A Spectrum of Modes of Knowledge Sharing between Agents

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Motivation of the author:

- The modal logic $S5_n$ has been used to model knowledge in multi-agent systems (MAS) for some years now, which expresses the private knowledge of perfect reasoners.
- A peculiarity of the logic $S5_n$, is that there is no *a priori* relationship between the knowledge of the various agents. In some applications, however, this might not be what is desired.

More than $S5_n$ is needed

For example, if agents have computation capabilities that can be ordered. If the agents are executing the same program on the same data then it is reasonable to model the MAS by enriching the logic $S5_n$ by:

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For example, if agents have computation capabilities that can be ordered. If the agents are executing the same program on the same data then it is reasonable to model the MAS by enriching the logic $S5_n$ by:

$$\Box_i p \to \Box_j p; \ i \prec j$$

where $\prec$ expresses the order in the computational power at disposal of the agents.
In this case some information is being shared among the agents of the group.

More than $S5_n$ is needed

A second example of sharing is the axiom

$$\Diamond_i \Box_j p \to \Box_j \Diamond_i p;\ i \neq j$$

which says that:

More than $S5_n$ is needed

A second example of sharing is the axiom

$$\Diamond_i \Box_j p \rightarrow \Box_j \Diamond_i p;\ i \neq j$$

which says that:*if agent i considers possible that agent j knows p then agent j must know that agent i considers possible that p is the case.*

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saying that *the agents have precisely the same knowledge (total sharing)*.

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The two examples mentioned above exist somewhere in the (partially ordered) spectrum between these two extremes.

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$$\boxdot p \to \boxdot p$$
$$\boxdot p \to \boxdot \boxdot p$$
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where each occurrence of $\boxdot$ is in the set $\{\Box_1, \Box_2, \Diamond_1, \Diamond_2\}$.

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- Technically we will prove correspondence properties and completeness for extensions of $S5_2$ with axioms of these forms.
- They are sufficient for expressing how knowledge and facts considered possible are related to each other up to a level of nesting of two, which is already significant for human intuition.

$S5_2$ system

Our syntax is the standard bi-modal language $\mathcal{L}$, defined from a set P of propositional variables:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \Box_i \phi$$

where $p \in P, i \in \{1, 2\}$.

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where $p \in P, i \in \{1, 2\}$.
As standard, we use Kripke frames and model to interpret the language $\mathcal{L}$. Interpretation,satisfaction and validity are defined as standard.

$S5_2$ system

System $S5$

Axioms		Rules	
TAUT	all the instances of tautologies	MP	$\dfrac{\phi, \phi \rightarrow \psi}{\psi}$
DISTK	$\Box_i(p \rightarrow q) \rightarrow (\Box_i p \rightarrow \Box_i q)$	NEC	$\dfrac{\phi}{\Box_i \phi}$
T	$\Box_i p \rightarrow p$	SUB	$\dfrac{\phi}{\phi[p/\psi]}$
4	$\Box_i p \rightarrow \Box_i \Box_i p$		
5	$\neg \Box_i p \rightarrow \Box_i \neg \Box_i p$		

Some useful descriptions about $S5_2$ system

Theorem

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Lemma

For any $\phi \in \mathcal{L}$, we have $\vdash \Box_i \phi \leftrightarrow \Box_i \Box_i \phi \leftrightarrow \Diamond_i \Box_i \phi$ and $\vdash \Diamond_i \phi \leftrightarrow \Box_i \Diamond_i \phi \leftrightarrow \Diamond_i \Diamond_i \phi$ where $i \in A$.

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Lemma

For any $\phi, \psi \in \mathcal{L}$, we have $\vdash \phi \to \psi$ implies $\Box_i\phi \to \Box_i\psi$ and $\Diamond_i\phi \to \Diamond_i\psi$.

Interaction Axioms of the Form $\boxdot p \to \boxdot p$

We start with extensions of $S5_2$ with respect to interaction axioms that can be expressed as:

$$\boxdot \phi \to \boxdot \phi, \text{ where } \boxdot \in \{\Box_1, \Box_2, \Diamond_1, \Diamond_2\}. \tag{1}$$

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$$\boxdot\phi \to \boxdot\phi, \text{ where } \boxdot \in \{\Box_1, \Box_2, \Diamond_1, \Diamond_2\}. \tag{1}$$

There are 16 axioms of this form; factoring 1-2 symmetries reduces this number to 8, of which 4 are already consequences of $S5_2$ and therefore do not generate proper extensions. The remaining 4 are proper extensions of $S5_2$ and give rise to correspondence properties. All the possibilities are described in Figure 1.

Interaction Axioms	*Completeness*	*Lemmas of reference*	*Notes*
$\Box_1 p \Rightarrow \Box_1 p$	—	–	–
$\Box_1 p \Rightarrow \Diamond_1 p$	—	–	$\vdash p \Rightarrow \Diamond_1 p$
$\Box_1 p \Rightarrow \Box_2 p$	$\sim_2 \subseteq \sim_1$	4.1 and 4.2	–
$\Box_1 p \Rightarrow \Diamond_2 p$	—	–	$\vdash p \Rightarrow \Diamond_2 p$
$\Diamond_1 p \Rightarrow \Box_1 p$	$\sim_1 = id_W$	4.3 and 4.4	–
$\Diamond_1 p \Rightarrow \Diamond_1 p$	—	–	–
$\Diamond_1 p \Rightarrow \Box_2 p$	$\sim_1 = \sim_2 = id_W$	4.5 and 4.6	–
$\Diamond_1 p \Rightarrow \Diamond_2 p$	$\sim_1 \subseteq \sim_2$	4.7 and 4.8	–
$\Box_2 p \Rightarrow \Box_1 p$	$\sim_1 \subseteq \sim_2$	4.7 and 4.8	–
$\Box_2 p \Rightarrow \Diamond_1 p$	—	–	$\vdash p \Rightarrow \Diamond_1 p$
$\Box_2 p \Rightarrow \Box_2 p$	—	–	–
$\Box_2 p \Rightarrow \Diamond_2 p$	—	–	$\vdash p \Rightarrow \Diamond_2 p$
$\Diamond_2 p \Rightarrow \Box_1 p$	$\sim_1 = \sim_2 = id_W$	4.5 and 4.6	–
$\Diamond_2 p \Rightarrow \Diamond_1 p$	$\sim_2 \subseteq \sim_1$	4.1 and 4.2	–
$\Diamond_2 p \Rightarrow \Box_2 p$	$\sim_2 = id_W$	4.5 and 4.10	–
$\Diamond_2 p \Rightarrow \Diamond_2 p$	—	–	–

Figure 1: An exhaustive list of interaction axioms generated by (1).

Several examples

- $\Box_1 p \to \Box_2 p$

Lemma

$F \models \Box_1 p \to \Box_2 p$ if and only if F is such that $\sim_2 \subseteq \sim_1$.

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Proof.

From right to left; consider any model M such that $\sim_2 \subseteq \sim_1$ and a point w such that $M \models_w \Box_1 p$. So, for every point w' such that $w \sim_1 w'$ we have $M \models_{w'} p$. But $[w]_{\sim_2} \subseteq [w]_{\sim_1}$ and we have $M \models_w \Box_2 p$.

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For the converse, suppose $w \sim_2 w'$ on a frame F, such that $F \models \Box_1 p \to \Box_2 p$; it remains to prove that $w \sim_1 w'$. Consider a valuation $\pi(p) = \{w'\}$. So $(F, \pi) \models_w \Diamond_2 p$, but then $(F, \pi) \models_w \Diamond_1 p$, and so, since w' is the only point in which p is satisfied we have $w \sim_1 w'$. □

Lemma

The logic $S5_2 + \{\Box_1 p \to \Box_2 p\}$ is sound and complete with respect to equivalence frames such that $\sim_2 \subseteq \sim_1$.

Proof.

Soundness was proven in the first part of the previous lemma.

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Proof.

Soundness was proven in the first part of the previous lemma. Consider the canonical model $M = (W, \sim_1, \sim_2, \pi)$ for the logic $S5_2 + \{\Box_1 p \to \Box_2 p\}$. We know that $S5_2$ is canonical, i.e. the frame underlying M is an equivalence frame. We prove that the extension $S5_2 + \{\Box_1 p \to \Box_2 p\}$ is also canonical.

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Suppose $w \sim_2 w'$, with $w, w' \in W$; it remains to show that $w \sim_1 w'$. For this, it suffices to prove that there is a consistent set

$$\{\alpha_1, \ldots, \alpha_m\} \cup \{\beta_i \mid \Box_1 \beta_i \in w\}$$

For if that is the case by the maximal extension lemma there exists a point in the canonical model M that contains those formulas. $\Box$

Proof.

By contradiction assume this is not the case; then we can choose some $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ such that $\vdash \neg(\alpha_1, \ldots, \alpha_m \wedge \beta_1, \ldots, \beta_n)$. Call $\alpha = \wedge_{i=1}^{m} \alpha_i$ and $\beta = \wedge_{i=1}^{n} \beta_i$. So $\vdash \neg\alpha \vee \neg\beta$, i.e. $\vdash \beta \to \neg\alpha$.

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So the set $\{\alpha_1, \dots, \alpha_m\} \cup \{\beta_i \mid \Box_1 \beta_i \in w\}$ has to be consistent and there is on the canonical model a point w' such that $w \sim_1 w'$. By canonicity the logic $S5_2 + \{\Box_1 p \to \Box_2 p\}$ is then complete with respect to this class of frames. $\Box$

- $\Diamond_1 p \to \Box_2 p$

Lemma

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Proof.

From left to right. We prove that it cannot be that $\sim_1 \neq id_W$; the proof for the other relation is equivalent by using the contrapositive of the axiom. Suppose there exist two points $w, w' \in W$ on a frame F such that $w \sim_1 w'$ and consider a valuation π such that $\pi(p) = \{w'\}$. We have $(F, \pi) \vDash_w \Diamond_1 p$. Then $(F, \pi) \vDash_w \Box_2 p$, and since F is reflexive this implies that $(F, \pi) \vDash_w p$, which is absurd, unless $w = w'$.

- $\Diamond_1 p \to \Box_2 p$

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$F \vDash \Diamond_1 p \to \Box_2 p$ if and only if F is such that $\sim_1 = \sim_2 = id_W$.

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From right to left. Consider any equivalence model M such that $M \vDash_w \Diamond_1 p$. Then there exists a point $w' \in W$ such that $w \sim_1 w'$ and $M \vDash_{w'} p$. But since $\sim_1 = \sim_2 = id_W$, then it must be that $w = w'$ and so $M \vDash_w \Box_2 p$. $\square$

Lemma

The logic $S5_2 + \{\Diamond_1 p \to \Box_2 p\}$ is sound and complete with respect to equivalence frames such that $\sim_1 = \sim_2 = id_W$.

Proof.

Soundness was proven in the second part of the previous lemma.

Lemma

The logic $S5_2 + \{\Diamond_1 p \to \Box_2 p\}$ *is sound and complete with respect to equivalence frames such that* $\sim_1 = \sim_2 = id_W$.

Proof.

Soundness was proven in the second part of the previous lemma. We prove that the logic $S5_2 + \{\Diamond_1 p \to \Box_2 p\}$ is canonical. Consider the canonical model M and suppose, by contradiction, that $\sim_1 \neq id_W$ on the canonical frame. So there exist two points $w, w' \in W$ such that there is at least a formula $\alpha \in \mathcal{L}$ such that $\alpha \notin w, \alpha \in w'$ and $w \sim_1 w'$. So we have $M \vDash_w \Diamond_1 \alpha$, and then by $\vdash \Diamond_1 p \to \Box_2 p$ we have $M \vDash_w \Box_2 \alpha$.

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Discussion

We have showed that out of 16 possible interaction axioms of the form of Equation (1) only 5 of them lead to a different proper extension of $S5_2$. In particular since all the logics were proven to be canonical we have the more general result.

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Theorem

All the logics $S5_2 + \{\phi\}$, where ϕ is the conjunction of formula expressible as Equation (1) are complete with respect to the intersection of the respective classes of frames.

Proof.

It follows from all the canonicity results. Proving the relation between the logics is straightforward. $\square$

Relations among logics

Figure 2 shows the relations between all the logics discussed in this section.

- the logic $S5_2 + \{\Box_1 p \leftrightarrow \Box_2 p\}$ that can be obtained by taking the union of $S5_2 + \{\Diamond_1 p \to \Diamond_2 p\}$ and $S5_2 + \{\Diamond_2 p \to \Diamond_1 p\}$.

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- The lines in the figure represent set inclusion between logics, i.e. the logics are ordered in terms of how many formulas they contain. For example it is straightforward to prove that if $\vdash_{S5_2+\{\Diamond_1 p \to \Box_1 p\}} \phi$ then $\vdash_{S5_2+\{\Diamond_1 p \to \Box_2 p\}} \phi$.

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- The pictured relations between the logics are reflexive and transitive.

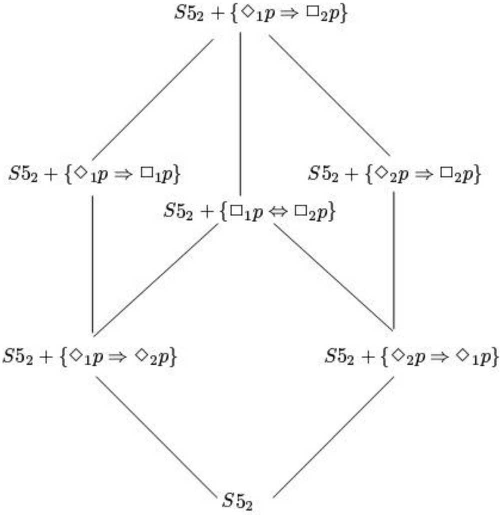

Figure 2: The proper extensions of $S5_2$ that can be obtained by adding axioms of the shape of Formula (1).

Discussion

- The most important logic is probably the one that forces the knowledge of an agent to be a subset of the knowledge of another. The logic $S5_2 + \{\Box_1 p \leftrightarrow \Box_2 p\}$ means that both agents have exactly the same knowledge base.

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- Stronger logics such as $\Diamond_1 p \to \Box_1 p$ can be defined by assuming that the modal component for one agent collapses onto the propositional calculus. We are in a situation in which "being possible according to one agent" is equivalent to "being known" and this in turn is equivalent to "being true".

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- Stronger logics such as $\Diamond_1 p \rightarrow \Box_1 p$ can be defined by assuming that the modal component for one agent collapses onto the propositional calculus. We are in a situation in which "being possible according to one agent" is equivalent to "being known" and this in turn is equivalent to "being true".
- The strongest consistent logic is $Triv_2$ that can be defined from $S5_2$ by adding the axiom $\Diamond_1 p \rightarrow \Box_2 p$ to $S5_2$ or equivalently by adding both $\Diamond_1 p \rightarrow \Box_1 p$ and $\Diamond_2 p \rightarrow \Box_2 p$. In this logic the two agents have equal knowledge that is equivalent to the truth on the world of evaluation.

Interaction Axioms of the Form $\boxdot p \to \boxdot \boxdot p$

There are 64 axioms of the shape

$$\boxdot p \to \boxdot \boxdot p \text{ where } \boxdot \in \{\Box_1, \Box_2, \Diamond_1, \Diamond_2\}. \quad (2)$$

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There are 64 axioms of the shape

$$\boxdot p \to \boxdot \boxdot p \text{ where } \boxdot \in \{\Box_1, \Box_2, \Diamond_1, \Diamond_2\}. \quad (2)$$

Factoring 1-2 symmetries reduces this number to 32. Again, many of these (14 in number) do not generate proper extensions of $S5_2$. For the remaining 18, the completeness results for the extension they generate are more complicated than the ones in the previous section.

Interaction axioms of the form $\Diamond_1 p \to \boxdot \boxdot p$

Interaction Axioms	*Completeness*	*Lemmas of Reference*	*Notes*
$\Diamond_1 p \Rightarrow \Diamond_1 \Box_1 p$	$\sim_1 = id_W$	4.3 and 4.4	$\vdash \Box_1 p \Leftrightarrow \Diamond_1 \Box_1 p$
$\Diamond_1 p \Rightarrow \Diamond_1 \Box_2 p$	$\sim_2 = id_W$	A.1 and A.2	–
$\Diamond_1 p \Rightarrow \Diamond_1 \Diamond_1 p$	—	–	$\vdash \Diamond_1 p \Leftrightarrow \Diamond_1 \Diamond_1 p$
$\Diamond_1 p \Rightarrow \Diamond_1 \Diamond_2 p$	—	–	$\vdash p \Rightarrow \Diamond_2 p$
$\Diamond_1 p \Rightarrow \Box_1 \Box_1 p$	$\sim_1 = id_W$	4.3 and 4.4	$\vdash \Box_1 p \Leftrightarrow \Box_1 \Box_1 p$
$\Diamond_1 p \Rightarrow \Box_1 \Box_2 p$	$\sim_1 = \sim_2 = id_W$	A.3 and A.4	–
$\Diamond_1 p \Rightarrow \Box_1 \Diamond_1 p$	—	–	$\vdash \Diamond_1 p \Leftrightarrow \Box_1 \Diamond_1 p$
$\Diamond_1 p \Rightarrow \Box_1 \Diamond_2 p$	$\sim_1 \subseteq \sim_2$	A.5 and A.6	–
$\Diamond_1 p \Rightarrow \Diamond_2 \Box_1 p$	$\sim_1 = id_W$	4.14 and 4.15	–
$\Diamond_1 p \Rightarrow \Diamond_2 \Box_2 p$	$\sim_1 = \sim_2 = id_W$	4.5 and 4.6	$\vdash \Box_2 p \Leftrightarrow \Diamond_2 \Box_2 p$
$\Diamond_1 p \Rightarrow \Diamond_2 \Diamond_1 p$	—	–	$\vdash p \Rightarrow \Diamond_2 p$
$\Diamond_1 p \Rightarrow \Diamond_2 \Diamond_2 p$	$\sim_1 \subseteq \sim_2$	4.7 and 4.8	$\vdash \Diamond_2 p \Leftrightarrow \Diamond_2 \Diamond_2 p$
$\Diamond_1 p \Rightarrow \Box_2 \Box_1 p$	$\sim_2 = \sim_1 = id_W$	A.7 and A.8	–
$\Diamond_1 p \Rightarrow \Box_2 \Box_2 p$	$\sim_1 = \sim_2 = id_W$	4.5 and 4.6	$\vdash \Box_2 p \Leftrightarrow \Box_2 \Box_2 p$
$\Diamond_1 p \Rightarrow \Box_2 \Diamond_1 p$	$\sim_2 \subseteq \sim_1$	A.9 and A.10	–
$\Diamond_1 p \Rightarrow \Box_2 \Diamond_2 p$	$\sim_1 \subseteq \sim_2$	4.7 and 4.8	$\vdash \Diamond_2 p \Leftrightarrow \Box_2 \Diamond_2 p$

Figure 3: An exhaustive list generated by (2) when the antecedent is $\Diamond_1 p$.

Important theorem

Theorem

All the logics in Figure are sound and complete with respect to the class of equivalence frames satisfying the corresponding property.

Proof.

Soundness can be checked straightforwardly.

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Proof.

Soundness can be checked straightforwardly.
For completeness, consider any logic $S5_2 + \{\phi\}$, where ϕ is an axiom in the previous figure. We have two cases.
To be continued *dots* □

Proof.

- $\vdash_{S5_2} \phi$. In this case, we obviously have that $S5_2 + \{\phi\}$ is equivalent to $S5_2$ and so the completeness of the logic $S5_2 + \{\phi\}$ with respect to equivalence frames follows.

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- $\vdash_{S5_2} \phi$. In this case, we obviously have that $S5_2 + \{\phi\}$ is equivalent to $S5_2$ and so the completeness of the logic $S5_2 + \{\phi\}$ with respect to equivalence frames follows.
- $\nvdash_{S5_2} \phi$. although the logic $S5_2 + \{\phi\}$ is a proper extension of $S5_2$, it can be proven equivalent to a logic $S5_2 + \{\psi\}$ for some axiom ψ examined in the previous section. The equivalence between $S5_2 + \{\phi\}$ and $S5_2 + \{\psi\}$, i.e. that $\vdash_{S5_2+\{\phi\}} \alpha$ if and only if $\vdash_{S5_2+\{\psi\}} \alpha$, follows once we have $\vdash_{S5_2+\{\phi\}} \psi$ and $\vdash_{S5_2+\{\psi\}} \phi$; in fact in this case any proof of α in one logic can be repeated in the other.

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- $\vdash_{S5_2} \phi$. In this case, we obviously have that $S5_2 + \{\phi\}$ is equivalent to $S5_2$ and so the completeness of the logic $S5_2 + \{\phi\}$ with respect to equivalence frames follows.
- $\nvdash_{S5_2} \phi$. although the logic $S5_2 + \{\phi\}$ is a proper extension of $S5_2$, it can be proven equivalent to a logic $S5_2 + \{\psi\}$ for some axiom ψ examined in the previous section. The equivalence between $S5_2 + \{\phi\}$ and $S5_2 + \{\psi\}$, i.e. that $\vdash_{S5_2+\{\phi\}} \alpha$ if and only if $\vdash_{S5_2+\{\psi\}} \alpha$, follows once we have $\vdash_{S5_2+\{\phi\}} \psi$ and $\vdash_{S5_2+\{\psi\}} \phi$; in fact in this case any proof of α in one logic can be repeated in the other. Now since $S5_2 + \{\psi\}$ was proven complete with respect to equivalence frames satisfying property P_ψ, the completeness of $S5_2 + \{\phi\}$ with respect to equivalence P_ψ frames also follows.

$\square$

An interesting example $\Diamond_1 p \to \Diamond_2 \Box_1 p$.

Lemma

$F \models \Diamond_1 p \to \Diamond_2 \Box_1 p$ if and only if F is such that $\sim_1 = id_W$.

Proof.

From left to right. Suppose there exist two points $w, w' \in W$ such that $w \sim_1 w'$. Consider a valuation π such that $\pi(p) = \{w'\}$. We have $(F, \pi) \models_w \Diamond_1 p$. Then $(F, \pi) \models_w \Diamond_2 \Box_1 p$, and since p is true only at w', which is related to w by relation $\sim_1$, then it must be that $[w]_{\sim_1} = \{w\}$.

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From left to right. Suppose there exist two points $w, w' \in W$ such that $w \sim_1 w'$. Consider a valuation π such that $\pi(p) = \{w'\}$. We have $(F, \pi) \models_w \Diamond_1 p$. Then $(F, \pi) \models_w \Diamond_2 \Box_1 p$, and since p is true only at w', which is related to w by relation $\sim_1$, then it must be that $[w]_{\sim_1} = \{w\}$.
From right to left. Consider any equivalence model M such that $M \models_w \Diamond_1 p$. Then there exists a point $w' \in W$ such that $w \sim_1 w'$ and $M \models_{w'} p$. But since $\sim_1 = id_W$, we have $w = w'$. So $M \models_w \Box_1 p$ and so $M \models_w \Diamond_2 \Box_1 p$. □

Completeness by equivalence

Lemma

$\vdash_{S5_2+\{\Diamond_1 p \to \Diamond_2 \Box_1 p\}} \Diamond_1 p \to \Box_1 p$ *and*
$\vdash_{S5_2+\{\Diamond_1 p \to \Box_1 p\}} \Diamond_1 p \to \Diamond_2 \Box_1 p.$

Completeness by equivalence

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$\vdash_{S5_2+\{\Diamond_1 p \to \Diamond_2 \Box_1 p\}} \Diamond_1 p \to \Box_1 p$ *and*
$\vdash_{S5_2+\{\Diamond_1 p \to \Box_1 p\}} \Diamond_1 p \to \Diamond_2 \Box_1 p$.

Proof.

First part. Suppose $\Diamond_1 p \to \Diamond_2 \Box_1 p$; so $\Box_2 \Diamond_1 p \to \Box_1 p$. Substitute the term $(p \to \Box_1 p)$ for p uniformly in the axiom above; we obtain $\Box_2 \Diamond_1 (p \to \Box_1 p) \to \Box_1 (p \to \Box_1 p)$. We prove that the antecedent of this formula is a theorem of $S5_2$. In fact we have $\neg \Box_1 p \vee \Box_1 p$ so we have $\Diamond_1 \neg p \vee \Diamond_1 \Box_1 p$. Now since, as it can easily be verified, diamond distributes over logical or, we have $\Diamond_1 (\neg p \vee \Box_1 p)$, which by necessitating by $\Box_2$ leads to $\Diamond_2 \Diamond_1 (\neg p \vee \Box_1 p)$. So it follows that $\Box_1 (p \to \Box_1 p)$, which gives $p \to \Box_1 p$. Then we can get $\Diamond_1 p \to \Diamond_1 \Box_1 p$, which is equivalent to $\Diamond_1 p \to \Box_1 p$.

Completeness by equivalence

Lemma

$\vdash_{S5_2+\{\Diamond_1 p \to \Diamond_2 \Box_1 p\}} \Diamond_1 p \to \Box_1 p$ and
$\vdash_{S5_2+\{\Diamond_1 p \to \Box_1 p\}} \Diamond_1 p \to \Diamond_2 \Box_1 p$.

Proof.

First part. Suppose $\Diamond_1 p \to \Diamond_2 \Box_1 p$; so $\Box_2 \Diamond_1 p \to \Box_1 p$. Substitute the term $(p \to \Box_1 p)$ for p uniformly in the axiom above; we obtain $\Box_2 \Diamond_1 (p \to \Box_1 p) \to \Box_1 (p \to \Box_1 p)$. We prove that the antecedent of this formula is a theorem of $S5_2$. In fact we have $\neg \Box_1 p \vee \Box_1 p$ so we have $\Diamond_1 \neg p \vee \Diamond_1 \Box_1 p$. Now since, as it can easily be verified, diamond distributes over logical or, we have $\Diamond_1 (\neg p \vee \Box_1 p)$, which by necessitating by $\Box_2$ leads to $\Diamond_2 \Diamond_1 (\neg p \vee \Box_1 p)$. So it follows that $\Box_1 (p \to \Box_1 p)$, which gives $p \to \Box_1 p$. Then we can get $\Diamond_1 p \to \Diamond_1 \Box_1 p$, which is equivalent to $\Diamond_1 p \to \Box_1 p$.
Second part. Suppose $\Diamond_1 p \to \Box_1 p$. By the axiom T we then obtain $\Diamond_1 p \to \Diamond_2 \Box_1 p$. $\square$

Interaction axioms of the form $\Box_1 p \rightarrow \boxdot \boxdot p$

Interaction Axioms	*Completeness*	*Lemmas of Ref.*	*Notes*
$\Box_1 p \Rightarrow \Diamond_1 \Box_1 p$	—	–	$\vdash \Diamond_1 \Box_1 p \Leftrightarrow \Box_1 p$
$\Box_1 p \Rightarrow \Diamond_1 \Box_2 p$	$\forall w \exists w' \in [w]_{\sim_1} : [w']_{\sim_2} \subseteq [w]_{\sim_1}$	4.18 and 4.19	–
$\Box_1 p \Rightarrow \Diamond_1 \Diamond_1 p$	—	–	$\vdash \Diamond_1 p \Leftrightarrow \Diamond_1 \Diamond_1 p$
$\Box_1 p \Rightarrow \Diamond_1 \Diamond_2 p$	—	–	$\vdash p \Rightarrow \Diamond_2 p$
$\Box_1 p \Rightarrow \Box_1 \Box_1 p$	—	–	$\vdash \Box_1 p \Leftrightarrow \Box_1 \Box_1 p$
$\Box_1 p \Rightarrow \Box_1 \Box_2 p$	$\sim_2 \subseteq \sim_1$	A.11 and A.12	–
$\Box_1 p \Rightarrow \Box_1 \Diamond_1 p$	—	–	$\vdash \Diamond_1 p \Leftrightarrow \Box_1 \Diamond_1 p$
$\Box_1 p \Rightarrow \Box_1 \Diamond_2 p$	—	–	$\vdash p \Rightarrow \Diamond_2 p$
$\Box_1 p \Rightarrow \Diamond_2 \Box_1 p$	—	–	$\vdash p \Rightarrow \Diamond_2 p$
$\Box_1 p \Rightarrow \Diamond_2 \Box_2 p$	$\sim_2 \subseteq \sim_1$	4.1 and 4.2	$\vdash \Diamond_2 p \Leftrightarrow \Diamond_2 \Box_2 p$
$\Box_1 p \Rightarrow \Diamond_2 \Diamond_1 p$	—	–	$\vdash p \Rightarrow \Diamond_2 p$
$\Box_1 p \Rightarrow \Diamond_2 \Diamond_2 p$	—	–	$\vdash \Diamond_2 p \Leftrightarrow \Diamond_2 \Diamond_2 p$
$\Box_1 p \Rightarrow \Box_2 \Box_1 p$	$\sim_2 \subseteq \sim_1$	A.13 and A.14	–
$\Box_1 p \Rightarrow \Box_2 \Box_2 p$	$\sim_2 \subseteq \sim_1$	4.1 and 4.2	$\vdash \Box_2 p \Leftrightarrow \Box_2 \Box_2 p$
$\Box_1 p \Rightarrow \Box_2 \Diamond_1 p$	$\sim_2 \subseteq \sim_1$	A.15 and A.16	–
$\Box_1 p \Rightarrow \Box_2 \Diamond_2 p$	—	–	$\vdash \Diamond_2 p \Leftrightarrow \Box_2 \Diamond_2 p$

Figure 4: An exhaustive list of interaction axioms generated by (2) in the case the antecedent is equal to $\Box_1 p$.

Taking $\Box_1 p \to \Diamond_1 \Box_2 p$ for example

Lemma

$F \models \Box_1 p \to \Diamond_1 \Box_2 p$ if and only if F is such that
$\forall w \exists w' \in [w]_{\sim_1} : [w']_{\sim_2} \subseteq [w]_{\sim_1}$.
($[w]_{\sim_1}$ is the $\sim_1$-equivalence class of w.)

Taking $\Box_1 p \to \Diamond_1 \Box_2 p$ for example

Lemma

$F \vDash \Box_1 p \to \Diamond_1 \Box_2 p$ if and only if F is such that
$\forall w \exists w' \in [w]_{\sim_1} : [w']_{\sim_2} \subseteq [w]_{\sim_1}$.
($[w]_{\sim_1}$ is the $\sim_1$-equivalence class of w.)

Proof.

From right to left; consider any model M and a point w in it such that $M \vDash_w \Box_1 p$. So, for every point w' such that $w \sim_1 w'$ we have $M \vDash_{w'} p$. But, by assumption, there exists a point $w' \in [w]_{\sim_1}$ such that $[w']_{\sim_2} \subseteq [w]_{\sim_1}$. So, p holds at any point of the equivalence class $[w']_{\sim_2}$, and so $M \vDash_{w'} \Box_2 p$. Therefore $M \vDash_w \Diamond_1 \Box_2 p$. □

Proof.

For the converse, suppose the relational property above does not hold. Then there exists a frame F and a point w in F such that for any $w' \in [w]_{\sim_1}$ we have $[w']_{\sim_2} \nsubseteq [w]_{\sim_1}$, i.e. we have the existence of a point $w'' \in [w']_{\sim_2}$ such that $w'' \notin [w]_{\sim_1}$. Consider a valuation π such that $\pi(p) = \{w' \mid w \sim_1 w'\}$. We have $(F, \pi) \models_w \Box_1 p$ and $(F, \pi) \not\models_{w''} p$. So $(F, \pi) \not\models_{w'} \Box_2 p$. So we have $(F, \pi) \not\models_w \Diamond_1 \Box_2 p$ which is absurd. $\square$

Completeness

Lemma

The logic $S5_2 + \{\Box_1 p \to \Diamond_1 \Box_2 p\}$ *is sound and complete with respect to equivalence frames satisfying the property* $\forall w \exists w' \in [w]_{\sim_1} : [w']_{\sim_2} \subseteq [w]_{\sim_1}$.

Proof.

Soundness was proven in first part of the previous Lemma.

Completeness

Lemma

The logic $S5_2 + \{\Box_1 p \to \Diamond_1 \Box_2 p\}$ *is sound and complete with respect to equivalence frames satisfying the property* $\forall w \exists w' \in [w]_{\sim_1} : [w']_{\sim_2} \subseteq [w]_{\sim_1}$.

Proof.

Soundness was proven in first part of the previous Lemma.
For completeness we prove that the logic $S5_2 + \{\Box_1 p \to \Diamond_1 \Box_2 p\}$ is canonical. In order to do that, suppose, by contradiction, that the frame of the canonical model does not satisfy the relational property above. Then, it must be that there exists a point w such that:

$$\forall w' \in [w]_{\sim_1} \exists w'' : w' \sim_2 w'' \text{ and } w \not\sim_1 w''.$$

□

Proof.

Call $w'_1, \ldots, w'_n, \ldots$ the points in $[w]_{\sim_1}$, and w''_i the point in $[w'_i]_{\sim_2}$ such that $w \not\sim_1 w''_i$; $i = 1, \ldots, n, \ldots$.

Proof.

Call $w'_1, \ldots, w'_n, \ldots$ the points in $[w]_{\sim_1}$, and w''_i the point in $[w'_i]_{\sim_2}$ such that $w \not\sim_1 w''_i$; $i = 1, \ldots, n, \ldots$. Recall that $w \sim_1 w'$ on the canonical model is defined as $\forall \alpha \in \mathcal{L}(\Box_i \alpha \in w$ implies $\alpha \in w')$; $w \not\sim_j w'$ is defined as $\exists \alpha \in \mathcal{L}(\Box_j \alpha \in w$ and $\neg \alpha \in w')$.

Proof.

Call $w'_1, \ldots, w'_n, \ldots$ the points in $[w]_{\sim_1}$, and w''_i the point in $[w'_i]_{\sim_2}$ such that $w \not\sim_1 w''_i$; $i = 1, \ldots, n, \ldots$. Recall that $w \sim_1 w'$ on the canonical model is defined as $\forall \alpha \in \mathcal{L}(\Box_i \alpha \in w$ implies $\alpha \in w')$; $w \not\sim_j w'$ is defined as $\exists \alpha \in \mathcal{L}(\Box_j \alpha \in w$ and $\neg\alpha \in w')$. So we can find some formulas $\alpha_i \in \mathcal{L}$; $i = 1, \ldots, n, \ldots$ such that $\Box_1 \alpha_i \in w, \alpha_i \in w'_i, \neg\alpha_i \in w''_i$; $i = 1, \ldots, n, \ldots$. Call $\alpha = \wedge_{i=1}^{n} \alpha_i$; we have $\Box_1 \alpha_i \in w$; $i = 1, \ldots, n, \ldots$. So $\Box_1 \alpha \in w$. But $\neg\alpha \in w''_i$, $i = 1, \ldots, n, \ldots$. So $\Diamond_2 \neg\alpha \in w'_i$ for every i in $\{1, \ldots, n, \ldots\}$. So $\Box_1 \Diamond_2 \neg\alpha \in w$ i.e. $\neg\Diamond_1 \Box_2 \alpha \in w$. But $\Box_1 \alpha \in w$ and $\Box_1 \alpha \to \Diamond_1 \Box_2 \alpha$, so w would be inconsistent. Therefore the canonical frame must satisfy the property above and the logic is complete with respect to equivalence frames satisfying the property $\forall w \exists w' \in [w]_{\sim_1} : [w']_{\sim_2} \subseteq [w]_{\sim_1}$. $\Box$

Discussion

Given the fact that all the logics were proven to be canonical we have the general result:

Theorem

All the logics $S5_2 + \{\phi\}$, where ϕ is the conjunction of formula expressible as Equation (2) are complete with respect to the intersection of the respective classes of frames.

Proof.

It follows from all the canonicity results. □

Discussion

- Among all these axioms, the most intuitive ones in terms of knowledge are probably $\Box_1 p \to \Box_2 \Box_1 p$ and its "dual" $\Box_2 p \to \Box_1 \Box_2 p$, representing scenarios in which agent 1 knows that agent 2 knows something every time this happens to be the case.

Discussion

- Among all these axioms, the most intuitive ones in terms of knowledge are probably $\Box_1 p \to \Box_2 \Box_1 p$ and its "dual" $\Box_2 p \to \Box_1 \Box_2 p$, representing scenarios in which agent 1 knows that agent 2 knows something every time this happens to be the case.
- A more subtle, independent axiom expressed by Equation (2) is the formula $\Box_1 p \to \Diamond_1 \Box_2 p$, which reads "*If agent 1 knows p, then he considers possible that agent 2 also knows p*". The above is an axiom that regulates a natural kind of "prudence" assumption of agent 1 in terms of what knowledge agent 2 may have.

Interaction axioms of the form $\boxdot \boxdot p \to \boxdot \boxdot p$

We now discuss the extensions of $S5_2$ with interaction axioms expressible as:

$$\boxdot \boxdot p \to \boxdot \boxdot p \text{ where } \boxdot \in \{\Box_1, \Box_2, \Diamond_1, \Diamond_2\}. \quad (3)$$

Interaction axioms of the form $\boxdot \boxdot p \to \boxdot \boxdot p$

We now discuss the extensions of $S5_2$ with interaction axioms expressible as:

$$\boxdot \boxdot p \to \boxdot \boxdot p \text{ where } \boxdot \in \{\Box_1, \Box_2, \Diamond_1, \Diamond_2\}. \tag{3}$$

Equation (3) expresses $4 \times 4 \times 4 \times 4 = 256$ different formulas; we lose half by 1-2 symmetry; of the remaining 128, 64 of them begin with $\boxdot_i \boxdot_j$ with $i = j$, which, by well known $S5_2$ equibalences collapse to a case of the previous section. The remaining 64 axioms divide into 26 which do not induce proper extensions of $S5_2$ and 38 axioms which do.

Interaction Axioms	*Completeness*
$\Box_1\Diamond_2 p \Rightarrow \Box_1\Diamond_1 p$	$\forall w \exists w' \in [w]_{\sim_1} : [w']_{\sim_2} \subseteq [w]_{\sim_1}$
$\Box_1\Diamond_2 p \Rightarrow \Diamond_1\Diamond_1 p$	$\forall w \exists w' \in [w]_{\sim_1} : [w']_{\sim_2} \subseteq [w]_{\sim_1}$
$\Box_1\Diamond_2 p \Rightarrow \Diamond_1\Box_2 p$	$? \forall w \exists w' \in [w]_{\sim_1} : [w']_{\sim_2} = \{w'\}$
$\Diamond_1\Diamond_2 p \Rightarrow \Diamond_2\Box_1 p$	$\sim_1 = id_W$
$\Diamond_1\Box_2 p \Rightarrow \Box_1\Box_1 p$	$\sim_1 \subseteq \sim_2$
$\Diamond_1\Box_2 p \Rightarrow \Diamond_1\Box_1 p$	$\sim_1 \subseteq \sim_2$
$\Diamond_1\Box_2 p \Rightarrow \Box_1\Box_2 p$	$\sim_1 \subseteq \sim_2$
$\Diamond_1\Box_2 p \Rightarrow \Box_1\Diamond_2 p$	$\sim_1 \subseteq \sim_2$
$\Diamond_1\Box_2 p \Rightarrow \Box_2\Box_1 p$	$\sim_1 \subseteq \sim_2$
$\Diamond_1\Box_2 p \Rightarrow \Box_2\Box_2 p$	$\sim_1 \subseteq \sim_2$
$\Diamond_1\Box_2 p \Rightarrow \Box_2\Diamond_2 p$	$\sim_1 \subseteq \sim_2$
$\Diamond_1\Box_2 p \Rightarrow \Diamond_2\Box_1 p$	$\sim_1 \subseteq \sim_2$
$\Diamond_1\Box_2 p \Rightarrow \Diamond_2\Box_2 p$	$\sim_1 \subseteq \sim_2$
$\Diamond_1\Box_2 p \Rightarrow \Diamond_2\Diamond_2 p$	$\sim_1 \subseteq \sim_2$
$\Diamond_1\Diamond_2 p \Rightarrow \Box_1\Diamond_2 p$	$\sim_1 \subseteq \sim_2$
$\Diamond_1\Diamond_2 p \Rightarrow \Box_2\Diamond_2 p$	$\sim_1 \subseteq \sim_2$
$\Diamond_1\Diamond_2 p \Rightarrow \Diamond_2\Diamond_2 p$	$\sim_1 \subseteq \sim_2$

$\Diamond_1\Diamond_2 p \Rightarrow \Box_1\Box_1 p$	$\sim_1=\sim_2= id_W$
$\Diamond_1\Diamond_2 p \Rightarrow \Box_1\Box_2 p$	$\sim_1=\sim_2= id_W$
$\Diamond_1\Diamond_2 p \Rightarrow \Box_2\Box_1 p$	$\sim_1=\sim_2= id_W$
$\Diamond_1\Diamond_2 p \Rightarrow \Box_2\Box_2 p$	$\sim_1=\sim_2= id_W$
$\Diamond_1\Diamond_2 p \Rightarrow \Diamond_1\Box_1 p$	$\sim_1=\sim_2= id_W$
$\Diamond_1\Diamond_2 p \Rightarrow \Diamond_2\Box_2 p$	$\sim_1=\sim_2= id_W$
$\Box_1\Diamond_2 p \Rightarrow \Box_1\Box_1 p$	$\sim_2= id_W$
$\Box_1\Diamond_2 p \Rightarrow \Diamond_1\Box_1 p$	$\sim_2= id_W$
$\Box_1\Diamond_2 p \Rightarrow \Box_2\Diamond_1 p$	$\sim_2\subseteq\sim_1$
$\Diamond_1\Diamond_2 p \Rightarrow \Box_1\Diamond_1 p$	$\sim_2\subseteq\sim_1$
$\Diamond_1\Diamond_2 p \Rightarrow \Box_2\Diamond_1 p$	$\sim_2\subseteq\sim_1$
$\Diamond_1\Diamond_2 p \Rightarrow \Diamond_1\Diamond_1 p$	$\sim_2\subseteq\sim_1$
$\Box_1\Diamond_2 p \Rightarrow \Box_1\Box_2 p$	$\sim_2= id_W$
$\Box_1\Diamond_2 p \Rightarrow \Box_2\Box_1 p$	$\sim_2= id_W$
$\Box_1\Diamond_2 p \Rightarrow \Box_2\Box_2 p$	$\sim_2= id_W$
$\Box_1\Diamond_2 p \Rightarrow \Diamond_2\Box_2 p$	$\sim_2= id_W$
$\Diamond_1\Diamond_2 p \Rightarrow \Diamond_1\Box_2 p$	$\sim_2= id_W$

$\Box_1\Box_2 p \Rightarrow \Box_2\Box_1 p$	$w \sim_1 w_1,\ w \sim_2 w_2 \Rightarrow \exists\overline{w}:$ $w_1 \sim_2 \overline{w},\ w_2 \sim_1 \overline{w}$
$\Diamond_1\Diamond_2 p \Rightarrow \Diamond_2\Diamond_1 p$	$w \sim_1 w_1,\ w \sim_2 w_2 \Rightarrow \exists\overline{w}:$ $w_1 \sim_2 \overline{w},\ w_2 \sim_1 \overline{w}$
$\Diamond_1\Box_2 p \Rightarrow \Box_2\Diamond_1 p$	$w \sim_1 w_1,\ w \sim_2 w_2 \Rightarrow \exists\overline{w}:$ $w_1 \sim_2 \overline{w},\ w_2 \sim_1 \overline{w}$
$\Box_1\Diamond_2 p \Rightarrow \Diamond_2\Box_1 p$	? Either $\sim_1 = id_W$ or $\sim_2 = id_W$

Figure 5: Proper extensions of $S5_2$ generated by axioms of the form $\boxdot \boxdot p \to \boxdot \boxdot p$.For axioms listed with “?” correspondence is proved but completeness is only conjectured.

Taking $\Box_1 \Diamond_2 p \to \Diamond_1 \Box_2 p$ for example

Definition

A point $w \in W$ is called an i-dead-end if for all $w' \in W$ we have $w \sim_i w'$ implies $w = w'$.

Taking $\Box_1 \Diamond_2 p \to \Diamond_1 \Box_2 p$ for example

Definition

A point $w \in W$ is called an i-dead-end if for all $w' \in W$ we have $w \sim_i w'$ implies $w = w'$.

Lemma

Given a frame F and a point w on it, w is an i-dead-end if and only if for any valuation π, we have $(F, \pi) \vDash_w p \to \Box_i p$.

Lemma

$F \models \Box_1 \Diamond_2 p \to \Diamond_1 \Box_2 p$ if and only if F is such that every point w is related by relation 1 to a 2-dead-end; i.e. for all $w \in W$ there exists a $w' \in W, w \sim_1 w'$ such that $[w']_{\sim_2} = \{w'\}$.

Lemma

$F \models \Box_1 \Diamond_2 p \to \Diamond_1 \Box_2 p$ if and only if F is such that every point w is related by relation 1 to a 2-dead-end; i.e. for all $w \in W$ there exists a $w' \in W, w \sim_1 w'$ such that $[w']_{\sim_2} = \{w'\}$.

Proof.

From right to left; consider any model M such that every point sees via 1 a 2-dead-end. Suppose $M \models_w \Box_1 \Diamond_2 p$; so for every point w' such that $w \sim_1 w'$ we have that there must be a w'' such that $w' \sim_2 w''$ and $M \models''_w p$. But by assumption one of the w' is a 2-dead-end, so we have the existence of a point $\overline{w} \in [w]_{\sim_1}$ such that $M \models_{\overline{w}} \Box_2 p$. Then $M \models_w \Diamond_1 \Box_2 p$. □

Proof.

For the converse, consider any equivalence frame F, such that $F \models \Box_1 \Diamond_2 p \to \Diamond_1 \Box_2 p$ and suppose by contradiction that the property above does not hold.

Proof.

For the converse, consider any equivalence frame F, such that $F \models \Box_1 \Diamond_2 p \to \Diamond_1 \Box_2 p$ and suppose by contradiction that the property above does not hold. Consider the set $X = [w]_{\sim_1}$, the equivalence relation $\sim = \sim_1 \cap \sim_2$ and the quotient set $X/\sim$.

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For the converse, consider any equivalence frame F, such that $F \models \Box_1 \Diamond_2 p \to \Diamond_1 \Box_2 p$ and suppose by contradiction that the property above does not hold. Consider the set $X = [w]_{\sim_1}$, the equivalence relation $\sim = \sim_1 \cap \sim_2$ and the quotient set $X/\sim$. Consider now the set Y constructed by taking one and only one representative w for each class $[w]_{\sim}$ in $X/\sim$. Consider a valuation $\pi(p) = Y$ and consider the model $M = (W, \sim_1, \sim_2, \pi)$.

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For the converse, consider any equivalence frame F, such that $F \vDash \Box_1\Diamond_2 p \to \Diamond_1\Box_2 p$ and suppose by contradiction that the property above does not hold. Consider the set $X = [w]_{\sim_1}$, the equivalence relation $\sim = \sim_1 \cap \sim_2$ and the quotient set $X/\sim$. Consider now the set Y constructed by taking one and only one representative w for each class $[w]_\sim$ in $X/\sim$. Consider a valuation $\pi(p) = Y$ and consider the model $M = (W, \sim_1, \sim_2, \pi)$. By construction we have $M \vDash_w \Box_1\Diamond_2 p$. Then by our assumption we also have $M \vDash_w \Diamond_1\Box_2 p$. So there must be a point w' such that $w \sim_1 w'$ such that $M \vDash_{w'} \Box_2 p$.

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Completeness

Completeness for the above remains an open problem.

Conjecture 1: The logic $S5_2 + \{\Box_1\Diamond_2 p \to \Diamond_1\Box_2 p\}$ is sound and complete with respect to equivalence frames such that every point is related by relation 1 to a 2-dead-end; i.e. for all $w \in W$ there exists a $w' \in W, w \sim_1 w'$ such that $[w']_{\sim_2} = \{w'\}$.

the axiom $\Box_1\Diamond_2 p \to \Diamond_2\Box_1 p$

Lemma

$F \models \Box_1\Diamond_2 p \to \Diamond_2\Box_1 p$ if and only if in every connected sub-frame either $\sim_1 = id_W$ or $\sim_2 = id_W$.

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From left to right. This part of the proof is structured as follows:

1. We prove that $F \models \Box_1\Diamond_2 p \to \Diamond_2\Box_1 p$ implies that any point $w \in W$ either sees via 1 a 2-dead-end, or the point w sees via 2 a 1-dead-end.

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3. The two facts above together prove that if $F \models \Box_1\Diamond_2 p \to \Diamond_2\Box_1 p$, then in every connected sub-frame either $\sim_1 = id_W$ or $\sim_2 = id_W$.

Proof.

1. By contradiction, consider any connected equivalence frame F, in which a $w \in W$ does not see via i any j-dead end, i.e. $\forall w' \in [w]_{\sim_i}, [w']_{\sim_j} \neq \{w'\}, i \neq j, i, j \in \{1, 2\}$; we prove that $F \not\models \Box_1 \Diamond_2 p \to \Diamond_2 \Box_1 p$.

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Proof.

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Proof.

From right to left. Consider any equivalence model M whose underlying frame satisfies the property above and suppose that $M \vDash_w \Box_1 \Diamond_2 p$.

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Suppose $\sim_1 = id_W$ and $M \vDash_w \Box_1 \Diamond_2 p$, so there is a $w' \in [w]_{\sim_2}$, such that $M \vDash_{w'} p$. But since $\sim_1 = id_W$ on the connected part, we also have $M \vDash_{w'} \Box_1 p$. So $M \vDash_{w'} \Diamond_2 \Box_1 p$. Suppose now $\sim_2 = id_W$ and $M \vDash_w \Box_1 \Diamond_2 p$. So for every $w' \in [w]_{\sim_1}$ we have $M \vDash_{w'} p$. But then we also have $M \vDash_w \Diamond_2 \Box_1 p$. □

Completeness

Conjecture 2: The logic $S5_2 + \{\Box_1 \Diamond_2 p \to \Diamond_2 \Box_1 p\}$ is sound and complete with respect to equivalence frames such that either $\sim_1 = id_W$ or $\sim_2 = id_W$ on every connected sub-frame.

Discussion

Theorem

All the logics $S5_2 + \{\phi\}$, where ϕ is the conjunction of formulas expressible as expressible as Equation (3) except the two McKinsey style axioms are complete with respect to the intersection of the corresponding classes of frames given in the figure of this section.

Discussion

- We have identified a number of non-trivial single-axiom extensions of $S5_2$ which specify a mode of interaction between two agents, and proved correspondence, soundness and completeness with respect to the appropriate classes of frames.

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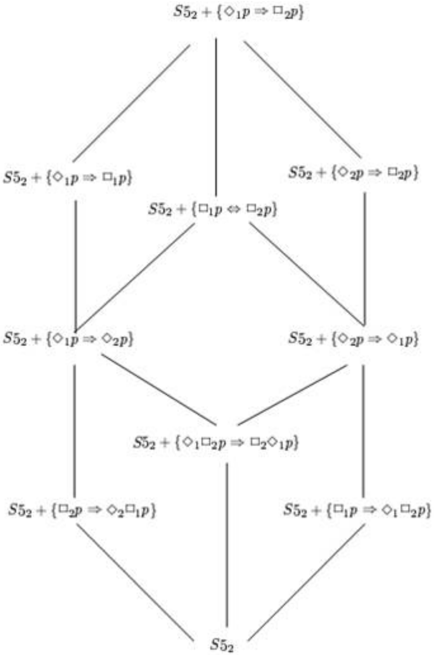
- We have identified a number of non-trivial single-axiom extensions of $S5_2$ which specify a mode of interaction between two agents, and proved correspondence, soundness and completeness with respect to the appropriate classes of frames.
- The main contribution of this paper lies in the identification of a spectrum of interactions above $S5_2$. The following figure represents graphically all the logics discussed so far together with the corresponding semantic classes. In the figure, the logics are ordered strength-wise.


$S5_2 + \{\Diamond_1 p \Rightarrow \Box_2 p\}$
$S5_2 + \{\Diamond_1 p \Rightarrow \Box_1 p\}$
$S5_2 + \{\Diamond_2 p \Rightarrow \Box_2 p\}$
$S5_2 + \{\Box_1 p \Leftrightarrow \Box_2 p\}$
$S5_2 + \{\Diamond_1 p \Rightarrow \Diamond_2 p\}$
$S5_2 + \{\Diamond_2 p \Rightarrow \Diamond_1 p\}$
$S5_2 + \{\Diamond_1 \Box_2 p \Rightarrow \Box_2 \Diamond_1 p\}$
$S5_2 + \{\Box_2 p \Rightarrow \Diamond_2 \Box_1 p\}$
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$S5_2$

Conclusions

- We examined all the interactions axioms that can be written as an implication expressing the fact that knowledge and facts considered possible are related to each other up to a level of nesting of two.

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Conclusions

- We examined all the interactions axioms that can be written as an implication expressing the fact that knowledge and facts considered possible are related to each other up to a level of nesting of two.
- A spectrum of degrees of knowledge sharing has emerged. Some meaningful logics in epistemic settings have emerged.
- The fairly exhaustive analysis carried out in this paper permits the AI-user with an interaction axiom in mind to refer to the above tables to identify the class of Kripke frames that gives completeness.

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Thank you!