure* of f, denoted by $\mathrm{PC}(f)$, is the smallest set $g \supseteq f$ such that:

- If $(X \rightarrow Y) \in g$, then $Y \in g$.
- If $[\alpha]X \in g$, then $X \in g$.
- If $[\alpha;\beta]X \in g$, then $[\alpha][\beta]X \in g$.
- If $[\alpha \cup \beta]X \in g$, then $[\alpha]X \in g$ and $[\beta]X \in g$.
- If $[\alpha^*]X \in g$, then $[\alpha][\alpha^*]X \in g$.

Note: Whenever f is finite, $\mathrm{PC}(f)$ is also finite.

Nice property 1

In certain proofs, $\mathrm{PC}(\cdot)$ is preserved in the following sense.

Lemma 4.2.1 (Leivant 1981, revision Venema 2014)
If P proves $f \vdash [\beta_1] \dots [\beta_k][\alpha]^m q$ from $\{f_i \vdash q\}_i$ where $q \notin L(f)$, all β_is are subprograms of α, $r < m$ and $f' \vdash [\alpha]^r q$ is a sequent in P (under a non-initial leaf) then $PC(f') \subseteq PC(f)$.

The case we need is $k = 0$.

Nice property 2

Definition
Let $P[X/q]$ be the result of substituting X for q in P.

Lemma 4.2.2 (Leivant 1981)
Suppose P proves $f \vdash [\alpha]^r X$ from $\{f_i \vdash X\}_i$ where $X \notin PC(f)$. Then there is a proof P' of $f \vdash [\alpha]^r q$ from $\{f_i' \vdash q\}_i$ such that $P = P'[X/q]$.

Intuitively, this means that P does not take X "apart":

$$\frac{\dfrac{\{f_i \vdash X\}_i}{\vdots}}{f \vdash [\alpha]^r X} = \left(\frac{\dfrac{\{f_i \vdash q\}_i}{\vdots}}{f \vdash [\alpha]^r q} \right) [X/q]$$

Nice property 3: Step by Step

Suppose P is a CD-proof of $f \vdash [\alpha]^n X$. Then P consists of proof parts $P_0, \ldots, P_n$ which build up the $[\alpha]$s "step by step":

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{P_0}{\{f_j \vdash X\}_{j \in I_0}}}{P_1}}{\{f_j \vdash [\alpha]X\}_{j \in I_1}}}{P_2}}{\{f_j \vdash [\alpha]^2 X\}_{j \in I_2}}}{\vdots}}{\cfrac{\{f_j \vdash [\alpha]^{n-1} X\}_{j \in I_{n-1}}}{\cfrac{P_n}{f \ \vdash \ [\alpha]^n X}}}$$

NB: This looks more linear than it actually is!

Linear Transformations

Think of programs and formulas as a vector space:

$$(\beta)\vec{Y} = \begin{pmatrix} \beta_{1,1} & \cdots & \beta_{1,k} \\ \vdots & \ddots & \vdots \\ \beta_{k,1} & \cdots & \beta_{k,k} \end{pmatrix} \begin{pmatrix} Y_1 \\ \vdots \\ Y_k \end{pmatrix} := \begin{pmatrix} [\beta_{1,1}]Y_1 \wedge \cdots \wedge [\beta_{1,k}]Y_k \\ \vdots \\ [\beta_{k,1}]Y_1 \wedge \cdots \wedge [\beta_{k,k}]Y_k \end{pmatrix}$$

Lemma
For every $k \times k$ matrix (β) there exists a (γ) such that

$$(\gamma) \equiv (\beta)^* = (\beta)(\beta)(\beta)\ldots$$

Linear Transformations: Example

Let $\vec{Y} = \langle p, q \rangle$ and $(\beta) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $(\beta)\vec{Y} = \begin{pmatrix} [a]p \wedge [b]q \\ [c]p \wedge [d]q \end{pmatrix}$

and $(\beta)(\beta)\vec{Y} = \begin{pmatrix} [a]([a]p \wedge [b]q) \wedge [b]([c]p \wedge [d]q) \\ [c]([a]p \wedge [b]q) \wedge [d]([c]q \wedge [d]q) \end{pmatrix} \ldots$

Let $\gamma := \begin{pmatrix} (a \cup (b; (d^*; c)))^* & (a^*; b)((c; a^*; b) \cup d)^* \\ (d^*; c)(a \cup (b; (d^*; c)))^* & ((c; a^*; b) \cup d)^* \end{pmatrix}$

Then $(\gamma) \equiv (\beta)^*$ and $(\beta)^* \vec{Y} \equiv (\gamma)\vec{Y}$.

This γ can be found systematically. Moreover, it is useful:

$$p \wedge [a]p \wedge ([a]p \wedge [b]q) \wedge ([a]([a]p \wedge [b]q) \wedge [b]([c]p \wedge [d]q)) \wedge \ldots$$

$$\equiv \quad [(a \cup (b; (d^*; c)))^*]p \wedge [(a^*; b)((c; a^*; b) \cup d)^*]q$$

Putting it all together: Back to the evil $*$ case

Now we can deal with this:

$$\dfrac{\dfrac{P_0}{f \;\vdash\; X} \qquad \dfrac{P_1}{f \;\vdash\; [\alpha]X} \qquad \cdots \qquad \dfrac{P_M}{f \;\vdash\; [\alpha]^M X}}{f \;\vdash\; [\alpha^*]X}\;(*\mathrm{R})$$

Fix a ridiculously large $h := s + v + d$ where

- d such that $[\alpha]^d X \notin \mathrm{PC}(f)$
- $v := 2^{|\mathrm{PC}(f)|} \cdot 2^{|f|} + 1$
- $s := 1$ (for now).

Apply the step by step property to P_h:

$$\dfrac{\dfrac{Q_i}{\{f_i^-; f_i^+ \vdash [\alpha]^d X\}_{i \in I_d}}}{\dfrac{\vdots}{f^-; f^+ \vdash [\alpha]^{d+v+s} X}}$$

Putting it all together: Finding a repetitive pattern

Now P_h has to look like this:

$$\dfrac{\dfrac{\dfrac{\dfrac{Q_{j,i}}{\{f_i^-; f_i^+ \vdash [\alpha]^d X\}_{i \in I_d}}}{R_j'[[\alpha]^d X/q]}}{\{f_j^-; f_j^+ \vdash [\alpha]^{d+v} X\}_{j \in I_{d+v}}}}{\dfrac{U'[[\alpha]^{d+v} X/q]}{f^-; f^+ \vdash [\alpha]^{d+v+s} X}}$$

For all $c \leq v$, $j \in I_{d+c}$: $f_i \subseteq \mathrm{PC}(f_i) \subseteq \mathrm{PC}(f)$ and $|\mathcal{P}(f_j)| \leq |\mathcal{P}(f)$

Hence $| \cup \{\mathcal{P}(f_j) \mid c \leq v, j \in I_{d+c}\} \leq |\mathcal{P}(\mathrm{PC}(f)| \cdot |\mathcal{P}(f)|$

$= 2^{|\mathrm{PC}(f)|} \cdot 2^{|f|} = v - 1 < v$.

Repetitive Pattern
For some $m \neq n$ we have $\{f_j^+; f_j^- \mid j \in I_m\} = \{f_j^+; f_j^- \mid j \in I_n\}$.

Furthermore, we can assume $d < m < n < d + v$ and $I_m = I_n$.

Putting it all together: Applying the induction hypothesis

Let r be such that $n = m + r$. Now P_h can be divided as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{Q_i}{\{f_i^-; f_i^+ \vdash [\alpha]^m X\}_{i \in I}}}{R'_j[[\alpha]^m X / q]}}{\{f_j^-; f_j^+ \vdash [\alpha]^{m+r} X\}_{j \in I}}}{U'[[\alpha]^{m+r} X / q]}}{f_i^-; f_i^+ \vdash [\alpha]^{m+r+s} X}$$

IH(i) yields $\vec{K}$ such that K_i interpolates $f_i^-; f_i^+ \vdash [\alpha]^m X$.

Using IH(ii) r times: If $\vec{M}$ contains interpolants for $f_i^-; f_i^+ \vdash Y$, then there is a matrix (β) such that $((\beta)M)_i$ interpolates $f_i^-; f_i^+ \vdash [\alpha]^r Y$.

Thus, for all n, by applying the latter to the former n times:

$$f_i^- \vdash ((\beta)^n K)_i \quad \text{and} \quad f_i^+, ((\beta)^n K)_i \vdash [\alpha]^m [\alpha]^{r \times n} X$$

Putting it all together: Done, repeat.

By linear transformations there is a γ such that:

$$f_i^- \vdash ((\gamma)K)_i \quad \text{and} \quad f_i^+, ((\gamma)K)_i \vdash [\alpha]^m[(\alpha^r)^*]X$$

Now apply IH(ii) to all the $((\gamma)K)_i$s and U'.

This yields an interpolant H_s for $f^-; f^+ \vdash [\alpha]^s[\alpha]^m[(\alpha^r)^*]X$.

Repeat all of the above to obtain $H_1, \ldots, H_{v+d}$.

Finally, let $K := \bigwedge_{s \leq v+d} H_s$. This interpolates $f^-; f^+ \vdash [\alpha^*]X$. ☺

Lemma
$\vdash_{\mathrm{CD}} \bigwedge_{k<w} [\alpha^k][(\alpha^w)^*]X \to [\alpha^*]X$

Putting it all together: This is the end, I promise.

Theorem 5.3.2 (i) (Leivant 1981)
PDL has Craig Interpolation.

Proof. Take any $\models X \to Y$. D is complete, hence $\vdash_D X \to Y$.
Then $\vdash_{CD} X^o \to Y^o$ and thus $X^o \vdash_{CD} Y^o$.

Partition-interpolation of $X^o; \varnothing \vdash Y^o$ yields Z such that

- $L(Z) \subseteq L(X^o) \cap L(\varnothing, Y^o)$,
- $X^o \to Z \in$ **PDL** and $Z \to Y^o \in$ **PDL**

By $X^o \equiv X$, $Y^o \equiv Y$, $L(X^o) = L(X)$ and $L(Y^o) = L(Y)$:

- $L(Z) \subseteq L(X) \cap L(Y)$,
- $X \to Z \in$ **PDL** and $Z \to Y \in$ **PDL**

Hence Z is an interpolant for $X \to Y$.

Criticism

Marcus Kracht: *Tools and techniques in modal logic*. (1999)
Chapter *10.6. The Unanswered Question*:

> *"[T]he problem of interpolation for **PDL** is one of the major open problems in this area. Twice a solution has been announced [. . .], but in neither case was it possible to verify the argument.*
> *The argument of Leivant makes use of the fact that if $\varphi \vdash_{\textbf{PDL}} \psi$ then we can bound the size of a possible countermodel so that the star α^* only needs to search up to a depth d which depends on φ and ψ."*

[8, p. 493]

Criticism

Marcus Kracht (continued):

> *"The argument of Leivant makes use of the fact that if $\varphi \vdash_{\textbf{PDL}} \psi$ then we can bound the size of a possible countermodel so that the star α^* only needs to search up to a depth d which depends on φ and ψ. Once that is done, we have reduced* ***PDL*** *to* ***EPDL****, which definitely has interpolation because it is a notational variant of polymodal* ***K****. However, this is tantamount to the following. Abbreviate by* ***PDL****^{n} the strengthening of* ***PDL*** *by axioms of the form $[a^*]p \leftrightarrow [a^{\leq n}]p$ for all a. Then, by the finite model property of* ***PDL***, ***PDL*** *is the intersection of the logics* ***PDL****^{n} . Unfortunately, it is not so that interpolation is preserved under intersection."*

[8, p. 493]

PDL and $\mathbf{PDL}^n$

Definition

Semantic closure $\mathrm{SCL}(A) := \{\varphi \mid A \vDash \varphi\}$

$[\alpha^{\leq n}]\varphi := \varphi \wedge [\alpha]\varphi \wedge [\alpha;\alpha]\varphi \wedge \cdots \wedge [\alpha^n]\varphi$

$\mathbf{PDL}^n := \mathrm{SCL}\left(\mathbf{PDL} \cup \{[\alpha^*]p \leftrightarrow [\alpha^{\leq n}]p \mid \alpha \in PROG, p \in \mathfrak{P}\}\right)$

Theorem

$$\mathbf{PDL}^0 \supseteq \mathbf{PDL}^1 \supseteq \mathbf{PDL}^2 \supseteq \cdots \supseteq \mathbf{PDL} = \bigcap_n \mathbf{PDL}^n$$

Idea / Question

Is there an n, depending on $|\varphi \to \psi|$ such that any $\mathbf{PDL}^n$-interpolant for $\varphi \to \psi$ is also a **PDL**-interpolant?

Refuting the Criticism

But this is not what Leivant was doing:

$$(*\mathrm{R})\ \frac{f \ \vdash \ \varphi \qquad f \vdash [\alpha]\varphi \qquad \cdots \qquad f \vdash [\alpha]^k\varphi}{f \vdash [\alpha^*]\varphi}$$

where $k = 2^{|f|+|\varphi|}$ and therefore depends on f and φ.

Theorem: Finite-Model Property
If φ is satisfiable, then there is a model $\mathcal{M} = (W, \mathcal{R}, V)$ and a world $w \in W$ such that $\mathcal{M}, w \vDash \varphi$ and $|W| \leq 2^{\mathsf{size}(\varphi)}$.

Lemma
If $\vDash \bigwedge f \to [\alpha]^n\varphi$ for all $n \leq k = 2^{|f|+|\varphi|}$, then $\vDash \bigwedge f \to [\alpha^*]\varphi$.

Theorem
The finitary rule is admissible.

Conclusion (for now . . .)

- There is a finitary sequent calculus for **PDL**.
  (In particular, Kracht's criticism does not apply.)
- This system has the "step by step" property.
- Therefore we can:
  - find a repetitive pattern in long enough proofs.
  - use linear transformations to build $*$ interpolants.
- This extends Maehara's method to show Craig Interpolation.

All this [c $\cup$ sh]ould have been known since 1981.

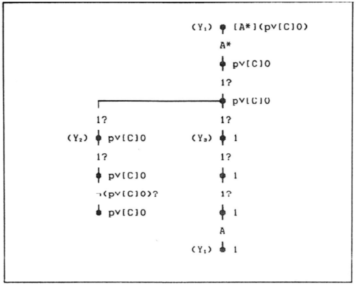
Please: We need more people to look at this proof.

Epilogue

Kracht: “**Twice** a solution has been announced . . . ”

Borzechowski 1988: unpublished, unknown and unread?

There is hope!



Ergebnis der Konstruktion ist also, daß I_0=[A*](p∨[C]0) ein Interpoland für die Formeln [(A;A)*](p∧[A;(B∪C)]0) und [A*](p∨[C]q) ist. Auf Grund der vorgenommenen Überlegungen, einen nicht unnötig großen Interpolanden zu konstruieren, ist I_0 sogar frei von Tests. Dieses Ergebnis ist jedoch nicht immer erreichbar.

Thank You for Listening!

And special thanks to Yanjing for hosting me at PKU!

`malvin@w4eg.de`

`https://w4eg.de/malvin`

(also online: note on PDL and C-I.)

References I

- Giovanna D'Agostino, Marco Hollenberg: *Logical Questions Concerning the μ-Calculus: Interpolation, Lyndon and Łoś-Tarski*. In: *Journal of Symbolic Logic*, Volume 65, Issue 1, March 2000, pp 310-332.
- Johan van Benthem: *The many faces of interpolation*. In: *Synthese*, Volume 164, 2008, pp 451–460.
- Manfred Borzechowski: *Tableau–Kalkül für PDL und Interpolation*. Diplomarbeit, Department of Mathematics, FU Berlin, 1988.
- Michael J. Fischer, Richard E. Ladner: *Propositional dynamic logic of regular programs*. In: *Journal of Computer and System Sciences*, Volume 18, Issue 2, April 1979, pp 194-211.
- Melvin Fitting: *First-Order Logic and Automated Theorem Proving*. Springer-Verlag, New York 1996.
- Tomasz Kowalski: *PDL has Interpolation*. In: *Journal of Symbolic Logic*, Volume 67, Issue 3, September 2002, pp 933-946.
- Tomasz Kowalski: *Retraction note for »PDL has interpolation«*. In: *Journal of Symbolic Logic*, Volume 69, Issue 3, September 2004, p 935.

References II

Marcus Kracht: *Tools and techniques in modal logic*.
Elsevier, 1999.

Dexter Kozen: *A Completeness Theorem for Kleene Algebras and the Algebra of Regular Events*. In: *Information and Computation*, Volume 110, Issue 2, 1994, pp 366–390.

Daniel Leivant: *Proof theoretic methodology for propositional dynamic logic*. In: *Lecture Notes in Computer Science*, Volume 107, 1981, pp 356-373.

Judit Madarász: *The Craig interpolation theorem in multi-modal logics*. In: *Bulletin of the Section of Logic*, Volume 3.24, 1995, pp 147-151.

Krister Segerberg: *A completeness theorem in the modal logic of programs*. In: *Notices American Mathematical Society*, Volume 24, 1977, A522. Republished in: *Banach Center Publications*, Volume 9.1, 1982, pp 31-46.

Gaisi Takeuti: *Proof Theory*. North-Holland, Amsterdam/Oxford 1975.