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A THEORY OF HYPERMODAL LOGICS: MODE SHIFTING IN MODAL LOGIC

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ABSTRACT. A hypermodality is a connective $\Box$ whose meaning depends on where in the formula it occurs. The paper motivates the notion and shows that hypermodal logics are much more expressive than traditional modal logics. In fact we show that logics with very simple **K** hypermodalities are not complete for any neighbourhood frames.

KEY WORDS: modal logic, combining logic, hypermodality, frame completeness

1. Introduction

Hypermodal logics belong to a special kind of modal logics which appear in a variety of forms in many applications. These logics will be motivated and studied in later sections, but if I were to characterise them in one sentence I would say that these are modalities $\Box$ whose nature changes depending on where they appear in the wff. Thus, for example, when we write the formula $\Box(\Box A \rightarrow A)$, the outer modality can be a **T** modality while the inner modality can be **K** modality.[1]

Semantically this means that evaluating $t \vDash \Box A$ in a Kripke model depends not only on t and on the meaning of $\Box$ (the kind of modality $\Box$ is), but also on how we got to t in the process of evaluation.[2] These distinctions will become clear later on in the paper. Proof theoretically this kind of dependence means that the set of available proof rules changes as we progress through the proof.

Traces of these ideas have been around for a while. However, this is the first time that they are all addressed in one framework the concepts of modes and of *hypermodality* is coined and their formal properties studied.

Since our main example is a system involving **K** and **T** modalities, let us remind ourselves of some convenient semantic and proof theoretical formulations of these systems.

DEFINITION 1.1 (Semantics for strict implication). (1) A Kripke propositional model has the form $\mathbf{m} = (S, R, a, h)$, where S is the set of possible

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worlds and $R \subseteq S^2$ is the accessibility relation. $a \in S$ is the actual world and h is the assignment, giving for each atomic q a subset $h(q) \subseteq S$.

(2) Satisfaction for $t \in S$ is defined inductively as follows:

- $t \vDash q$ iff $t \in h(q)$ for q atomic,
- $t \vDash A \wedge B$ iff $t \vDash A$ and $t \vDash B$,
- $t \nvDash \bot$,
- $t \vDash A \Rightarrow B$ iff $\forall s(tRs$ and $s \vDash A$ imply $s \vDash B)$,
- $\mathbf{m} \vDash A$ iff $a \vDash A$.

Here $\Rightarrow$ is strict implication and $\Box A$ can be defined as $(T \Rightarrow A)$.

(3) We have completeness as follows, see [1].

- $\mathbf{K} \vDash A$ iff in all models $\mathbf{m} = (S, R, a, h)$ we have $\mathbf{m} \vDash A$,
- $\mathbf{T} \vDash A$ iff in all models $\mathbf{m}$ with R reflexive (i.e. $(\forall x)(xRx)$ holds) we have $\mathbf{m} \vDash A$,
- $\mathbf{S4} \vDash A$ iff in all models in which R is reflexive and transitive we have $\mathbf{m} \vDash A$.

DEFINITION 1.2 (Proof theory for strict implication). Consider for simplicity the language with $\Rightarrow$ only.

(1) A data-structure is a list of sets of sentences of the form $(\Delta_1, \dots, \Delta_n)$. Our proof rules define the consequence relation $(\Delta_1, \dots, \Delta_n) \vdash B$. The semantic meaning of $\vdash$ is as follows:

- $(\Delta_1, \dots, \Delta_n) \vDash_{\mathbf{L}} B$ iff for all Kripke models $\mathbf{m}$ of the logic $\mathbf{L}$ and all points $t_1, \dots, t_n$ in S such that $t_1Rt_2, t_2Rt_3, \dots, t_{n-1}Rt_n$ and such that $t_i \vDash \Delta_i$, we have that $t_n \vDash B$.

(2) To see what kind of proof rules we can expect, consider the following.

Suppose $A \Rightarrow B \in \Delta_i$ and $A \in \Delta_j$. This means semantically that $t_i \vDash A \Rightarrow B$ and $t_j \vDash A$.

If in the model t_iRt_j holds, then also $t_j \vDash B$. If $\mathbf{L} = \mathbf{K}$ then t_iRt_j is assured to hold only if $j = i + 1$. If $\mathbf{L} = \mathbf{T}$, then t_iRt_j holds necessarily only if $i \leq j \leq i+1$. If $\mathbf{L} = \mathbf{S4}$ then t_iRt_j is assured to hold only if $i \leq j$.

We thus get the following rule of $\Rightarrow E$:

$$(\Rightarrow E) \quad \frac{\Delta_1, \dots, \Delta_i \cup \{A \Rightarrow B\}, \dots, \Delta_j \cup \{A\}, \dots, \Delta_n \vdash D}{\Delta_1, \dots, \Delta_i \cup \{A \Rightarrow B\}, \dots, \Delta_j \cup \{A, B\}, \dots, \Delta_n \vdash D}.$$

Provided $j = i + 1$ for $\mathbf{K}$, $i \leq j \leq i + 1$ for $\mathbf{T}$ and $i \leq j$ for $\mathbf{S4}$.

(3) What would be the form of $\Rightarrow I$, that is the $\Rightarrow$ introduction rule? Suppose we want to show that $\Delta_1, \dots, \Delta_n \vdash A \Rightarrow B$. This means that in *any* model $\mathbf{m}$, and $t_1, \dots, t_n$ as before, $t_n \vDash A \Rightarrow B$. This means that for

any t_{n+1} such that $t_n R t_{n+1}$ and $t_{n+1} \vDash A$ we must have $t_{n-1} \vDash B$. But this simply means $\Delta_1, \ldots, \Delta_n, \{A\} \vdash B$.

We thus get the rule

$(\Rightarrow I)$ $\quad (\Delta_1, \ldots, \Delta_n) \vdash A \Rightarrow B$ iff $(\Delta_1, \ldots, \Delta_n, \{A\}) \vdash B$.

For completeness of these rules, see [7].

(4) Let us now look at the proof rules from the Δ_j point of view (i.e. as if we are 'living' in the world t_j). Suppose we have argued successfully and shown A is available to us in world t (i.e. in Δ_j). We want to use an $A \Rightarrow B \in \Delta_i$, $i \leq j$, and perform modus ponens and get $B \in \Delta_j$. We first have to ask in which logic we are operating? If the logic is **K**, then $A \Rightarrow B \in \Delta_i$ is usable to us iff $j = i + 1$. If the logic is **T**, we can use $A \Rightarrow B$ also if it is in Δ_j and if the logic is **S4**, we can use any $A \Rightarrow B$ in any Δ_i, for any $i \leq j$.

How do we show that $X \Rightarrow Y$ holds at Δ_j? We start a new theory Δ_{j+i}, assume X in there and try and prove Y. Again if the logic is **K**, only $A \Rightarrow B$ from Δ_j can be used in Δ_{j+1}. If the logic is **T**, we can also use $A \Rightarrow B$ which are proved in Δ_{j+1}. If logic is **S4**, we can use any $A \Rightarrow B$ from $\Delta_1, \ldots, \Delta_{j+1}$.

In this context it is very easy to define the notion of *mode of proof*. The *mode* of proof as applied to proofs in Δ_{j+1} tells us from which Δ_i we can import and use wffs $A \Rightarrow B$.

(5) We can now talk about *changing modes*. We can have a rule of the following form

- To show $A \Rightarrow B$ in Δ_j, when we are in mode x, assume $\Delta_{j+1} = \{A\}$ and try and prove B in a new mode $y = \varepsilon(x)$, where ε is the mode changing function.

The exact meaning of the above will be explained in the next example and beyond.

Let us consider some natural examples involving modes.

EXAMPLE 1.3 (Modes of Proof with **K** and **T**). This example illustrates how proof theory can change modes. The modal logics **K** and **T** are used. Consider modal logic formulated with strict implication $\Rightarrow$, cojunction $\wedge$, $\top$ and $\bot$.

Focus on a natural deduction proof theory for $\Rightarrow$ with a view of making $\Rightarrow$ either a **K** or **T** implication. Consider the following candidate formula to be proved:

$$\alpha = A \wedge (A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \wedge (C \Rightarrow D) \Rightarrow D).$$

Consider the following proof of α in **T**:[3]

(1) We show α, from the subproof in lines 1.1–1.4

1.1. A, assumption,
1.2. $A \Rightarrow (B \Rightarrow C)$, assumption,
1.3. $B \Rightarrow C$, from 1.1 and 1.2. This step is allowed since the logic is **T**. It is not allowed if the logic is **K**.
1.4. We show $B \wedge (C \Rightarrow D) \Rightarrow D$, from the subproof in lines 1.4.1–1.4.4.

   1.4.1. B, assumption,
   1.4.2. $C \Rightarrow D$, assumption,
   1.4.3. C, from 3 and 1.4.1. This step is available for **K** and for **T**.
   1.4.4. D, from 1.4.2 and 1.4.3. This step is correct if the logic is **T** but not if the logic is **K**.

Thus, a can be proved in the logic **T**.

Imagine now that we have $\Rightarrow$ as a hypermodality where the outermost $\Rightarrow$ is a **K** modality and the inner $\Rightarrow$ is a **T** modality and then the **T** and **K** modalities keep on alternating with the nesting of $\Rightarrow$ inside each other. We can rewrite α more explicitly as α'

$$\alpha' = A \wedge (A \Rightarrow_{\mathbf{T}} (B \Rightarrow_{\mathbf{K}} C)) \Rightarrow_{\mathbf{K}} (B \wedge (C \Rightarrow_{\mathbf{K}} D) \Rightarrow_{\mathbf{T}} D).$$

Let us now rewrite the proof above, annotating the modalities and proof rules available at each subproof. Our outer proof procedure is a **K** proof procedure. This makes the proof procedure of any $X \Rightarrow Y$ subproof (i.e. of the form: to show $X \Rightarrow Y$: assume X show Y) a **T** procedure.[4]

Show α' using **K** rules. This follows from subproof 1.1–1.4 which is carried out using **T** rules:

1.1. A, assumption,
1.2. $A \Rightarrow_{\mathbf{T}} (B \Rightarrow_{\mathbf{K}} C)$, assumption.
1.3. $B \Rightarrow_{\mathbf{K}} C$, from 1.1–1.2. We can execute modus ponens, since we are using **T** rules which include reflexivity.
1.4. We show $B \wedge (C \Rightarrow_{\mathbf{K}} D) \Rightarrow_{\mathbf{T}} D$ from subproof 1.4.1–1.4–4, using **K** rules (we are alternating the use of **K**, **T** rules as we go through the nesting).

   1.4.1. B, assumption.
   1.4.2. $C \Rightarrow_{\mathbf{K}} D$, assumption.
   1.4.3. C, from 3 and 1.4.1. This step is available using **K** rules.
   1.4.4. We are not allowed to get D from 1.4.2 and 1.4.3, since we are following **K** rules here!

So the proof fails.

The structure of the paper is as follows: Section 2 will introduce hypermodalities from a variety of semantical points of view. Section 3 will semantically present a case study. Section 4 will discuss translations. Section 5 will axiomatise the case study, and Section 6 will conclude the paper.

2. HYPERMODALITIES

This section introduces hypermodality first intuitively and then more formally.

2.1. *Intuitive Presentation of Hypermodality*

The traditional semantics for propositional modal logic with $\Box$ (and $\neg$, $\vee$, $\wedge$, $\to$) offers Kripke models of the form (S, R, a, h), where S is the set of possible worlds, $R \subseteq S^2$ is the accessibility relation, $a \in S$ is the actual world and h is the *assignment*, associating with each atomic q a subset $h(q) \subseteq S$. (S, R, a) is called the *frame* of the model.

Completeness of a logical system **L** relative to a class of Kripke structures $\mathcal{K}$ is defined as follows:

- Actual world completeness:
$$\mathbf{L} \vdash A \text{ iff for all } (S, R, a, h) \in \mathcal{K} \quad \text{we have } a \vDash A.$$

Note that we require satisfaction in the actual world only. This may turn out to be important later.

A class $\mathcal{K}$ of models is said to be a class of frames defined by a frame formula $\varphi(a, R)$ (in a possibly higher-order language with a and R) iff we have

- $(S, R, a, h) \in \mathcal{K}$ iff $(S, R, a) \vDash \varphi(a, R)$.

For example we may have $\varphi(a, R) \equiv aRa$. The logic of this class is not normal, (because $\Box A \to A$ is a theorem but $\Box(\Box A \to A)$ is not).

This semantics offers the following truth conditions for $\Box A$ at a world t.

- $t \vDash \Box A$ iff for all s such that tRs, $s \vDash A$.

Let us refer to the above as the *mode of evaluation* of $\Box$ at a possible world.

We now continue with our explanation of hypermodalities.

Consider the configuration in Figure 1 in which we have aRt, aRs and sRt all hold.

Let us now evaluate at a the two wffs $A_0 = \Box\Box\Box q$ and $A_1 = \Box\Box q$.

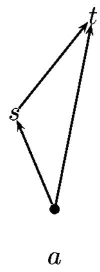

Figure 1.

- $a \vDash A_0$ ff $\forall xy(aRx \wedge xRy \rightarrow y \vDash \Box q)$.
- $a \vDash A_1$ iff $\forall y(aRy \rightarrow y \vDash \Box q)$.

In the course of evaluating $a \vDash A_0$, we come to evaluate $t \vDash \Box q$, after having 'passed through' the points s and a.

In the course of evaluating $a \vDash A_1$, we come to evaluate $t \vDash \Box q$, after having 'passed through' the point a only.

The traditional evaluation of $t \vDash \Box q$ does not care about 'how we got to t' and how many points we 'passed through'.

When evaluating $t \vDash \Box A$ for a hypermodality $\Box$, however, we do care about the 'history' of previous evaluation points.

The *history* of points one passes through in the course of evaluation can be taken as a *historical mode parameter* $\alpha = (a, x_1, \ldots, x_n)$, where $aRx_1 \wedge x_1Rx_2 \wedge \cdots \wedge x_{n-1}Rx_n$ holds. We have

- $x_n \vDash_\alpha \Box A$ iff $\forall x_{n+1}(x_nRx_{n+1} \rightarrow x_{n+1} \vDash_{\alpha*(x_{n+1})} A)$, where $*$ is concatenation of sequences.

In traditional modal logic the evaluation at x_n does not depend on α, i.e. $\vDash_\alpha$ equals $\vDash$ for all α.

To give some intuition as to how the evaluation might depend on α, consider the connective *Soon* A. Consider a flow of time $\{1, 2, 3, 4, \ldots, \}$ and let us understand *Soon* A to mean that A will be true soon. The meaning of 'soon' depends on the moment of time. So assume that there exists a function f such that

- $m \vDash$ *Soon* A iff $n \vDash A$ for some n such that $m \leq n \leq f(m)$.

We can assume that f is a function such that $m < f(m)$ for all m. The interval $[m, f(m)]$ is the interval of time considered 'soon' from the point of view of world m. The above gives us a traditional temporal logic, with temporal accessibility relation R where

$$mRn \text{ iff } n \in [m, f(m)].$$

Let us examine the truth value of *Soon*^{2} A at time 1. We have:

- $1 \vDash Soon^2\, A$ iff $\exists n_1, n_2(1 \leq n_1 \leq f(1)$ and $n_1 \leq n_2 \leq f(n_1)$ and $n_2 \vDash A)$.

It may be the case that $n_2 < f(1)$, in which case $n_2 \vDash A$ is a witness for the truth of $1 \vDash Soon\, A$ as well for the truth of $1 \vDash Soon^2\, A$.

Our evaluation of $n_2 \vDash A$ in traditional temporal logic is independent of whether we come to consider it in the course of evaluating $1 \vDash Soon\, A$ or $1 \vDash Soon^2\, A$.

Let us now add a twist to our story. Imagine a time traveller who is trying to evaluate $t \vDash Soon\, A$ by actually travelling over the time points $t, t+1, t+2, t+3, \ldots, t+f(t)$. The more he travels, the more impatient he becomes. Suppose in the course of evaluation of $1 \vDash Soon^3\, q$ he gets to the point of evaluating $n_2 \vDash Soon\, q$ and he is so impatient that he decides to look for q at the points n_2 and $n_2 + 1$ only. Thus we have two modes of evaluation for $n \vDash Soon\, q$, *impatient mode* and *patient mode*. We have

- $t \vDash_p Soon\, q$ iff $\exists n(t \leq n \leq f(t)$ and $n \vDash q)$.
- $t \vDash_i Soon\, q$ iff $t \vDash q$ or $t + 1 \vDash q$.

To summarise:

The mode of evaluation does not depend only on t but also on how (long) it took to get to t.

Let us now go back to our background Kripke semantics; to models of the form (S, R, a, h). We are going to introduce evaluation modes into such semantics. It is convenient to regard the relation xRy as a classical formula $\Psi_{\mathbf{K}}(x, R, a, y)$ in the language of the relation R, the individual variables x, y and actual world constant a as follows

- $\Psi_{\mathbf{K}}(x, R, a, y) =_{\text{def}} xRy$

we have

- $t \vDash \Box A$ iff $\forall s(\Psi_{\mathbf{K}}(t, s) \to s \vDash A)$.

We can refer to $\Psi_{\mathbf{K}}$ as the mode of evaluation for $\Box$. It is fixed in the semantics and does not change. Intuitively it tells us, for a world x, how to evaluate $\Box A$ at x, namely where to look for worlds y where $y \vDash A$ must hold. The subscript **K** indicates that this formula is used in the case of **K** modality.

We can think of different formulas Ψ for the mode. Consider, for example (see [1], **KB** is also known as **B**, the logic for R symmetric).

- $\Psi_{\mathbf{T}}(x, R, a, y) =_{\text{def}} xRy \vee x = y$.
- $\Psi_{\mathbf{K4}}(x, R, a, y) =_{\text{def}} (\exists n \geq 1)xR^n y$.

Where $xR^n y$ is defined by induction as:

– xR^0y iff $x = y$,
– $xR^{n+1}y$ iff $\exists z(xRz \wedge zR^n y)$.

- $\Psi_{\mathbf{KB}}(x, R, a, y) =_{\text{def}} (xRy \vee yRx)$.

Clearly $\Psi_{\mathbf{K4}}(x, R, a, y)$ is not a first-order formula. It defines the transitive closure of R.

One can think of Ψ as changing the accessibility relation from R to $\lambda x \lambda y \Psi(x, y)$. Another way of looking at Ψ is that it gives us a new mode of how to use R in evaluating the truth value of $\Box A$. The latter view is more convenient to use because we will be shifting modes during the evaluation.

Let us write $\vDash_i$, to mean that the mode Ψ_i is used in the evaluation. Then $\vDash_{\mathbf{K}}$ for arbitrary frames (S, R, a) yields the logic **K**, $\vDash_{\mathbf{T}}$ yields the logic **T**, $\vDash_{\mathbf{KB}}$ yields the logic **KB** and $\vDash_{\mathbf{K4}}$ yields the logic **K4**.

Note that our starting point is a frame (S, R, a) with an arbitrary R. We define $\Psi_i(x, R, a, y)$ as a binary relation and use it to evaluate $\Box$. Thus in traditional terms the frame we are using is (S, Ψ_i, a) not (S, R, a). When we shift modalities, i.e. change from $t \vDash_i \Box A$ to $s \vDash_j \Box A$ it is like shifting from (S, Ψ_i, a) to (S, Ψ_j, a).

We need to be able to recognise the logic $\mathbf{L}_i$, defined by all frames of the form (S, Ψ_i, a), so that we can say that in mode i we let $\Box_i$ be this logic. This presents us with a technical problem. Let **L** be a familiar logic, can it be characterised by a Ψ as the logic of all frames of the form (S, Ψ, a)? We discuss this problem in Section 6.

We now give another example, an historical mode dependent modality.

DEFINITION 2.1 (Historical hypermodality). Let (S, R, a, h) be a Kripke model.

(1) A historical mode parameter α is any sequence of points of the form $\alpha = (t_0 = a, t_1, t_2, \dots, t_n)$, $n \geq 0$ such that for $i = 0, \dots, n-1$, $t_i R t_{i+1}$ hold.

(2) For each α, let $\Psi_\alpha(x, R, a, y)$ be a (possibly second-order) formula with two free variables x, y and constants R and a.Ψ_α is called a *mode formula*.

(3) Satisfaction has the form $t \vDash_\alpha A$, and satisfies the following clauses

- $t \vDash_\alpha q$ iff $t \in h(q)$, for q atomic.
- $t \vDash_\alpha A \wedge B$ iff $t \vDash_\alpha A$ and $t \vDash_\alpha B$.
- $t \vDash_\alpha \neg A$ iff $t \nvDash_\alpha A$.
- $t \vDash_\alpha \Box A$ iff $\forall s(\Psi_\alpha(t, R, a, s) \to s \vDash_{\alpha^*(s)} A)$.

We say A α-holds in the model if $a \vDash_\alpha A$.

REMARK 2.2. The above definition is an extreme case, where different historical parameters α yield different modes Ψ_α. We can regard the family

$$\{\Psi_{(a,t_1,\ldots,t_k,t)}(t, R, a, y)\}$$

as a relation $\Psi(\alpha, y)$ between finite sequences α and points y, and consider the model (S, Ψ, a, h). We have

- $\alpha \vDash \Box A$ iff $\forall s(\Psi(\alpha, s) \rightarrow \alpha^*(s) \vDash A)$.

Such models arise in connection with Generalised quantifiers and automated deduction for modal logic. See [2, 4, 5]. The respective logic is decidable and axiomatisable and has the f.m.p.

EXAMPLE 2.3. Consider the modes

$$\begin{aligned}\Psi_1(x, y) &= \Psi_{\mathbf{K}}(x, y) \equiv xRy,\\ \Psi_0(x, y) &= \Psi_{\mathbf{T}}(x, y) \equiv xRy \vee x = y.\end{aligned}$$

Let us shift alternately between the two modes, namely

- $t \vDash_i \Box A$ iff $\forall s(\Psi_i(t, s) \rightarrow s \vDash_{1-i} A)$

Let $\mathbf{K}[\Psi_{\mathbf{T}}, \Psi_{\mathbf{K}}] = \{A \mid a \vDash_0 A$, in any model $(S, R, a, h)\}$.
To see what our logic does, consider the formula $\Box(\Box A \rightarrow A)$. We have

- $a \vDash_0 \Box(\Box A \rightarrow A)$ iff $\forall x(aRx \vee a = x \rightarrow x \vDash_1 (\Box A \rightarrow A))$.
- $x \vDash_1 \Box A \rightarrow A$ iff $\forall y(xRy \rightarrow y \vDash_0 A) \rightarrow x \vDash_1 A$.

EXAMPLE 2.4 (Quantifiers). If we apply the same ideas to quantifiers we get that an expression of the form $QxQyQzR(x, y, z)$ has the following optional meaning.

(1) Generalised quantifiers, where the range of $QuA(u, \vec{w})$ depends on the free variables $\vec{w}$ in the matrix $A(u, \vec{w})$. Denoting the range by $V_{\vec{w}}$ we get

$$QxQyQzR(x, y, z) = \forall x \in V_\emptyset \forall y \in V_x \forall z \in V_{x,y} R(x, y, z).$$

This reading corresponds to the hypermodal logic of Remark 2.2.

(2) If we want the meaning of Q to alternate between $\forall$ and $\exists$ we get

$$QxQyQzR(x, y, z) \equiv \forall x \exists y \forall z R(x, y, z).$$

This corresponds to Example 2.3 where $\Box$ alternates between a **T** and **K** modality.

This kind of Q quantifier is reminiscent of the natural language quantifier 'any', where its meaning (as $\forall$ or $\exists$) depends on its position in the sentence.

2.2. *Formal Presentation of Hypermodality*

This subsection gives a formal definition of hypermodality. It also develops some machinery for later sections.

We treat the simple case is where the number of modes is a finite set μ and there is a function ε for shifting modes. This case is given in the next definition.

DEFINITION 2.5 (Mode shifting). Let $\mu = \{\Psi_0, \dots, \Psi_k\}$ be a set of modes and let ε be a function assigning to each $0 \leq i \leq k$ a value $0 \leq \varepsilon(i) \leq k$.

Let (S, R, a, h) be a Kripke model. We define the following (μ, ε) satisfaction in the model

- $t \vDash_i \Box A$ iff $\forall s(\Psi_i(t, s) \to s \vDash_{\varepsilon(i)} A)$.
- We say A is true in the model if $a \vDash_0 A$.

DEFINITION 2.6. (1) Let $\mathcal{K}$ be a class of models of the form (S, R, a, h). Let (μ, ε) be a mode system. We write $\mathcal{K} \vDash_{(\mu,\varepsilon)} A$ iff for every model (S, R, a, h) in $\mathcal{K}$ we have $a \vDash_0 A$.

(2) Let **L** be a logic complete for a class $\mathcal{K}$ of Kripke models of the form (S, R, a, h). Let $\mathcal{K}[\mu, \varepsilon]$ be $\{A \mid \mathcal{K} \vDash_{(\mu,\varepsilon)} A\}$. We sometimes write $\mathbf{L}[\mu, \varepsilon]$ for $\mathcal{K}[\mu, \varepsilon]$, when the implicit dependence on $\mathcal{K}$ is clear.

Obviously the nature of hypermodal logic depends on (μ, ε) and its abstract properties and also on the class $\mathcal{K}$ of models chosen.

A general mode shifting system (μ, ε) is a sort of abstract non-deterministic automaton, shifting from mode to mode. When used to shift in any concrete model (S, R, a), it also interacts with the properties of R. In order to identify those abstract properties of this automaton which are independent of the particular (S, R, a), we can 'run' it on a 'free' model (S, R, a) and see what it does. To give an example of what we mean, consider the condition

- $t \vDash_m \Box A$ iff $\forall s(tRs \to s \vDash_n A)$.

This abstract condition does not imply that for some s, evaluation of a $\Box$ can occur both in m and in n modes, since in a general free model (S, R, a), tRs implies $s \neq t$. However, if the m mode is reflexive (as in the case of $\Psi_{\mathbf{T}}(t, s)$) then we know that such an s exists, even in the free (S, R, a) model.

We need some definitions.

In a free model of a binary relation of the form (S, R, a), we need to have infinitely different non-related successors of $a, a_1, a_2, a_3, \ldots$, each a_i of which has infinitely many all different successors $a_{i,1}, a_{i,2}, \ldots$ and so on. Similarly for predecessors. Here is the definition:

DEFINITION 2.7 (Free model of a binary relation). (1) Let a be constant and let $\mathbf{a}_n, \mathbf{b}_n$, $n = 1, 2, \ldots$ be formal pairwise different unary functions. Let $\mathbb{S}$ be the set of all Skolem terms constructed using these functions. Namely we have

- $a \in \mathbb{S}$.
- If $x \in \mathbb{S}$ then $\mathbf{a}_n(x) \in \mathbb{S}$ and $\mathbf{b}_n(x) \in \mathbf{S}$.

Define a relation $\mathbb{R}$ on $\mathbb{S}$ by

- $x\mathbb{R}y$ iff for some n either $y = \mathbf{a}_n(x)$ or $x = \mathbf{b}_n(y)$.

(2) Let $\mathbb{R}^*$ be the reflexive and transitive closure of $\mathbb{R}$. Let for $t \in \mathbb{S}$, $\mathbb{S}_t = \{x \in \mathbb{S} \mid t\mathbb{R}^*x\}$.

For ordinary modality $\Box$, we evaluate $t \vDash \Box A$ at points of $\mathbb{S}_t$.

We now need some machinery for future use. Given a mode system (μ, ε), and a model (S, R, a, h), we begin evaluating any wff A at mode Ψ_0, i.e. $a \vDash_0 A$. Thus we can ask for a $t \in S$, what possible modes can be evaluated at t, if we try all evaluations $a \vDash_0 A$ for all A? The next definition deals with this.

DEFINITION 2.8 (Worlds where evaluation is at mode m). Let (S, R, a) be a frame, let (μ, ε) be a mode system, with $\mu = \{\Psi_0, \ldots, \Psi_k\}$ and ε the mode shift function as defined in Definition 2.5.

We first define by induction the (not necessarily disjoint!) sets $V_m \subseteq S$, $0 \leq m <\leq k$ as follows:

(1) $a \in V_0$ (O is the start mode).
(2) If $t \in V_m$ and $\Psi_m(t, s)$ holds and $m' \in \varepsilon(m)$ then $s \in V_{m'}$.

For each $t \in S$ and mode m define the set S_t^m by induction on m. S_t^m is the set of all worlds that may be involved in any evaluation of the form $t \vDash_m A$, for any A.

(3) Let $\varepsilon^0(m) = \{m\}$, $\varepsilon^1(m) = \varepsilon(m)$, $\varepsilon^{k+1}(m) = \bigcup_{y \in \varepsilon^k(m)} \varepsilon(y)$.

(4) Let $S^m_{t,0} = \{t\}$ if $t \in V_m$, and $\emptyset$ otherwise.
(5) If $x \in S^m_{t,k}$ we can inductively assume that $x \in V_n$, $n \in \varepsilon^k(m)$.
Let y be any point such that $\Psi_n(x, y)$ holds and let $n' \in \varepsilon(n)$, (then $n' \in \varepsilon^{k+1}(m)$ and $y \in V_{n'}$ by our definitions above), we let $y \in S^m_{t,k+1}$.
(6) Let $S^m_t = \bigcup_k S^m_{t,k}$.

The previous definition can be used to define properties of ε as it applies to μ.

DEFINITION 2.9 (Coherence of a mode system). (1) Let (μ, ε) be a mode system and let $(\mathbb{S}, \mathbb{R}, a)$ be the free model. Consider the sets V_m as defined on the free model. Let $\mu' = \{\Psi_{m_1}, \dots, \Psi_{m_n}\} \subseteq \mu$ be a maximal set of modes such that $\bigcap_j V_{m_j} \neq \emptyset$.

We say (μ, ε) is coherent iff for any such maximal $\mu' = \{\Psi_{m_1}, \dots, \Psi_{m_n}\}$ we have that $\lambda y \bigwedge_j \Psi_{m_j}(x, y)$ is consistent as a property of x.[5]

(2) In any model (S, R, a), let $V_{\mu'} = \bigcap_{m \in \mu'} V_m$. Note that any $t \in S$ is a member of a single $V_{\mu'}$. I.e. for $\mu' \neq \mu''$ we have $V_{\mu'} \cap V_{\mu''} = \emptyset$.

DEFINITION 2.10. Let (μ, ε) be a coherent mode shifting system. Let $I_\mu = \{\mu' \mid \mu'$ a maximal subset of models such that the intersection $V_{\mu'} = \cap_{m \in \mu'} V_m \neq \emptyset$ holds on the free model of Definition 2.7.

Define a relation $<_\varepsilon$ on I_μ by

- $\mu' <_\varepsilon \mu''$ iff for some modes $\Psi_{m'} \in \mu'$, $\Psi_{m''} \in \mu''$ we have $m'' \in \varepsilon(m')$.

3. Case Study: Shifting between **K** and **T** modalities

The language in this section contains one hypermodal symbol $\Box$.

This section will study in detail the logic $\mathbf{K}[\Psi_{\mathbf{T}}, \Psi_{\mathbf{K}}]$ of Example 2.3 (named in this section as $\mathbf{H}^S_1$ in short).

We shall see that this logic is not complete for any Kripke or neighbourhood frames. We shall also axiomatise it. This will give us a feel for what hypermodal logics can do.

DEFINITION 3.1 (Semantical definition of the hypermodal logic $\mathbf{H}^S_1$). Consider the family of Kripke models of the form $\mathbf{m} = (S, R, a, h)$ where (S, R, a) is a tree.[6]

We define two satisfaction relations $t \vDash_0 A$ and $t \vDash_1 A$ inductively as follows:

(1) $t \vDash_i q$ iff $t \in h(q)$, for q atomic and $i = 0, 1$.

(2) $t \vDash_i A \wedge B$ iff $t \vDash_i A$ and $t \vDash_i B$, $t \vDash_i \neg A$ iff $t \nvDash_i A$.
(3) $t \vDash_0 \Box A$ iff $t \vDash_1 A$ and $\forall s(tRs \to s \vDash_1 A)$.
(4) $t \vDash_1 \Box A$ iff $\forall s(tRs \to s \vDash_0 A)$.
(5) We say $\mathbf{m} \vDash A$ iff $a \vDash_0 A$.
(6) Let $\mathbf{H}_1^S \vDash A$ mean that $\mathbf{m} \vDash A$ for all $\mathbf{m}$.

DEFINITION 3.2 (Translation of $\mathbf{H}_1^S$ into Standard **K**). We define two translations τ_0 and τ_1 from $\mathbf{H}_1^S$ to **K**.

(1) $\tau_i(A) = A$, for A without modality, $i = 0, 1$.
(2) $\tau_i(A \wedge B) = \tau i(A) \wedge \tau_i(B)$,
$\tau_i(\neg A) = \neg\tau_i(A)$.
(3) $\tau_0(\Box A) = \tau_1(A) \wedge \Box\tau_1(A)$.
(4) $\tau_1(\Box A) = \Box\tau_0(A)$.

LEMMA 3.3. *Let* $\mathbf{m} = (S, R, a, h)$ *and let* $t \in S$ *be arbitrary and let* A *be any wff. Let* $\vDash_0$, $\vDash_1$ *be as before and let* $\vDash$ *be the traditional* **K** *satisfaction relation. Then for* $i = 0, 1$ *we have:*

$(*)$ $t \vDash_i A$ *iff* $t \vDash \tau_i(A)$.

Proof. By structural induction.
(1) $(*)$ clearly holds for A without $\Box$.
(2) The cases of $\wedge$ and $\neg$ are immediate.
(3) $t \vDash_0 A$ iff $t \vDash_1 A$ and $\forall s(tRs \to s \vDash_1 A)$ iff (by the induction hypothesis) $t \vDash \tau_1(A) \wedge \forall s(tRs \to s \vDash \tau_1(A))$ iff $t \vDash \tau_1(A) \wedge \Box\tau_1(A)$ iff $t \vDash \tau_0(\Box A)$.
(4) $t \vDash_1 \Box A$ iff $\forall s(tRs \to s \vDash_0 A)$ iff (by the induction hypothesis) $\forall s(tRs \to s \vDash \tau_0(A))$ iff $t \vDash \Box\tau_0(A)$ iff $t \vDash \tau_1(\Box A)$. □

The following series of Lemmas, Examples and Definitions serve to acquaint us with the logic $\mathbf{H}_1^S$ as well as providing tools for axiomatising $\mathbf{H}_1^S$.

LEMMA 3.4. *Let A be any wff. Then*

(1) $\tau_0(\Box^0 A) = \tau_0(A)$
(2) $\tau_0(\Box A) = \tau_1(A) \wedge \Box\tau_1(A)$.
(3) $\tau_0(\Box^2 A) = \Box\tau_0(A) \wedge \Box^2\tau_0(A)$.
(4) $\tau_0(\Box^3 A) = \Box\tau_1(A) \wedge \Box^2\tau_1(A) \wedge \Box^3\tau_1(A)$.
(5) $\tau_0(\Box^4 A) = \Box^2\tau_0(A) \wedge \Box^3\tau_0(A) \wedge \Box^4\tau_0(A)$.
(6) $\tau_0(\Box^5 A) = \Box^2\tau_1(A) \wedge \Box^3\tau_1(A) \wedge \Box^4\tau_1(A) \wedge \Box^5\tau_1(A)$.
(7) *For* $n \geq 2$ *and* $0 \leq i \leq 1$.
(8) $\tau_0(\Box^{2n+i} A) = \bigwedge_{m=n}^{2n+i} \Box^m \tau_i(A)$.

Proof. (1) $\Box^0 A = A$.
(2) By definition.

$$\begin{array}{lll}
(3) & \tau_0(\Box^2 A) & = \tau_1(\Box A) \wedge \Box\tau_1(\Box A) \\
& & = \Box\tau_0(A) \wedge \Box\Box\tau_0(A), \\
& \tau_0(\Box^3 A) & = \tau_1(\Box^2 A) \wedge \Box\tau_1(\Box^2 A) \\
& & = \Box\tau_0(\Box A) \wedge \Box^2(\tau_0(\Box A)). \\
(4) & & = \Box(\tau_1(A) \wedge \Box\tau_1(A)) \wedge \Box^2(\tau_1(A) \wedge \Box\tau_1(A)) \\
& & = \Box\tau_1(A) \wedge \Box^2\tau_1(A) \wedge \Box^2\tau_1(A) \wedge \Box^3\tau_1(A) \\
& & = \Box\tau_1(A) \wedge \Box^2\tau_1(A) \wedge \Box^3\tau_1(A).
\end{array}$$

(5)–(6) check directly or use (7) below.
(7) We use induction on $n \geq 2$.

$$\begin{aligned}
\tau_0(\Box^2(\Box^{2n+i} A)) &= \Box(\tau_0(\Box^{2n+i} A) \wedge \Box^2\tau_0(\Box^{2n+i} A)) \\
&= \Box \bigwedge_{m=n}^{2n+i} \Box^m \tau_i(A) \wedge \Box^2 \bigwedge_{m=n}^{2n+i} \Box^m \tau_i(A) \\
&= \bigwedge_{m=n+1}^{2(n+1)+i-1} \Box^m \tau_i(A) \wedge \bigwedge_{m=n+2}^{2(n+1)+i} \Box^m \tau_i(A) \\
&= \bigwedge_{m=n+1}^{2(n+1)+i} \Box^m \tau_i(A).
\end{aligned}$$

$\Box$

EXAMPLE 3.5. For atomic q

(1) $\tau_0(\Box^2 q) = \Box q \wedge \Box\Box q$.
(2) $\tau_0(\Box^3 q) = \Box q \wedge \Box^2 q \wedge \Box^3 q$.
(3) $\tau_0(\Box^4 q) =$
(4) For $n \geq 2$, $\tau_0(\Box^{2n+i} q) = \bigwedge_{m=n}^{2n+i} \Box^m q$.
(5) $\tau_0(\neg q \wedge \Box^2 q) = \neg q \wedge \Box q \wedge \Box\Box q)$.
(6) $\tau_0(\neg q \wedge \Box^2 q \wedge \Diamond(\neg q \wedge A) = \neg q \wedge \Box q \wedge \Box^2 q \wedge \tau_1(A)$.
(7) $\tau_0(\neg q \wedge \Box^2 q \wedge \Diamond(\neg q \wedge \Box B) = \neg q \wedge \Box q \Box^2 q \wedge \Box\tau_0(B)$.

EXAMPLE 3.6. Let q^i_j be a double indexed sequence of atoms. Let

(1) $\beta^i_1(q^i_1) = \neg q^i_1 \wedge \Box^2(q^i_1)$.
(2) $\beta^i_2(q^i_1, q^i_2) = \neg q^i_2 \wedge \Box^2 q^i_2 \wedge \Diamond(\neg q^i_2 \wedge \Box\beta^i_1)$.
(3) $\beta^i_{n+1}(q^i_1, \ldots, q^i_{n+1}) = \neg q^i_{n+1} \wedge \Box^2 q^i_{n+1} \wedge \Diamond(\neg q^i_{n+1} \wedge \Box\beta^i_n)$.

(4) Note that:

$$\begin{aligned}\tau_0(\beta_1^i) &= \neg q_1^i \wedge \Box^2 q_1^i = \beta_1^i,\\ \tau_0(\beta_{n+1}^i) &= \neg q_{n+1}^i \wedge \Box^2 q_{n+1}^i \wedge \Box\tau_0(\beta_n^i).\end{aligned}$$

Using induction we get:

$$\tau_0(\beta_{n+1}^i) = \bigwedge_{j=0}^{n} \Box^j(\neg q_{n+1-j}^i \wedge \Box^2 q_{n+1-j}^i).$$

(5) We show that if $t_{n+1}Rt_n \wedge \cdots \wedge t_2Rt_1$ and $t_{n+1} \vDash_0 \beta_{n+1}^i$ then $t_1 \vDash_0 \beta_1^i$.

The proof is by induction
Case $n = 1$.

$$\begin{aligned}t_1 \vDash_0 \beta_2^i \quad &\text{iff } t_2 \vDash_0 \neg q_2^i \wedge \Box^2 q_2^i \wedge \Diamond(\neg q_2^i \wedge \Box(\neg q_1^i \wedge \Box^2 q_1^i))\\ &\text{iff } t_2 \vDash_0 \neg q_2^i \wedge \Box^2 q_2^i \wedge \forall t_1(t_2RT_1 \to t_1 \vDash_0 \neg q_1^i \wedge \Box^2 q_1^i).\end{aligned}$$

Hence if t_2Rt_1 and $t_2 \vDash_0 \beta_2^i$ then $t_1 \vDash_0 \beta_1^i$.
Case n.

$$\begin{aligned}t_{n+1} \vDash_0 \beta_{n+1}^i \quad &\text{iff } t_{n+1} \vDash_0 \neg q_{n+1}^i \wedge \Box^2 q_{n+1}^i \wedge \Diamond(\neg q_{n+1}^i \wedge \Box\beta_n^i)\\ &\text{iff } t_{n+1} \vDash_0 \neg q_{n+1}^i \wedge \Box^2 q_{n+1}^i \wedge \forall t_n(t_{n+1}Rt_n \to t_n \vDash \beta_n^i).\end{aligned}$$

Hence by induction for all $t_n, \ldots, t_1$ if $t_{n+1}Rt_n \wedge \cdots \wedge t_2Rt_1$ and $t_{n+1} \vDash_0 \beta_{n+1}^i$ we get $t_1 \vDash \beta_1^i$.

LEMMA 3.7. *Let* A *be such that*

$$\mathbf{H}_1^S \vDash A$$

then

$$\mathbf{H}_1^S \vDash \Box^{2n} A.$$

Proof. We have $\mathbf{K} \vdash \tau_0(A)$ by assumption.

$$\begin{aligned}\tau_0(\Box^2 A) &= \Box\tau_0(A) \wedge \Box^2\tau_0(A),\\ \tau_0(\Box^{2n} A) &= \bigwedge_{m=n}^{2n} \Box^m \tau_0(A).\end{aligned}$$

In both cases we have $\mathbf{K} \vdash \tau_0(\Box^{2n} A)$ and hence $\mathbf{H}_1^S \vDash A$. □

LEMMA 3.8. *Let* q *be an atom, then*

(1) $\mathbf{H}_1^S \nvDash \Box(\Box q \to q)$.

(2) $\mathbf{H}_1^S \vDash \Box q \to q$.
(3) $\mathbf{H}_1^S$ *is not closed under substitution of provably equivalent formulas.*

Proof. (1) $\tau_0(\Box(\Box q \to q)) = \tau_1(\Box q \to q) \wedge \Box\tau_1(\Box q \to q) = (\Box q \to q) \wedge \Box(\Box q \to q)$ and this is not a theorem of **K**.

(2) $\tau_0(\Box q \to q) = q \wedge \Box q \to q$ which is a theorem of **K**.

(3) We have $\mathbf{H}_1^S \vDash \top \leftrightarrow (\Box q \to q)$ but $\mathbf{H}_1^S \nvDash \Box\top \leftrightarrow \Box(\Box q \to q)$. $\Box$

LEMMA 3.9. $\mathbf{H}_1^S$ *is not complete for any class of Kripke frames.*

Proof. Assume otherwise and get a contradiction. Let (S, R, a) be a frame where all theorems of our logic hold under any assignment h and let h_0 be an assignment such that $a \vDash_{h_0} \Diamond(\neg q \wedge \Box q)$. Such a frame and an assignment must exist in view of previous lemmas.

Since every instance of $\Box q \to q$ must be valid at a under any assignment we must have that aRa holds. We also have since $a \vDash_{h_0} \Diamond(\neg q \wedge \Box q)$ that there exists an a_1 such that $a_1 \vDash_{h_0} \neg q$ and $a_1 \vDash_{h_0} \Box q$ and therefore a_1Ra_1 does not hold.

Consider now $\Box^2(\Box q \to q)$. This is a theorem of the logic and must therefore hold at a under h_0.

But how can it hold?, because we have $aRa \wedge aRa_1$ and $a_1 \vDash_{h_0} \Box q$ and $a_1 \vDash_{h_0} \neg q$.

A contradiction. $\Box$

The next lemma shows that our logic is not complete for any class of neighbourhood frames. Although the case of Kripke frames follows from that, we think including both proofs is instructive for the case of other logics.

DEFINITION 3.10. Neighbourhood models have the form $(S, \mathbb{F}, a, h)$ where S is the set of possible worlds, $a \in S$ is the actual world and $\mathbb{F}$ is a family of filters, associating with each $t \in S$ a filter $\mathbb{F}_t$.[7] h is the assignment to atoms. The following is the truth condition for $\Box$.

- $t \vDash_h \Box A$ iff $\|A\|_h \in \mathbb{F}_t$ where $\|A\|_h = \{x \mid x \vDash_h A\}$.
- We say A is true in the model iff $a \vDash_h A$.

LEMMA 3.11. $\mathbf{H}_1^S$ *is not complete for any class of neighbourhood frames.*

Proof. Assume otherwise, then since $\neg\Box(\Box q \to q)$ is consistent, there must exist a neighbourhood model $(S, \mathbb{F}, a, h_0)$ such that for any assignment h and any A

$$a \vDash_h \Box A \to A,$$
$$a \vDash_h \Box^2(\Box A \to A)$$

while for h_0 and q

$$a \not\models_{h_0} \Box(\Box q \to q).$$

Let $Q_0 = \|q\|_{h_0}$.

The above conditions mean the following second-order conditions hold for arbitrary $X \subseteq S$

(1) $\forall X[X \in \mathbb{F}_a \to a \in X]$.

(1) holds because $\Box A \to A$ mush hold at a under any assignment and any A.

Since $a \not\models_{h_0} \Box(\Box q \to q)$, this means that $\|\Box q \to q\|_{h_0} \notin \mathbb{F}_a$. This implies that $\{y \mid y \not\models_{h_0} \Box q \text{ or } y \models_{h_0} q\} \notin \mathbb{F}_a$, which further implies that

$$\overline{Q}_0 = \{y \mid Q_0 \notin \mathbb{F}_y \text{ or } y \in Q_0\} \notin \mathbb{F}_a.$$

We summarise the latter as condition (2):

(2) $\overline{Q}_0 = Q_0 \cup \{y \mid Q_0 \notin \mathbb{F}_y\} \notin \mathbb{F}_a$.

The third condition is obtained from the fact that for any h and any A, $a \models_h \Box\Box(\Box A \to A)$. This gives us the following condition for any atom x and h

$$\{z \mid z \models_h \Box(\Box x \to x)\} \in \mathbb{F}_a.$$

but

$$z \models_h \Box(\Box x \to x) \text{ iff } \|\Box x \to x\|_h \in \mathbb{F}_z.$$

Thus we get

$$\{z \mid \{y \mid \|x\| \notin \mathbb{F}_y \text{ or } y \models_h x\} \in \mathbb{F}_y\} \in \mathbb{F}_a.$$

Since h is arbitrary we get condition (3).

(3) $\forall X[\{z \mid X \cup \{y \mid X \notin \mathbb{F}_y\} \in \mathbb{F}_z\} \in F_a]$.

We can now proceed to get a contradiction:

Let $X = Q_0$ in (3). We therefore get that:

$$Y_0 = \{z \mid Q_0 \cup \{y \mid Q_0 \notin \mathbb{F}_y\} \in \mathbb{F}_z\} \in \mathbb{F}_a.$$

By (1) we get that $a \in Y_0$ and hence

$$Q_0 \cup \{y \mid Q_0 \notin \mathbb{F}_y\} \in \mathbb{F}_a$$

which contradicts (2). $\Box$

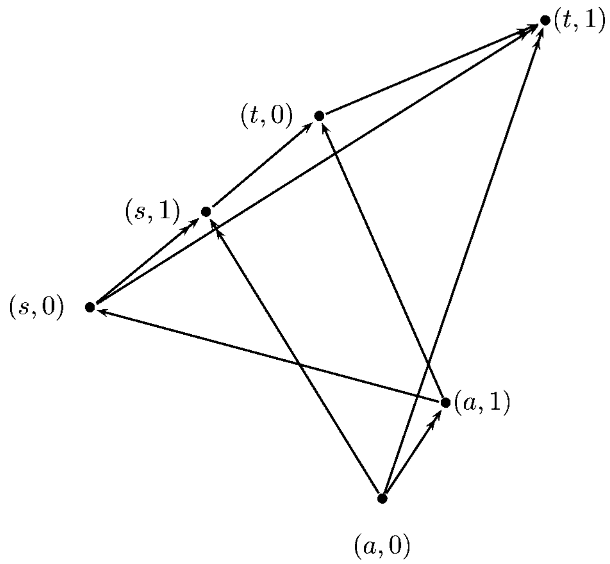

Figure 2.

REMARK 3.12. Notice that we have not used any special properties of the filters $\mathbb{F}_t$. This means that any extension of **K** which proves all instances of $\Box A \to A$, $\Box^2(\Box A \to A)$ but not $\Box(\Box A \to A)$ cannot be complete for any neighbourhood frames. For example we can take **K4.3** [**T**, **K**] (i.e. assume R is linear transitive and irreflexive).

4. Translations of Hypermodality

This section discusses the relationship of hypermodality and other kinds of modalities, namely many dimensional modal logics and multimodal logics.

We can view the system $\mathbf{H}_1^S$ as a two-dimensional system. Instead of writing $t \vDash_i A$ we write $(t, i) \vDash A$ and consider our set of possible worlds as $S \times \{0, 1\}$. We then have a relation R_2 defined as follows:

$$(t, 0)R_1(s, i) \text{ iff } i = 1 \wedge (t = s \vee tRs),$$
$$(t, 1)R_2(s, i) \text{ iff } i = 0 \wedge tRs.$$

Figure 1 will become Figure 2, where $\to$ now shows R_2 accessibility, and $\twoheadrightarrow$ shows R_1 accessibility.

The two-dimensional view requires the further restriction on the assignment h, namely that for all atomic q and $t \in S$ and $i, j \in \{0, 1\}$ we have:

$$(t, i) \in h(q) \text{ iff } (t, j) \in h(q).$$

Therefore this point of view does not yield completeness for frames!

We now turn to the connection of hypermodality and multimodal logics.

A multimodal logic with modalities $\Box_0$, $\Box_1$ can be characterised by Kripke models of the form (S, R_0, R_1, a, h) where $R_i \subseteq S^2$, $i = 0, 1$ and the truth table for $\Box_i$, $i = 0, 1$ is as follows:

- $t \vDash \Box_i A$ iff $\forall s(tR_i s \to s \vDash A)$.

If we want $\Box_0$ to be a **T** modality we can assume that R_0 is reflexive. We can view a hypermodal system with $\Box$ as a fragment of a multimodal logic, where $\Box$ can be $\Box_i$, depending on its position in the formula.

Let ν_i be two translations from the hypermodal language into the multimodal language with $\Box_0$ and $\Box_1$. We have

- $\nu_i(A) = A$, for A without $\Box$.
- $\nu_i(\neg A) = \neg\nu_i(A)$.
- $\nu_i(A \wedge B) = \nu_i(A) \wedge \nu_i(B)$.
- $\nu_0(\Box A) = \Box_0\nu_1(A)$.
- $\nu_1(\Box A) = \Box_1\nu_0(A)$.

Thus

- $\nu_0(\Box q \to q) = \Box_0 q \to q$.
- $\nu_0(\Box(\Box q \to q)) = \Box_0(\Box_1 q \to q)$.
- $\nu_0(\Box\Box(\Box q \to q)) = \Box_0\Box_1(\Box_0 q \to q)$.
- $\nu_0(\Box\Box q \to \Box q) = \Box_0\Box_1 q \to \Box_0 q$.

Our hypermodal logic for $\Box$ with modes Ψ_0, Ψ_1 based on the class of models $\{(S, R, a, h)\}$ is translated into the multimodal logic with $\Box_0$, $\Box_1$ based on the class of models $\{(S, \Psi_0, \Psi_1, a, h)\}$. Ψ_0, Ψ_1 are based on R. Thus our modal logic $\mathbf{H}_1^S$ when viewed as a multimodal logic is special; the relations used for $\Box_0$ and $\Box_1$ are both defined from a single underlying relation R.

We can define the notion of a multimodal frame (S, R_0, R_1, a, h) as *generated* from a frame (S, R, a, h) iff there exists generators $\Psi_i(x, R, a, y)$, $i = 0, 1$ such that $R_i = \lambda x \lambda y \Psi_i$, $i = 0, 1$.

This means for our case that we can take R_1 as our R and assume that R_0 is the reflexive closure of R_1. So let (S, R_0, R_1, a, h) be a bimodal logic where $R_1 = R$ and R_0 is the reflexive closure of R. Consider (S, R, a, h) as a monomodal model for $\mathbf{H}_1^S$.

LEMMA 4.1. *For the above translation we have:*

$(*)$ $\quad t \vDash_i A$ *iff* $t \vDash \nu_i(A)$.

Proof. This can be proved by structural induction on A.

The cases of $\Box$ free A and of classical connectives $\wedge$ and $\neg$ present no difficulties. Let us check the case of $\Box$:

$t \vDash_0 \Box A$ iff $t \vDash_1 A$ and $\forall s(tRs \to s \vDash_1 A)$ iff $t \vDash \nu_1(A)$ and $\forall s(tRs \to s \vDash \nu_1(A)$ iff $t \vDash \Box_0 \nu_1(A)$ iff $t \vDash \nu_0(\Box A)$.

Similarly $t \vDash_1 \Box A$ iff $\forall s(tRs \to s \vDash_0 A)$ iff $\forall s(tRs \to s \vDash \nu_0(A))$ iff $t \vDash \Box_1 \nu_0(A)$ iff $t \vDash \nu_1(\Box A)$. $\Box$

$(*)$ presents us with a conceptual query. We saw that $\mathbf{H}_1^S$ is not complete for any family Kripke frames. Is the bimodal logic with $\Box_0$, $\Box_1$ complete for a family of Kripke frames?

The answer is yes, the family of frames of the form (S, R_0, R_1, a) where R_0 is the reflexive closure of R_1.[8]

Does this not imply that $\mathbf{H}_1^S$ is complete for frames? The answer is no. When we examine $\Box q \to q$ in a frame (S, R, a), since it is a theorem of $\mathbf{H}_1^S$ for all q, we conclude that R must be reflexive. If we examine $\nu_0(\Box q \to q) = \Box_0 q \to q$ in models (S, R_0, R_1, a), we conclude R_0 is reflexive, which is OK.[9]

Compare, for example, $\mathbf{H}_1^S \vDash \Box q \to q$ but $\mathbf{H}_1^S \nvDash \Box\Box q \to \Box q$. The translation $\nu_0(\Box\Box q \to \Box q)$ is $\Box_0\Box_1 q \to \Box_0 q$ which does not hold in the bimodal logic.

The reader may be under the intuitive impression that somehow a hypermodal logic is a composition of several modalities $\Box_1, \ldots, \Box_n$ used according to position in the formula. Thus $\Box$ is sometimes $\Box_i$, sometimes $\Box_j$, etc. The reader may think that the study of hypermodality should be reduced to special cases of the study of multimodal logic. This is not the general case. The interaction between $\Box_1, \ldots, \Box_n$ may make $\Box$ a completely new modality.[10] Example 6.1 below illustrates this point.

5. AXIOMATISING $\mathbf{H}_1^S$

5.1. *Methodological Discussion*

Before giving syntactical axiomatisation for our logic, we need to discuss methodology. There are different kinds of proof methods around. There are Hilbert-type systems, Gentzen-type systems, tableaux systems, natural deduction systems and labelled deductive systems. The difference between these methods, when applied to modal logics, manifests itself in how much of the geometry of the Kripke structure (possible worlds semantics) is brought into the syntax.

We defined modal logic satisfaction in terms of truth in the actual world. Thus the set of theorems of a modal logic **L** is the set

$$\{A \mid a \vDash A, \text{ for all models } (S, R, a, h)\}.$$

Axiomatising the logic means generating this set. In the case of $\mathbf{H}_1^S$ it means generating the set of all formulas A such that $a \vDash_0 A$, in any model (S, R, a, h), see Definition 3.1.

We know, however, that $a \vDash_0 A$ may depend, in its semantic evaluation, on other points in the model; for example it may depend on whether a point b such that aRb, $a \neq b$ exists such that $b \vDash_1 B$, for some subformula B.

The Hilbert-type axiomatisation, because it generates only As such that $a \vDash_0 A$, cannot directly talk about such bs. We can indirectly write $a \vDash_0 \Diamond B$ but this means $\exists x[(x = a \vee aRx) \wedge x \vDash_1 B]$.

We cannot force the geometrical condition $x \neq a$.

There are other ways around the problem. Suppose we had an atomic q such that $a \vDash_0 \neg q$ and $\forall x(aRx \wedge a \neq x \rightarrow x \vDash_1 q)$, then we can say what we want by writing $a \vDash_0 \Diamond(q \wedge B)$.

Can we say that such a q exists? That depends on the properties of our logic. Indeed, we can write $a \vDash_0 \neg q \wedge \Box\Box q$. This will give us such a q. Thus to axiomatise our logic as a Hilbert system, we need to say more about other points and we need to use such qs to enable us to say what we need. This approach gives us a rather complicated Hilbert axiomatisation of our logic.

The source of the complication lies in the fact that a Hilbert system allows us to code information of the form $b \vDash_i B$, bRc, etc. only though $\vDash_0$ validity in the actual world and the use of the connectives $\Diamond$ and $\Box$. It makes the axioms complicated for the case of our logic because of the multiple modes available. Traditional Hilbert axiomatisations of **K** happen to be able to manage because there is only one mode.

It is not that our two-mode modal logic is complicated, it is that Hilbert style axiomatisation are geometrically poor.

Let us now turn to Gentzen style axioms. This style allows for a richer geometry. We can look at sequents of the form

$$A_0, A_1, \ldots, A_n \vDash B.$$

It is up to us to interpret this sequent semantically. Let us, for the case of ordinary modal logic, take the following interpretation

- $A_0, A_1, \ldots, A_n \vDash B$ iff in any Kripke model (S, R, a, h) and any $t_0 = a, t_1, \ldots, t_n$ such that $t_0Rt_1, t_1Rt_2, \ldots, t_{n-1}Rt_n$ and such that $t_i \vDash A_i$, $i = 0, \ldots, n$ we have $t_n \vDash B$.

This kind of geometry gives us better expressive power reflecting more of the possible world semantics in the syntax. In the case of two modes 0 and 1 we can venture and propose to amend the definition to:

- $A_0, A_1, \ldots, A_n \vDash B$ iff for all models and all $t_0 = aRt_1, \ldots, t_{n-1}Rt_n$ we have that if $t_i \vDash_i A_i$, $i = 0, \ldots, n$ then $t_n \vDash_n B$, where $\vDash_m$ is taken to be $\vDash_0$ if m is even and $\vDash_1$ if m is odd.

It is more likely that we find a simpler axiomatisation using the above sequents because of its better expressive power.

In fact, we can put more of the geometry into the syntax. Authors have proposed double sequents for modal and substructural logics; we can allow sequents of the form

$$(A_0, A'_0), (A_1, A'_1), \ldots, (A_n, A'_n) \vDash (B, B')$$

with the following validity definition

- $(A_0, A'_0), \ldots, (A_n, A'_n) \vDash (B, B')$ is valid iff for every model and every $t_0 = aRt_1, \ldots, t_{n-1}Rt_n$ such that for all $0 \le i \le 1$, $t_i \vDash A_i$ and $t_i \vDash_1 A'_i$ we have $t_n \vDash_0 B$ and $t_n \vDash_1 B'$.

It is obvious that the more semantical information we put into the syntax the easier it becomes to axiomatise. We therefore adopt the following approach. Rather than sneak in the information through the geometry of the sequents, double sequents, hyper-sequents, special connectives or whatever, we shall go all the way and use the theory of labelled deductive systems. It allows for part of the Kripke structure to be put in the syntax as labels. See [7]. Thus we have letters standing for labels (which stand for possible worlds) and a relation ρ on labels (standing for the accessibility R) and our theory (sequents) have the form ($m_t, n = 0$ or 1)

$$\Delta = \{(m_t, t) : A_t, t_i \rho t_j\} \vDash (n, s) : B.$$

Validity is defined as follows:

- $\Delta \vDash (n, s)$: B is valid iff for every Kripke model (S, R, a, h) and every function g assigning to labels t values $g(t) \in S$ such that if $t\rho t' \in \Delta$ then $g(t)Rg(t')$ holds, we have the following:
  If for all (m_t, t): $A_t \in \Delta$ we have $t \vDash_{m_t} A_t$ then also $s \vDash_n B$ holds.

In this paper we shall give two extreme axiomatisations. The Hilbert one and the labelled deductive systems one.

5.2. *Hilbert Axiomatisation*

We shall give a Hilbert axiomatisation of $\mathbf{H}_1^S$. We need first to adopt a different point of view of axiomatising the traditional logic **K**. We saw that in $\mathbf{H}_1^S$, $\Box q \to q$ for q atomic is a theorem, so is $\Box^{2n}(\Box q \to q)$ but not $\Box(\Box q \to q)$.

Also the rule of substitution of logical equivalents does not hold. We have $\mathbf{H}_1^S \vDash \top \leftrightarrow (\Box q \to q)$ but $\mathbf{H}_1^S \nvDash \Box\top \leftrightarrow \Box(\Box q \to q)$.

So we shall therefore formulate an axiomatisation of traditional **K** without necessitation and substitution.

DEFINITION 5.1 (Another formulation of modal logic **K**). Consider the following formulation of modal **K**.

Axioms

1. A if A is a substitution instance of a truth function tautology.
2. $\Box A$ if A is as in (1).
3. $\Box(A \to B) \to (\Box A \to \Box B)$, A arbitrary.
4. $\Box(\Box(A \to B) \to (\Box A \to \Box B))$, A arbitrary.

Rules

$$\frac{\vdash A, \vdash A \to B}{\vdash B},$$

$$\frac{\vdash A}{\vdash \Box\Box A}.$$

Note that we replaced necessitation

$$\frac{\vdash A}{\vdash \Box A}$$

by 2-necessitation

$$\frac{\vdash A}{\vdash \Box^2 A}.$$

To compensate for the loss of necessitation we had to take also as axiom $\Box A$ for any axiom A of **K**.

DEFINITION 5.2 (The modal logic $\mathbf{H}_1$). Let $\mathbf{H}_1$ be the system obtained by extending **K** under the formulation of the previous Definition 5.1 with the axiom

5. $\Box A \to A$, for A such that $\mathbf{K} \vdash \tau_1(A) \wedge \Box\tau_1(A) \to \tau_0(A)$.
   Note that we will have shown that $\Box(\Box A \to A)$ cannot be proved in $\mathbf{H}_1$.
   If it is added as an axiom we get the traditional logic **T**.

6. If $\mathbf{K} \vdash \tau_0(A) \to \tau_0(B)$ then $\mathbf{H}_1 \vdash A \to B$.

The above is not a good axiomatisation. It makes use of $\mathbf{K}$ provability (which is technically OK, since $\mathbf{K}$ is decidable) and of the translations τ_0 and τ_1. It is illuminating, however, and affords a neat, almost trivial completeness proof.

We shall give a better Hilbert axiomatisation later on.

We saw that $\mathbf{H}_1^S$ can be translated into a bimodal logic with $\Box_0$, $\Box_1$ and that the Hilbert axiomatisation of the bimodal logic is relatively easy. If we can define inside $\mathbf{H}_1^S$, using $\Box$ only, the two modalities $\Box_0$ and $\Box_1$, we could then give an axiomatisation of $\mathbf{H}_1^S$. The key to defining $\Box_0$, $\Box_1$ is irreflexivity. Let q be an atom and assume that $t \vDash_0 \neg q \wedge \Box\Box q$. This means that $t \vDash \neg q$ and $\forall x, y(tRx \wedge xRy \to x \vDash q$ and $y \vDash q$.

Now imagine that $t \vDash_0 \Diamond(\neg q \wedge A)$ holds. This forces $t \vDash_1 A$. In our logic, a theory Δ is $\{A \mid t \vDash_0 A\}$ for some t. We can get the $\vDash_1$ part of Δ by looking at $\Theta = \{A \mid \Delta \vdash \Diamond(\neg q \wedge A)\}$ provided q is such that $\Delta \vdash \neg q \wedge \Box\Box q$. Together (Δ, Θ) constitute a possible world t because they contain in them both $\vDash_0$ and $\vDash_1$ satisfaction.

$$\Delta = \{A \mid t \vDash_0 A\}, \quad \Theta = \{A \mid t \vDash_1 A\}.$$

So to give an effective axiomatisation we need irreflexivity rules involving $\neg q \wedge \Box\Box q$.

DEFINITION 5.3 (IRR Hilbert System for $\mathbf{H}_1^S$). The following axiomatisation defining the system $\mathbf{H}_1$ makes use of the well known Gabbay Irreflexivity Rule, (see [3]).

Axioms: (E, F are wffs without $\Box$).

1. $A \wedge \Box A$, where A is a substitution instance of a truth functional tautology.
2. $\Box(A \to B) \to (\Box A \to \Box B)$.
3. $\Box(\Box(A \to B) \to (\Box A \to \Box B))$.
4. $\Diamond\top$.
5. $\neg E \wedge \Box^2 E \wedge Y \to \Diamond(\neg E \wedge Y)$, where $Y = A$ or $Y = \Box A$, for A without $\Box$.
6. $\neg E \wedge \Box^2 E \wedge \Diamond(\neg E \wedge A) \wedge \Diamond(\neg E \wedge B) \to \Diamond(\neg E \wedge A \wedge B)$.
7. $\neg E \wedge \Box^2 E \to \Diamond(\neg E \wedge A) \vee \Diamond(\neg E \wedge \neg A)$.
8. $\Box A \wedge \neg E \wedge \Box^2 E \to \Diamond(\neg E \wedge A)$.
9. $\Box X \wedge \neg E \wedge \Box^2 E \to \Diamond(\neg E \wedge \Box(\neg\Box^2 F \to \Diamond\neg F \wedge X))$.
10. $\neg E \wedge \Box^2 E \wedge \Diamond A \wedge \Diamond(\neg E \wedge \Box Y \wedge \Box\Box Y' \wedge \neg A \wedge \Box\Box X) \to \Diamond(A \wedge X \wedge Y \wedge \Box Y')$ where Y, Y' are without $\Box$.
11. $\neg E \wedge \Box^2 E \wedge \Diamond(C \wedge E \wedge Y) \to \Diamond(\neg E \wedge \Diamond(Y \wedge E \wedge \Diamond(C \wedge E))$.

Rules

$$\text{MP}: \quad \frac{\vdash A; \vdash A \to B}{\vdash B}$$

$$\text{IRR}: \quad \frac{\vdash \neg q \wedge \Box^2 q \to A}{\vdash A}$$

where q is an atom not in A.

2-necessitation: $\frac{\vdash A}{\vdash \Box^2 A}$

$$\text{IRR}^n: \quad \frac{\vdash \bigwedge_{m=1}^{n} \beta_m^m \to A}{\vdash A}$$

where β_m^m are as in Example 3.6 and q_j^i are all not in A.

LEMMA 5.4. *The axioms and rules given are sound for the semantics.*
Proof. By direct verification. □

DEFINITION 5.5. A theory Δ is said to be 0-consistent (resp. 1-consistent) iff for no finite number of $A_i \in \Delta$ do we have $\vdash \neg \bigwedge A_i$ (resp. $\vdash \Box\neg \bigwedge A_i$).

LEMMA 5.6. *Any i-consistent theory can be extended to a complete i-consistent theory.*
Proof. We need to address the case $i = 1$.
Assume Δ_n is 1-consistent and let $\delta_{n=1}$ be the $n+1$ wff. Can $\Delta_n \cup \{\delta_{n+1}\}$ and $\Delta_n \cup \{\neg\delta_{n+1}\}$ be both 1-inconsistent?
Otherwise, we have for some A_i, B_j

$$\vdash \Box\Big(\bigwedge A_i \to \delta_{n+1}\Big),$$

$$\vdash \Box\Big(\bigwedge B_j \to \neg\delta_{n+1}\Big),$$

hence $\vdash \Box(\bigwedge A_i \wedge \bigwedge B_j \to \bot)$.
A contradiction because of axiom $\Diamond\top$. □

DEFINITION 5.7. A theory Δ is said to be an IRR-theory if it is 0-consistent and for some sequence of atoms q_j^i, $i, j = 1, 2, 3, \ldots$ and for each $\beta_n^n(q_1^n, \ldots, q_n^n)$ we have that $\beta_n^n \in \Delta$.

We shall see later how any 0-consistent formula A can be extended to a complete IRR-theory. At this stage, let us show that every IRR theory has a model.

DEFINITION 5.8. (Δ, Θ) is a 0-1 pair iff

(1) Δ is 0-consistent and complete and Θ is 1-consistent and complete.
(2) For any A, B:

$$\Box B \in \Delta \to B \in \Theta,$$

$$\Box^2 \neg A \wedge \Diamond A \in \Delta \to A \in \Theta.$$

(3) $A \in \Theta \to \Diamond A \in \Delta$, for all A.
(4) For any Y without $\Box$, $Y \in \Delta \to Y \in \Theta$.
(5) For all Y' without $\Box$:

$$\Box Y' \in \Delta \to \Box Y' \in \Theta.$$

LEMMA 5.9. *Let* (Δ) *be an IRR theory and assume* $\neg q_1^1 \wedge \Box^2 q_1^1 \in \Delta$. *Let* $\Theta = \{A \mid \Diamond(\neg q_1^1 \wedge A) \in \Delta\}$. *Then* Θ *is a complete* 1*-consistent theory such that* (Δ, Θ) *is a* 0-1 *pair.*

Proof. Axiom 7 gives us completeness. Assume Θ is not 1-consistent then $\vdash \Box(\neg q_1^1 \to \neg \bigwedge_i A_i)$, $A_i \in \Theta$. Hence $\Box(\neg q_1^1 \to \neg \bigwedge_i A_i) \in \Delta$. But from axiom 6 we get $\Diamond(\neg q_1^1 \wedge \bigwedge_i A_i) \in \Delta$, a contradiction.

The 0-1 pair property of (Δ, Θ) follows from axioms 5, 8. $\Box$

LEMMA 5.10. *Let* (Δ, Θ) *be a* 0-1 *pair, with* Δ *an IRR theory.*

Assume $\Diamond A \in \Theta$.

Let $\Delta'_{1,A} = \{B \mid \Box B \in \Theta\} \cup \{A\}$. *Then* $\Delta'_{1,A}$ *is* 0*-consistent and can be completed to a* 0*-consistent IRR theory* Δ_{1A}.

Proof. Otherwise

$$\vdash \bigwedge B_i \to \neg A, \Box B_i \in \Theta.$$

Hence $\vdash \Box^2(\bigwedge B_i \to \neg A)$.
Hence $\Box^2(\bigwedge B_i \to \neg A) \in \Delta$ and from axiom 8

$$\Box\Big(\bigwedge B_i \to \neg A\Big) \in \Theta.$$

Hence $\bigwedge \Box B_i \to \Box\neg A \in \Theta$ and so $\Box\neg A \in \Theta$, a contradiction.

We can complete $\Delta'_{1,A}$ to a 0-consistent complete theory $\Delta_{1,A}$.

Note that since Δ is an IRR theory, $\beta_{n+1}^{n+1} \in \Delta$, namely $\neg q_{n+1}^{n+1} \wedge \Box^2 q_{n+1}^{n+1} \wedge \Diamond(\neg q_{n+1}^{n+1} \wedge \Box \beta_n^{n+1})$ is in Δ and so $\beta_n^{n+1} \in \Delta_{1,A}$.

These considerations show that $\Delta_{1,A}$ is IRR. For let $\bar{q}_i^n$ be a renaming of $q_{i+1}^{n=1}$, $i = 1, \ldots, n$. Then $\beta_n^n(p_1^n, \ldots, p_n^n) = \beta_n^{n+1}$. $\Box$

LEMMA 5.11. *Let* (Δ, Θ), $\Delta_{1,A}$ *be as before. Let* $\Theta_{1,A} = \{A \mid \neg q_1^2 \wedge \Box^2 q_1^2 \wedge \Diamond(\neg q_1^2 \wedge A) \in \Delta_{1,A}\}$.

Then as proved before, $(\Delta_{1,A}, \Theta_{1,A})$ *are a* 0-1 *pair. Furthermore, for any* X*, if* $\Box X \in \Delta$*, then* $X \in \Theta_{1,A}$.

Proof. Assume $\Box X \in \Delta$, then from axiom 9 we get that $\Box(\neg q_1^2 \wedge \Box^2 q_1^2 \to \Diamond(\neg q_1^2 \wedge X)) \in \Theta$ and hence $X \in \Theta_{1,A}$. $\Box$

LEMMA 5.12. *Let* (Δ, Θ) *be as before and assume* $\neg A \in \Theta$ *and* $\Diamond A \in \Delta$*. Then* $\Theta'_{0,A} = \{A\} \cup \{B \mid \Box B \in \Delta\} \cup \{C \mid \Box\Box C \in \Theta\} \cup \{Y \wedge \Box Y' \mid \Box Y \wedge \Box^2 Y' \in \Theta, Y, Y'$ *without* $\Box\}$ *is* 1-*consistent and can be completed to a* 1-*consistent theory* $\Theta_{0,A}$.

Proof. Otherwise

$$\vdash \Box\Big(\bigwedge_i B_i \to \neg\Big(A \wedge Y \wedge \Box Y' \wedge \bigwedge_j C_j\Big)\Big)$$

hence $\Box\neg(A \wedge Y \wedge \Box Y' \wedge \bigwedge_j C_j \in \Delta$ but also $\Box\Box \bigwedge_j C_j \in \Theta$.

From axiom 10 we get that $\Diamond(A \wedge Y \wedge \Box Y' \wedge \bigwedge_j C_j) \in \Delta$, a contradiction. We can extend $\Theta'_{0,A}$ to a complete 1-consistent theory $\Theta_{0,A}$. $\Box$

LEMMA 5.13. *Let* (Δ, Θ), $\Theta_{0,A}$ *be as before. Then*

$$\Delta'_{0,A} = \{B \mid \Box B \in \Theta\} \cup \{\Diamond C \mid C \in \Theta_{0,A}\} \cup \{Y \mid Y \in \Theta_{0,A},\ Y \text{ without } \Box\}$$

is 0-*consistent.*

Proof. Otherwise let E be such that $\neg E \wedge \Box^2 E \in \Delta$, $\neg E \in \Theta$ and $E \in \Theta_{0,A}$. Such E exists by construction. We have

$$\vdash \bigwedge_i \Box B_i \to \Big(Y \wedge E \to \neg \bigwedge_j \Diamond(C_j \wedge E \wedge Y)\Big)$$

hence

$$\vdash \Box^2\Big(\bigwedge_i \Box B_i \to \Big(Y \wedge E \to \bigwedge_j \Diamond(C_j \wedge E \wedge Y)\Big)\Big)$$

hence

$$\Box\Big(\bigwedge_i \Box B_i \to \Big(Y \wedge E \to \neg \bigwedge_j \Diamond C_j \wedge E \wedge Y\Big)\Big) \in \Theta$$

hence

$$\Box\Big(Y \wedge E \to \neg \bigwedge_j \Diamond(C_j \wedge E \wedge Y)\Big) \in \Theta$$

but

$$\vdash \Box\Big(\neg \bigwedge \Diamond D_j \to \neg\Diamond \bigwedge D_j\Big)$$

hence

$$\Box\Big(\Diamond \bigwedge (C_j \wedge E \wedge Y\Big) \to \neg(E \wedge Y)) \in \Theta.$$

But by construction of $\Theta_{0,A}$ we have $\Diamond(\bigwedge_i C_i \wedge E \wedge Y) \in \Delta$. From axiom 11 we have $\Diamond(Y \wedge E \wedge \Diamond(\bigwedge C_i \wedge E \wedge Y)) \in \Theta$, a contradiction. □

DEFINITION 5.14. Let (Δ, Θ) and (Δ', Θ') be two 0-1 pairs. We say $(\Delta, \Theta) R (\Delta', \Theta')$ iff the following holds:

(1) If $\Box A \in \Theta$ then $A \in \Delta'$.
(2) If $\Box A \in \Delta$ then $A \in \Theta'$.

The previous series of lemmas proved the following:

LEMMA 5.15. *Let* (Δ, Θ) *be an IRR* 0-1 *pair then*

(1) *If* $\Diamond A \in \Theta$ *then there exists an IRR* 0-1 *pair* (Δ', Θ') *such that* $(\Delta, \Theta) R (\Delta', \Theta')$ *and* $A \in \Delta$.
(2) *If* $\Diamond A \in \Delta$ *and* $\neg A \in \Theta$ *then there exists an IRR* 0-1 *pair* (Δ', Θ') *such that* $(\Delta\Theta) R (\Delta', \Theta')$ *and* $A \in \Theta'$.

Proof. From previous lemmas. □

LEMMA 5.16. *Let A be a consistent wff. Then A has a model.*

Proof. Since A is consistent, so is $\Delta_0 = \{A\} \cup \{\beta_n^n \mid n = 1, 2, \ldots, \}$, for β_n^n using completely new atoms $\{q_j^i\}$. This follows from the IRR^n rule. We can complete Δ_0 to a complete IRR theory Δ and construct the IRR 0-1 pair (Δ, Θ). Let S be the set of all 0-1 IRR theories. Let R as before and let $a = (\Delta, \Theta)$. Let h be defined by

$$(\Delta, \Theta) \in h(q) \text{ iff } q \in \Delta$$

(we know that $q \in \Delta$ iff $q \in \Theta$ for an IRR 0-1 pair!).

We claim for any pair $(\Gamma_0, \Gamma_1) \in S$ and any A

$$(*) \qquad (\Gamma_0, \Gamma_1) \vDash_i A \text{ iff } A \in \Gamma_i,\ i = 0, 1.$$

This is proved by structural induction on A. □

5.3. *LDS Axiomatisation*

We now develop the basic concepts of an LDS theory and consequence for the same logic $\mathbf{H}_1^S$. The general theory of LDS is presented in [7]. See also Definition 1.2.

DEFINITION 5.17 (Labelled theories). (1) Consider a formal first-order language with atomic terms D= $\{t_1, t_2, \ldots\}$, a binary relation ρ, equality $=$ and a distinguished constant $\mathbf{d} \in D$. This is the language of the algebra of labels. D stands for a set of possible worlds, $\mathbf{d}$ stands for the actual world and ρ stands for the accessibility relation.

(2) A literal has the form $x = y$, $x \neq y$, $x\rho y$, $\neg x\rho y$ where $x, y \in D$.

(3) A declarative unit has the form $(m, t) : A$ where $m \in \{0, 1\}$, $t \in D$ and A is a wff of the modal language with $\Box$.

(4) A (labelled) theory is a set of declarative units and literals.

DEFINITION 5.18 (Labelled rules). This definition lists proof rules to be used to define a consequence relation for $\mathbf{H}_1^S$.

1. *Axioms*
   $(m, t) : A$, for A a substitution instance of a truth functional tautology.

2. $\dfrac{(m, t) : A;\ (m, t) : A \to B}{(m, t) : B}$.

3. (a) $\dfrac{(m, t) : \Box A;\ t\rho s}{(1 - m, s) : A}$,

   (b) $\dfrac{(1 - m, s) : \Box A;\ t\rho s}{(m, t) : \Diamond A}$.

4. (a) $\dfrac{(0, t) : \Box A}{(1, t) : A}$,

   (b) $\dfrac{(1, t) : A}{(0, t) : \Diamond A}$.

5. $\dfrac{(m, t) : \bot}{(n, s) : A}$.

6. $\dfrac{(1, t) : \Diamond A}{\text{create a } \textit{new } s,\ t\rho s \text{ with } (0, s) : A}$.

7. $\dfrac{(0, t) : \Diamond A;\ (1, t) : \neg A}{\text{create a } \textit{new } s,\ t\rho s \text{ with } (1, s) : A}$.

8. $\dfrac{(m, t) : A}{(1 - n, t) : A}$

   for A not containing $\Box$.

9. Rules 6–7 are referred to as *creative*, the others as *non-creative*.

DEFINITION 5.19 (Consequence). Let Δ be a theory and let $(m, t) : A$ be a declarative unit. We define the notion of $\Delta \vdash_n (m, t) : A$. n is the number of creation steps used.

(1) $\Delta \vdash_0 (m, t) : A$ if (m, t) appears in Δ and $(m, t) : A$ can be reached after a finite number of applications of the non-creative rules of the previous definitions to the units in Δ.

(2) $\Delta \vdash_0 \Delta'$ if $\Delta \vdash_0 (x, s) : B$ for all $(x, s) : B$ in Δ'.

(3) $\Delta \vdash_{n+1} (m, t) : A$ if Δ is obtained from a Δ', Δ'' such that $\Delta \vdash_0 \Delta'$ and Δ'' is obtained from Δ' by the application of a creative rule (and the new declarative unit added) such that $\Delta'' \vdash_n (m, t) : A$.

(4) $\Delta \vdash (m, t) : A$ if for some n, $\Delta \vdash_n (m, t) : A$.

(5) Δ is consistent if for no (m, t) do we have $\Delta \vdash (m, t) : \bot$.

DEFINITION 5.20. Let Δ be a theory and (S, R, a, h) a model. Let g be a function $g : \Delta \mapsto S$ such that for any t appearing in Δ, $g(t) \in S$.

Assume g satisfies the following:

- If $\pm t\rho s \in \Delta$ then $\pm g(t) R g(s)$ holds.
- If $t = s$ (resp. $t \neq s$) in Δ then $g(t) = g(s)$ (resp. $g(t) \neq g(s)$) holds.
- $g(\mathbf{d}) = a$.
- If $(m, t) : A \in \Delta$ then $g(t) \vDash_m A$ holds,

then we say that (S, R, a, h, g) is a model of Δ.

THEOREM 5.21. *If Δ is consistent then it has a model.*

Proof. The usual Henkin/tableaux construction. □

6. Conclusion: Hypermodality in Context

One can certainly look at the hypermodal framework as a way of significantly broadening the expressive power of modal logic in an interesting way. However, there is another, more general way of looking at hypermodality, as a manifestation of a more general proof theoretic mechanism. This point of view will now be explained. It is best done by an example.

Consier **S4** strict implication, $A \Rightarrow B$. We can understand it as having a future temporal meaning. Read $A \Rightarrow B$ as an *insurance policy*, say that whenever an accident happens (i.e. A holds), then remedies are guaranteed (B holds). It is clear that the accident must happen after the policy was taken out. Thus if we list all data avaialble according to the earlier–later temporal occurrence then modus ponens can be effected only in the form

$$A \Rightarrow B, A \vdash B$$

and not in the form

$$A, A \Rightarrow B \vdash ?B.$$

We can be even more careful and record that B does not come at the same time as A, but afterwards. (Compare with Definition 1.2, items 2 and 4.)

Now let us look at automobile insurance known as *Home Start*. This allows you to call the company any time if your car does not start and they will come and start/charge your battery, wherever you are. Arrival is guaranteed within the hour. There is a catch, however. You are allowed to use this service only a fixed number of times (I think it is 4 times). So let us see what this means in terms of modus ponens. We will have to write something like

$$(\text{mode } n)\ A \Rightarrow B, A \vdash B\ (\text{mode } n - 1),$$

where the (mode n) says how many times you can use $A \Rightarrow B$.

The proof theory will have to tell us how to change mode with every use of modus ponens.

This example is very real and can become very complex. I can now make my general point:

Logic is now evolving and becoming more and more complex in response to the needs of computer science and artificial intelligence. These disciplines are building devices which help/replace humans in their daily activities. To be successfully sold to the public, we need to properly model (*in logic*) *various aspects of human activity and use these logical models to help build these devices. The urgency of this need is fuelling the development of new concepts in logic. The process of modelling is the same as the intuitive analysis traditionally done by philosophical logic, only more urgent and more computational. The notion of mode is one such concept arising in this context.*

I believe it is going to be central in the modal logic of the future.

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NOTES

[1] The word 'hyper' was suggested to me by Michael Gabbay. It can be used to qualify the logical meaning of any symbol x and make it dependent on its occurrence in the wff. So, for instance, a *hypernominal* is a nominal whose meaning depends on its place in the wff. Similarly a *hyperSkolem-function*, *hyperquantifier*, etc.

[2] As we evaluate $s \vDash B$ inductively from the outside, the evaluation path leading to any $t \vDash \Box A$ indicates the position of this occurrence of $\Box A$ inside B.

[3] In the proof below think semantically. Let $\mathbf{m}$ be an arbitrary model, then $a \vDash \alpha$ iff for all t, aRt and $t \vDash$ (1.1 and 1.2 and 1.3) imply $t \vDash$ 1.4. But $t \vDash$ 1.4 iff for any s, tRs, if $s \vDash$ (1.4.1 and 1.4.2) then $s \vDash$ 1.4.4.

Thus proof lines 1.1–1.4 take place in t and 1.4.1–1.4.4 take place in s.

[4] In semantical terms in a model (S, R, a), we are evaluating as follows: $a \vDash_{\mathbf{K}} \alpha'$ iff $\forall s(aRs \wedge s \vDash_{\mathbf{T}} A \wedge (A \Rightarrow_{\mathbf{T}} (B \Rightarrow_{\mathbf{K}} C))$ implies $s \vDash_{\mathbf{T}} B \wedge (C \Rightarrow_{\mathbf{K}} D) \Rightarrow_{\mathbf{T}} D)$.

[5] By $\lambda z A(z)$ we mean the unary relation formed by the predicate $A(z)$. Some books use the notation $\hat{z} A(z)$ or $\{z \mid A(z)\}$ viewed as a one place relation.

[6] The modal logic $\mathbf{K}$ is complete for many classes of models. The widest class is where R is an arbitrary relation. A more strict class is where (S, R, a) is a tree. When we define modes on a class $\mathcal{K}$ of models we have to be careful which class to choose. We may get different hypermodal logics for different choices of classes of models, even though these classes characterise the same ordinary modal logic. Thus for $\mathbf{H}_1^S$ we start with trees as the class of models.

We now define our notion of a tree. Let W be the set of all finite sequences of the form $(-x_0, x_1, \ldots, x_m)$ where $x_0 \geq 0$ is a natural number and x_i, $i \geq 1$ are positive natural numbers. Let $\mathbf{p}$ be a function defined by $\mathbf{p}((-x_0, x_1, \ldots, x_n)) = (-x_0, x_1, \ldots, x_{n-1})$, for $n \geq 1$ and $\mathbf{p}((-x_0)) = (-x_0 - 1)$. Let, for t, s in W, tRs means $t = \mathbf{p}(s)$. The full tree model is $(W, R, 0)$. We can now be specific about our tree (S, R, a). It must satisfy the properties

- $a \in S \subseteq W$.
- $x \in S \wedge yRx \rightarrow y \in S$.

[7] A filter $\mathbb{F}_t$ is a family of subsets of S satisfying

(1) $S \in \mathbb{F}_t$.
(2) $\emptyset \notin \mathbb{F}_t$.
(3) $X \in \mathbb{F}_t \wedge X \subseteq Y$ imply $X \in \mathbb{F}_t$.
(4) $X \in \mathbb{F}_t$ and $Y \in \mathbb{F}_t$ imply $X \cap Y \in \mathbb{F}_t$.

[8] The proof is standard. We take as axioms and rules all $\mathbf{K}$ axioms for $\Box_1$ and $\mathbf{T}$ axioms for $\Box_0$, together with the additional interaction axioms $\Box_0 A \rightarrow \Box_1 A$ and $\Diamond_0 A \rightarrow A \vee \Diamond_1 A$.

To prove completeness, let S be the set of all complete consistent theories. Let $\Delta R_i \Theta$ be defined as 'for all $\Box_i A \in \Delta$ we have $A \in \Theta$'. For atomic q, let $\Delta \vDash q$ iff $\Delta \vdash q$ and show by induction that for all A, $\Delta \vDash A$ iff $\Delta \vdash A$. The axioms $\Box_0 A \rightarrow \Box_1 A$ and $\Diamond_0 A \rightarrow A \vee \Diamond_1 A$ ensure that R_0 is the reflexive closure of R_1.

[9] Another point to watch for is substitutivity of equivalents. $\Box q \rightarrow q$ is equivalent to $\top$ in $\mathbf{H}_1^S$ but $\Box\top$ is not equivalent to $\Box(\Box q \rightarrow q)$.

However, as a two-dimensional formula, $\Box q \rightarrow q$ is $\Box_0 q \rightarrow q$ and indeed $\Box_1 \top$ is equivalent to $\Box_1(\Box_0 q \rightarrow q)$.

The non-substitutivity of equivalents, in itself, does not imply non-completeness for a class of frames. Consider the class of all frames (S, R, a) such that R is transitive and aRa holds. Let the logic be the set of all wffs valid at the actual world a. This logic is finitely axiomatisable (with modus ponens only, without necessitation, since it is not normal) and it has no substitutivity of equivalents.

[10] The theory of Abelian groups, for example, is a special case of the theory of groups but it is so special that it is completely different.

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