Almost Necessary

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Abstract

A formula is contingent if it is possibly true and possibly false. A formula is non-contingent if it is not contingent, i.e., if it is necessarily true or necessarily false. In an epistemic setting, 'a formula is contingent' means that you are ignorant about its value, whereas 'a formula is non-contingent' means that you know whether it is true. Although non-contingency is definable in terms of necessity as above, necessity is not always definable in terms of non-contingency, as studied in the literature. We propose an 'almost-definability' schema AD for non-contingency logic, the logic with the non-contingency operator as the only modality, making precise when necessity is definable with non-contingency. Based on AD we propose a notion of bisimulation for non-contingency logic, and characterize non-contingency logic as the (non-contingency) bisimulation invariant fragment of modal logic and of first-order logic. A known pain for non-contingency logic is the absence of axioms characterizing frame properties. This makes it harder to find axiomatizations of non-contingency logic over given frame classes. In particular, no axiomatization over symmetric frames is known, despite the rich results about non-contingency logic obtained in the literature since the 1960s. We demonstrate that the 'almost-definability' schema AD can guide our search for proper axioms for certain frame properties, and help us in defining the canonical models. Following this idea, as the main result, we give a complete axiomatization of non-contingency logic over symmetric frames.

Keywords: Non-contingency, modal logic, completeness, definability, bisimulation

1 Introduction

Contingency is an important concept in philosophical logic; the notion goes back to Aristotle [2]. In [10], Montgomery and Routley define contingency in modal logic. A proposition φ is *non-contingent*, if it is necessary that φ or it

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is necessary that $\neg\varphi$. Otherwise, it is *contingent*. For 'φ is non-contingent' we write $\Delta\varphi$ and for 'φ is contingent' we write $\nabla\varphi$.

One theme is how to define necessity from non-contingency. Non-contingency is definable in terms of necessity as above, i.e., as $\Delta\varphi =_{df} \Box\varphi \vee \Box\neg\varphi$. But necessity cannot always be defined with non-contingency. In [10] it is proposed to define necessity as $\Box\varphi =_{df} \Delta\varphi \wedge \varphi$. Intuitively, necessity is non-contingent truth. However, this definition is only available in the systems containing $\Box\varphi \to \varphi$ [13, page 128]. When else is necessity definable in terms of non-contingency? In [3], it is shown that $\Box$ can only be defined in terms of Δ in the Verum system (i.e. the minimal modal logic extended with $\Box\varphi$), or the systems containing $\Box\varphi \to \Diamond\varphi$.

To provide the definability of $\Box$ in the general case, researchers extend the language with the introduction of extra operators. In [11], based on the postulate that some proposition is contingent, the author uses propositional quantifiers when defining necessity: $\Box\varphi =_{df} \forall p(\Delta(p \wedge \varphi) \to \Delta p)$. This says that a proposition is necessary if adding it cannot change the contingency of any contingent proposition. In the subsequent papers such as [12] the author introduces a *propositional constant* τ instead of propositional quantifiers to define necessity based on the axiom $\nabla\tau$: φ is necessary, if it is non-contingent, and it is non-contingently implied by τ, formally, $\Box\varphi =_{df} \Delta\varphi \wedge \Delta(\tau \to \varphi)$. In [17], inspired by the similarity between the definition of canonical relation in the completeness proof for the minimal non-contingency logic and that for the minimal modal logic, the author defines an infinitary operator in terms of Δ, and shows that this new operator behaves like, but differs from $\Box$. Such methods are compared in detail in [8]. In this paper we propose the 'almost-definability' schema $\nabla\psi \to (\Box\varphi \leftrightarrow \Delta\varphi \wedge \Delta(\psi \to \varphi))$: necessity is definable by non-contingency on a world, when some contingent proposition holds on that world. Note that we do not require $\nabla\psi$ to be valid. This schema also guides our proposals for bisimulation for non-contingency logic, and to characterize it within modal logic and within first-order logic using this new notion of bisimulation.

Another theme is axiomatizing the logic with the non-contingency operator as the only modality. A well-known difficulty is the absence of axioms characterizing frame properties in this logic, which makes it highly non-trivial to find axiomatizations of non-contingency logics over given frame classes. An unpublished axiomatization for non-contingency-based **S5** was proposed by Lemmon and Gjertsen in 1959 [7, note 10]. The non-contingency logics over reflexive frames and its extensions are axiomatized in [10]. In [6], Humberstone presents an infinite axiomatization for non-contingency logic over arbitrary frames and over serial frames. A finite axiomatization is given in [9]. This also provides a finite axiomatization for transitive non-contingency logic. In [16], an axiomatization for Euclidean non-contingency logic is proposed. However, to our knowledge, the axiomatization for non-contingency logic over symmetric frames is still open, due to technical difficulties. In this paper we solve this open problem by using the 'almost-definability' schema as a guiding clue.

Non-contingency logic also arose in the area of epistemic logic but with different terminology: 'φ is non-contingent' there means 'the agent knows whether φ', so that 'φ is contingent' means 'the agent is ignorant about φ'. Apparently unaware of the non-contingency logic literature, in [4] the author provides an axiomatization on **S5** frames. In [15] *a logic of ignorance* is presented and this logic is axiomatized over arbitrary frames. In [14] a topological completeness on the class of **S4** models is shown for the logic of ignorance. In [5], knowing whether logic is axiomatized over transitive frames and other frame classes (except symmetric frames), employing other than the traditional methods in the non-contingency literature. A novel result in [5] is the extension of knowing whether logic with public announcements, and its axiomatization.

As the main technical contributions of this work, we characterize the non-contingency logic within modal logic and within first-order logic using a novel notion of bisimulation, and give a complete axiomatization of non-contingency logic over symmetric frames. Both results are inspired by the almost-definability schema.

In Section 2 we define non-contingency logic and the almost-definability schema. In Section 3 we propose a notion of bisimulation on Kripke models that is suitable for non-contingency logic (called Δ-bisimulation), and also a suitable notion of bisimulation contraction. These are non-trivially different from standard bisimulation and contraction. In Section 4 we then characterize non-contingency logic as the Δ-bisimulation invariant fragment of modal logic and of first-order logic. Section 5 axiomatizes non-contingency logic over the class of symmetric frames. We conclude with some discussions in Section 6.

2 Non-contingency logic and almost-definability

Let us first recall the language and semantics of non-contingency logic as a fragment of the following logical language with both the necessity operator and the non-contingency operator:

Definition 2.1 (Logical languages NCL□, NCL and ML) *Given a set* $\boldsymbol{P}$ *of propositional variables, the logical language* $\boldsymbol{NCL}\Box$ *is defined as:*

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Delta\varphi \mid \Box\varphi$$

where $p \in \boldsymbol{P}$*. Without the* $\Box\varphi$ *construct, we have the* language **NCL** of non-contingency logic. *Without the* $\Delta\varphi$ *construct, we have the* language **ML** of modal logic. *If* $\varphi \in \boldsymbol{NCL}$*, then we say* φ *is an* ***NCL****-formula, Similarly we say* φ *is an* ***ML****-formula for* $\varphi \in \boldsymbol{ML}$*.*

In the rest of the paper, we will be mostly focusing on **NCL** which has Δ as the only primitive modality.

The formula $\Box\varphi$ says 'it is necessary that φ' and $\Delta\varphi$ expresses 'it is non-contingent that φ'.[4] As usual, we define $\bot$, $(\varphi \vee \psi)$, $(\varphi \to \psi)$, $(\varphi \leftrightarrow \psi)$,

[4] In [6], Humberstone suggested to rephrase it as 'it is non-contingent *whether* φ' to avoid

$\nabla\varphi$ and $\Diamond\varphi$ as the abbreviations of, respectively, $\neg\top$, $\neg(\neg\varphi \wedge \neg\psi)$, $(\neg\varphi \vee \psi)$, $((\varphi \to \psi) \wedge (\psi \to \varphi))$, $\neg\Delta\varphi$ and $\neg\Box\neg\varphi$. We omit parentheses from formulas unless confusion. Note that $\nabla\varphi$ is *not* the dual but the negation of $\Delta\varphi$, which expresses 'it is contingent that φ'.

Definition 2.2 (Model) *A* model *is a triple $\mathcal{M} = \langle S, R, V\rangle$ where S is a non-empty set of possible worlds, R is a binary relation over S, and V is a valuation function assigning a set of worlds $V(p) \subseteq S$ to each $p \in \boldsymbol{P}$. Given a world $s \in S$, the pair $(\mathcal{M}, s)$ is a* pointed model. *We will omit parentheses around pointed models $(\mathcal{M}, s)$ whenever convenient. A* frame *is a pair $\mathcal{F} = \langle S, R\rangle$, i.e. a model without a valuation.*

Definition 2.3 (Semantics) *Given a model $\mathcal{M} = \langle S, R, V\rangle$, the semantics of* ***NCL***$\Box$ *is defined as follows:*

$\mathcal{M}, s \models \top$	$\Leftrightarrow$	*true*
$\mathcal{M}, s \models p$	$\Leftrightarrow$	$s \in V(p)$
$\mathcal{M}, s \models \neg\varphi$	$\Leftrightarrow$	$\mathcal{M}, s \not\models \varphi$
$\mathcal{M}, s \models \varphi \wedge \psi$	$\Leftrightarrow$	$\mathcal{M}, s \models \varphi$ *and* $\mathcal{M}, s \models \psi$
$\mathcal{M}, s \models \Delta\varphi$	$\Leftrightarrow$	*for any* t_1, t_2 *such that* sRt_1, sRt_2 : $(\mathcal{M}, t_1 \models \varphi \Leftrightarrow \mathcal{M}, t_2 \models \varphi)$
$\mathcal{M}, s \models \Box\varphi$	$\Leftrightarrow$	*for all* t *such that* $sRt : \mathcal{M}, t \models \varphi$

If $\mathcal{M}, s \models \varphi$ we say that φ is true in $(\mathcal{M}, s)$, and sometimes write $s \models \varphi$ if $\mathcal{M}$ is clear; if for all s in $\mathcal{M}$ we have $\mathcal{M}, s \models \varphi$ we say that φ is valid on *$\mathcal{M}$ and write $\mathcal{M} \models \varphi$; if for all $\mathcal{M}$ based on $\mathcal{F}$ with $\mathcal{M} \models \varphi$ we say that φ is* valid on *$\mathcal{F}$ and write $\mathcal{F} \models \varphi$; if for all $\mathcal{F}$ with $\mathcal{F} \models \varphi$, φ is* valid *and we write $\models \varphi$. If there exists an $(\mathcal{M}, s)$ such that $\mathcal{M}, s \models \varphi$, then φ is* satisfiable. *Given any two pointed models $(\mathcal{M}, s)$ and $(\mathcal{N}, t)$, we say they are Δ-equivalent, notation: $(\mathcal{M}, s) \equiv_\Delta (\mathcal{N}, t)$, if they satisfy the same* ***NCL****-formulas; we say they are $\Box$-equivalent, notation: $(\mathcal{M}, s) \equiv_\Box (\mathcal{N}, t)$, if they satisfy the same* ***ML****-formulas.*

We are now ready to propose the *almost-definability* of the necessity operator.

Definition 2.4 *Let $\varphi, \psi \in$* ***NCL***. *Almost-definability is the schema $\nabla\psi \to (\Box\varphi \leftrightarrow \Delta\varphi \wedge \Delta(\psi \to \varphi))$ for which we write AD.*

Proposition 2.5 *Almost-definability AD is a validity of* ***NCL***$\Box$.[5]

Proof Given any pointed model $(\mathcal{M}, s)$, suppose that $\mathcal{M}, s \models \nabla\psi$. We need to show $\mathcal{M}, s \models \Box\varphi \leftrightarrow \Delta\varphi \wedge \Delta(\psi \to \varphi)$.

First, assume that $\mathcal{M}, s \models \Box\varphi$. It follows that for all t such that sRt, we have $t \models \varphi$ (thus $t \models \psi \to \varphi$). Then $\mathcal{M}, s \models \Delta\varphi$ and $\mathcal{M}, s \models \Delta(\psi \to \varphi)$, and thus $\mathcal{M}, s \models \Delta\varphi \wedge \Delta(\psi \to \varphi)$.

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ambiguity. Here we follow the traditional reading of $\Delta\varphi$ in the literature.

[5] From Proposition 2.5 it follows "$\models \nabla\psi$ implies $\models \Box\varphi \leftrightarrow (\Delta\varphi \wedge \Delta(\psi \to \varphi))$". This validates Pizzi's definition of $\Box\varphi$ using the new proposition constant τ (let ψ be that τ, see the Introduction). But note that there are no ψ for which $\nabla\psi$ is valid.

Next, assume that $\mathcal{M}, s \vDash \Delta\varphi \wedge \Delta(\psi \to \varphi)$. From the supposition that $\mathcal{M}, s \vDash \nabla\psi$, it follows that there exist t_1 and t_2 such that sRt_1 and sRt_2 and $t_1 \vDash \psi$ and $t_2 \vDash \neg\psi$. By $t_2 \vDash \neg\psi$, it is clear that $t_2 \vDash \psi \to \varphi$. Then using the fact that $s \vDash \Delta(\psi \to \varphi)$, sRt_1 and sRt_2, we obtain that $t_1 \vDash \psi \to \varphi$, thus $t_1 \vDash \varphi$. Since $s \vDash \Delta\varphi$, we have $u \vDash \varphi$ for each u in $\mathcal{M}$ such that sRu. Therefore $s \vDash \Box\varphi$. $\Box$

With the almost-definability schema, we are able to find the proper notions of bisimulation and of bisimulation contraction for non-contingency logic, as shown in the next section. Also, almost-definability can guide us to search for proper axioms for certain frame properties, and help us in defining the canonical models. This will be seen more clearly in Section 5.

3 Bisimulation

The standard notion of bisimulation ($\Box$-bisimulation) is too refined for non-contingency logic. In this section we propose a suitable weaker notion of Δ-bisimulation.

Definition 3.1 ($\Box$-Bisimulation) *Let $\mathcal{M} = \langle S, R, V\rangle$ and $\mathcal{M}' = \langle S', R', V'\rangle$ be two models. A binary relation Z is a* $\Box$-bisimulation *between $\mathcal{M}$ and $\mathcal{M}'$, if Z is non-empty and whenever sZs':*

(Invariance) s and s' satisfy the same propositional variables;

($\Box$-Zig) if sRt, then there is a t' in $\mathcal{M}'$ such that $s'R't'$ and tZt';

($\Box$-Zag) if $s'R't'$, then there is a t in $\mathcal{M}$ such that sRt and tZt'.

We say that $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ are $\Box$-bisimilar*, if there is a $\Box$-bisimulation linking two states s in $\mathcal{M}$ and s' in $\mathcal{M}'$, and we write $(\mathcal{M}, s) \leftrightarroweq_\Box (\mathcal{M}', s')$.*

Example 3.2 The models $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ below satisfy the same **NCL**-formulas but they are not $\Box$-bisimilar.

$$\mathcal{M}: \quad s:p \longrightarrow t:p \qquad\qquad \mathcal{M}': \quad s':p \longrightarrow t':\neg p$$

Inspired by the almost-definability schema, we can obtain the notion of Δ-bisimulation by revising the $\Box$-Zig and $\Box$-Zag conditions in the definition of $\Box$-bisimulation. Recall that almost-definability says $\Box$ is definable in terms of Δ, given a condition $\nabla\psi$ for some ψ. This condition corresponds to a precondition that the current world can see two non-**NCL**-equivalent successors. Note that for technical convenience, we define Δ-bisimulation within a single model, since the new Zig and Zag conditions require a precondition about 'sibling' worlds, i.e. the structural counterpart of non-**NCL**-equivalency. Based on Δ-bisimulation we can define Δ-bisimilarity between different models.

Definition 3.3 (Δ-Bisimulation) *Let $\mathcal{M} = \langle S, R, V\rangle$ be a model. A binary relation Z over S is a* Δ-bisimulation *on $\mathcal{M}$, if Z is non-empty and whenever sZs':*

(Invariance) s and s' satisfy the same propositional variables;

(Δ-Zig) if there are two successors t_1, t_2 of s such that $(t_1, t_2) \notin Z$ and sRt for some t, then there is a t' such that $s'Rt'$ and tZt';

(Δ-Zag) if there are two successors t'_1, t'_2 of s' such that $(t'_1, t'_2) \notin Z$ and $s'Rt'$ for some t', then there is a t such that sRt and tZt'.

We say $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ are Δ-bisimilar, *notation: $(\mathcal{M}, s) \leftrightarroweq_\Delta (\mathcal{M}', s')$, if there is a Δ-bisimulation linking s and s' in the disjoint union of $\mathcal{M}$ and $\mathcal{M}'$.*

We observe (without proof) that the notion of Δ-bisimilarity is an equivalence relation.

The following result indicates the relationship between the notion of Δ-bisimilarity and that of $\Box$-bisimilarity: Δ-bisimilarity is strictly weaker than that of $\Box$-bisimilarity. This corresponds to the fact that non-contingency logic is strictly weaker than modal logic.

Proposition 3.4 *$(\mathcal{M}, s) \leftrightarroweq_\Box (\mathcal{N}, t)$ implies $(\mathcal{M}, s) \leftrightarroweq_\Delta (\mathcal{N}, t)$ for any pointed models $(\mathcal{M}, s)$ and $(\mathcal{N}, t)$, but the converse is not true.*

Proof (Sketch) Collect all the $\Box$-bisimilar pairs (s, t) with s in $\mathcal{M}$ and t in $\mathcal{N}$ to construct a relation Z. We can check that Z is a Δ-bisimulation on the disjoint union of $\mathcal{M}$ and $\mathcal{N}$. For the converse, Example 3.2 yields two Δ-bisimilar but not $\Box$-bisimilar pointed models $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$. $\Box$

The following result says that **NCL**-formulas are invariant under Δ-bisimilarity.

Proposition 3.5 *Let $\mathcal{M} = \langle S, R, V \rangle$ and $\mathcal{M}' = \langle S', R', V' \rangle$. Then, for every $s \in S$ and $s' \in S'$, if $(\mathcal{M}, s) \leftrightarroweq_\Delta (\mathcal{M}', s')$, then $(\mathcal{M}, s) \equiv_\Delta (\mathcal{M}', s')$. In other words, Δ-bisimilarity implies Δ-equivalence.*

Proof Assume that $(\mathcal{M}, s) \leftrightarroweq_\Delta (\mathcal{M}', s')$. We need to show that for any $\varphi \in$ **NCL**, we have $\mathcal{M}, s \vDash \varphi$ iff $\mathcal{M}', s' \vDash \varphi$.

By induction on φ. The non-trivial case is $\Delta\varphi$.

Suppose $\mathcal{M}, s \nvDash \Delta\varphi$. Then there exist t_1, t_2 such that sRt_1, sRt_2 and $t_1 \vDash \varphi$ and $t_2 \nvDash \varphi$. As $(\mathcal{M}, s) \leftrightarroweq_\Delta (\mathcal{M}', s')$, there exists a Δ-bisimulation Z linking s and s'. By the fact that $t_1 \vDash \varphi$ and $t_2 \nvDash \varphi$ and the induction hypothesis, $(t_1, t_2) \notin Z$. From sRt_1 we obtain by (Δ-Zig) that there exists t'_1 such that $s'R't'_1$ and $t_1 Z t'_1$, thus $(\mathcal{M}, t_1) \leftrightarroweq_\Delta (\mathcal{M}', t'_1)$. Similarly, from sRt_2 we have that there exists t'_2 such that $s'R't'_2$ and $(\mathcal{M}, t_2) \leftrightarroweq_\Delta (\mathcal{M}', t'_2)$. From $t_1 \leftrightarroweq_\Delta t'_1$ and $t_1 \vDash \varphi$, by the induction hypothesis, $t'_1 \vDash \varphi$. Analogously, we can get $t'_2 \nvDash \varphi$. Therefore $\mathcal{M}', s' \nvDash \Delta\varphi$. For the other direction use (Δ-Zag).

$\Box$

The notion of Δ-bisimulation has many applications. First, it can be used to show that some properties of models definable in **ML** cannot be defined in **NCL**; second, it can show undefinability for usual frame properties; moreover, it can help to show that **NCL** is less expressive than **ML** on symmetric (and many other) models. For the definitions of expressivity and definability, we refer the reader to, e.g. [1].

Proposition 3.6 *The property "is an endpoint" is undefinable in **NCL**, while it can be defined in **ML**.*

Proof The property "is an endpoint" is defined by $\Box\bot$.

For the other part, consider pointed models $(\mathcal{M}, s)$ and $(\mathcal{N}, t)$, where s is an endpoint, t has only one successor, and s, t agree on proposition variables. If the property in question were defined by a set of **NCL**-formulas, say Φ. Since s is an endpoint, we have $\mathcal{M}, s \vDash \Phi$. Moreover, $(\mathcal{M}, s) \underline{\leftrightarrow}_\Delta (\mathcal{N}, t)$. By Proposition 3.5 we obtain $\mathcal{N}, t \vDash \Phi$, thus t is an endpoint, contradiction. $\Box$

The undefinability results below were presented in the literature ([6,16,5]). With Δ-bisimulation, we can give them simpler proofs.

Proposition 3.7 *The frame properties of seriality, reflexivity, transitivity, symmetry, and Euclidicity are not definable in **NCL**.*

Proof Consider the following frames:

$$\mathcal{F}_1: \quad s_1 \longrightarrow t \longrightarrow u \qquad\qquad \mathcal{F}_2: \quad \begin{tikzcd} s_2 \arrow[loop above] \end{tikzcd}$$

We first show that, for any $\varphi \in$ **NCL**, $\mathcal{F}_1 \models \varphi$ iff $\mathcal{F}_2 \models \varphi$. Fix a φ. If $\mathcal{F}_1 \nvDash \varphi$, then there exists $\mathcal{M}_1 = \langle \mathcal{F}_1, V_1 \rangle$ and s in $\mathcal{M}_1$ such that $\mathcal{M}_1, s \nvDash \varphi$. Let V_2 be a valuation based on $\mathcal{F}_2$ such that $p \in V_2(s_2)$ iff $p \in V_1(s)$ for all $p \in \mathbf{P}$. By definition of Δ-bisimilarity, $(\mathcal{M}_1, s) \underline{\leftrightarrow}_\Delta (\mathcal{M}_2, s_2)$ where $\mathcal{M}_2 = \langle \mathcal{F}_2, V_2 \rangle$. From Proposition 3.5 follows that $\mathcal{M}_2, s_2 \nvDash \varphi$, thus $\mathcal{F}_2 \nvDash \varphi$. The converse is similar.

If seriality were to be defined by a set of **NCL**-formulas, say Γ, then since $\mathcal{F}_2$ is serial, we have $\mathcal{F}_2 \vDash \Gamma$. Then we should also have $\mathcal{F}_1 \vDash \Gamma$, i.e., $\mathcal{F}_1$ should also be serial, contradiction. The proof for other properties are similar. $\Box$

Proposition 3.8 was also shown in [5]. With Δ-bisimulation, we get a simpler proof.

Proposition 3.8 ***NCL** is less expressive than **ML** on the class of symmetric models.*

Proof Since $\Delta\varphi =_{df} \Box\varphi \vee \Box\neg\varphi$, **ML** is at least as expressive as **NCL**. Consider the following symmetric models which can be distinguished by $\Box p$:

$$\underset{(\mathcal{M}, s)}{s : p \longleftrightarrow p} \qquad\qquad \underset{(\mathcal{N}, t)}{t : p \longleftrightarrow \neg p}$$

However, by definition of Δ-bisimilarity, $(\mathcal{M}, s) \underline{\leftrightarrow}_\Delta (\mathcal{N}, t)$. Due to Proposition 3.5, $(\mathcal{M}, s) \equiv_\Delta (\mathcal{N}, t)$, thus no **NCL**-formulas can distinguish the two. $\Box$

A model $\mathcal{M}$ is said to be ***NCL**-saturated*, if given any s in $\mathcal{M}$, and any set $\Sigma \subseteq$ **NCL**, if every finite subset of Σ is satisfiable in the set of successors of s, then Σ is satisfiable in the set of successors of s. In what follows, we show that $\equiv_\Delta$ and $\underline{\leftrightarrow}_\Delta$ coincide on **NCL**-saturated models. The proof is similar to its modal counterpart but it makes a crucial use of the **NCL**-formulas.

Proposition 3.9 *For any **NCL**-saturated pointed models $(\mathcal{M}, s)$ and $(\mathcal{N}, t)$, $(\mathcal{M}, s) \equiv_\Delta (\mathcal{N}, t)$ iff $(\mathcal{M}, s) \underline{\leftrightarrow}_\Delta (\mathcal{N}, t)$.*

Proof Based on Proposition 3.5, we only need to show the direction from left to right. Let $\mathcal{M} = \langle S, R, V\rangle$ and $\mathcal{N} = \langle S', R', V'\rangle$ be two **NCL**-saturated models. Suppose $(\mathcal{M}, s) \equiv_\Delta (\mathcal{N}, t)$, we need to show $\equiv_\Delta$ is a Δ-bisimulation on the disjoint union of $\mathcal{M}$ and $\mathcal{N}$, which entails $(\mathcal{M}, s) \leftrightarroweq_\Delta (\mathcal{N}, t)$. It suffices to show the condition (Δ-Zig) holds, as the proof for (Δ-Zag) is similar.

For this, assume that there exist s_1, s_2 such that sRs_1, sRs_2 and $s_1 \not\equiv_\Delta s_2$, and assume sRs', to show there exists t' such that $tR't'$ and $s' \equiv_\Delta t'$. Let $\Sigma = \{\psi \in \mathbf{NCL} \mid s' \vDash \psi\}$. Clearly, $s' \vDash \Sigma$. Then for any finite $\Gamma \subseteq \Sigma$, $s' \vDash \bigwedge\Gamma$. If for all t' such that $tR't'$, $t' \nvDash \bigwedge\Gamma$, then $t \vDash \Delta\bigwedge\Gamma$, and by supposition we derive $s \vDash \Delta\bigwedge\Gamma$. In the meantime, as $s_1 \not\equiv_\Delta s_2$, there exists $\varphi \in \mathbf{NCL}$ such that $s_1 \vDash \varphi$ and $s_2 \nvDash \varphi$. Then by the fact that $s' \vDash \bigwedge\Gamma$ and $s \vDash \Delta\bigwedge\Gamma$, it is not hard to get $s_1 \vDash \bigwedge\Gamma \to \varphi$ and $s_2 \nvDash \bigwedge\Gamma \to \varphi$, and thus $s \nvDash \Delta(\bigwedge\Gamma \to \varphi)$. On the other hand, since for every t' such that $tR't'$ it holds that $t' \nvDash \bigwedge\Gamma$, we have that $t' \vDash \bigwedge\Gamma \to \varphi$, and thus $t \vDash \Delta(\bigwedge\Gamma \to \varphi)$, contradicting to $s \equiv_\Delta t$ and $s \nvDash \Delta(\bigwedge\Gamma \to \varphi)$. Hence there exists t_Γ such that $tR't_\Gamma$ and $t_\Gamma \vDash \bigwedge\Gamma$. By **NCL**-saturation, there exists t' with $tR't'$ and $t' \vDash \Sigma$. Moreover, $s' \equiv_\Delta t'$: given any $\psi \in \mathbf{NCL}$, if $s' \vDash \psi$, then $\psi \in \Sigma$, and thus $t' \vDash \psi$; if $s' \nvDash \psi$, then $s' \vDash \neg\psi$, and thus $\neg\psi \in \Sigma$, and hence $t' \vDash \neg\psi$, i.e., $t' \nvDash \psi$. $\Box$

If we remove the condition of **NCL**-saturation, then $\leftrightarroweq_\Delta$ does not coincide with $\equiv_\Delta$, as illustrated below.

Example 3.10 Consider two models $\mathcal{M} = \langle S, R, V\rangle$ and $\mathcal{M}' = \langle S', R', V'\rangle$, where $S = \mathbb{N} \cup \{s\}$, $R = \{(s, n) \mid n \in \mathbb{N}\}, V(p_n) = \{n\}$ and $S' = \mathbb{N} \cup \{s', \omega\}$, $R' = \{(s', n) \mid n \in \mathbb{N}\} \cup \{(s', \omega)\}$, and $V'(p_n) = \{n\}$. In pictures:

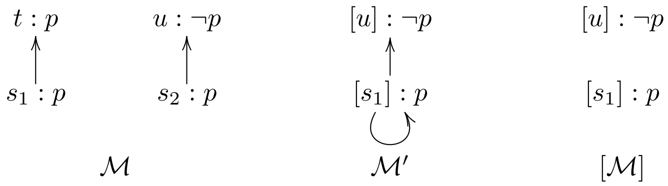
$$\begin{tikzcd} s \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] & & & \mathcal{M} \\ p_1 & p_2 & p_3 & \cdots \end{tikzcd} \qquad \begin{tikzcd} s' \arrow[r] \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] & \omega & & \mathcal{M}' \\ p_1 & p_2 & p_3 & \cdots \end{tikzcd}$$

Now $\mathcal{M}$ is not **NCL**-saturated. We can also check that $(\mathcal{M}, s) \equiv_\Delta (\mathcal{M}', s')$ but $(\mathcal{M}, s)$ is not Δ-bisimilar to $(\mathcal{M}', s')$.

Bisimulation contraction The $\Box$-bisimulation contraction is defined as a quotient model modulo $\Box$-bisimilarity (as the equivalence relation) such that one equivalence class is accessible from another equivalence class if a world in the first is accessible from a world in the second. However, if we just replace $\Box$-bisimilarity with Δ-bisimilarity in this definition, the contracted model may not be Δ-bisimilar to the original one, as the following example shows. Therefore we propose a novel notion of Δ-bisimulation contraction, which features an extra condition in the definition of the quotient relation.

Example 3.11 Model $\mathcal{M}'$ is the 'Δ-bisimulation contraction' of $\mathcal{M}$ if we just replace $\Box$-bisimilarity with Δ-bisimilarity as above, but $(\mathcal{M}', [s_1])$ is not Δ-

bisimilar to $(\mathcal{M}, s_1)$: for example, Δp is true in $(\mathcal{M}, s_1)$ but false in $(\mathcal{M}', [s_1])$.



To overcome the above problem, we propose a novel notion of Δ-bisimulation contraction. In particular, we would like to have $[\mathcal{M}]$ as the contracted model of $\mathcal{M}$ in the above example.

Definition 3.12 (Δ-Bisimulation Contraction) *Given a model $\mathcal{M} = \langle S, R, V\rangle$, recall that Δ-bisimilarity ($\mathrel{\underline{\leftrightarrow}}_\Delta$) within $\mathcal{M}$ is an equivalence relation. Let $[s]$ be the equivalence class of s w.r.t. $\mathrel{\underline{\leftrightarrow}}_\Delta$ within $\mathcal{M}$. The* Δ-bisimulation contraction *of $\mathcal{M}$ is the quotient structure $[\mathcal{M}] = \langle [S], [R], [V]\rangle$, where*

- $[S] = \{[s] \mid s \in S\}$;
- $[s][R][t]$ *iff there exist $s' \in [s]$ and $t' \in [t]$ such that $s'Rt'$ and there exist t_1, t_2 such that $s'Rt_1$ and $s'Rt_2$ and $t_1 \not\mathrel{\underline{\leftrightarrow}}_\Delta t_2$;*
- *For all propositional variables p, $[V](p) = \{[s] \mid s \in V(p)\}$.*

According to the above definition, we do get $[\mathcal{M}]$ in the earlier example. We show that the contracted model is Δ-bisimilar to the original model.

Proposition 3.13 *Let $\mathcal{M} = \langle S, R, V\rangle$ and $[\mathcal{M}]$ be the Δ-bisimulation contraction of $\mathcal{M}$. Then for all $s \in S$, we have $([\mathcal{M}], [s]) \mathrel{\underline{\leftrightarrow}}_\Delta (\mathcal{M}, s)$.*

Proof Define

$$Z = \{([s], s) \mid s \in S\} \cup \{(t, t') \mid t \in S,\ t' \in S \text{ and } t \mathrel{\underline{\leftrightarrow}}_\Delta t'\}.$$

We show Z is a Δ-bisimulation on the disjoint union of $[\mathcal{M}]$ and $\mathcal{M}$, which implies that $([\mathcal{M}], [s]) \mathrel{\underline{\leftrightarrow}}_\Delta (\mathcal{M}, s)$. First of all, Z is clearly non-empty due to the fact that S is non-empty.

- Invariance: by the definition of $[V]$.
- Δ-Zig: We prove a stronger version ($\Box$-Zig in fact) that for any $u, v \in [S]$, if $u[R]v$ and uZs (i.e. $s \in u$) then there exists $t \in S$ such that vZt (i.e. $t \in v$) and sRt. Now suppose $u[R]v$ and uZs, then according to the definition of $[R]$ there is an $s' \in u$ and a $t' \in v$ such that $s'Rt'$ and there are t_1 and t_2 such that $s'Rt_1$ and $s'Rt_2$ and $t_1 \not\mathrel{\underline{\leftrightarrow}}_\Delta t_2$. Since $s \in u$ and $s' \in u$, $s \mathrel{\underline{\leftrightarrow}}_\Delta s'$. Note that $\mathrel{\underline{\leftrightarrow}}_\Delta$ is also a Δ-bisimulation,[6] thus since $s'Rt'$ there is a t such that sRt and $t \mathrel{\underline{\leftrightarrow}}_\Delta t'$ thus $t \in v$.
- Δ-Zag: Suppose that sRt and there exist two R-successors t_1, t_2 of s such that $(t_1, t_2) \notin Z$. By the definition of Z, we have $t_1 \not\mathrel{\underline{\leftrightarrow}}_\Delta t_2$, thus $[s][R][t]$ by

[6] In fact, $\mathrel{\underline{\leftrightarrow}}_\Delta$ is the largest Δ-bisimulation.

the definition of $[R]$. Now since $t \in [t]$, we have $[t]Zt$. We have thus proved that there exists $[t] \in [S]$ such that $[s][R][t]$ and $[t]Zt$, as desired.

$\square$

4 Characterization results via Δ-bisimulation

The non-contingency logic **NCL** can be seen as a fragment of modal logic **ML**, as $\Delta\varphi =_{df} \Box\varphi \vee \Box\neg\varphi$. In this section we characterize the fragment of **NCL** within **ML** and within first-order logic. Given a model $\mathcal{M}$ and s in $\mathcal{M}$, we use $ue(\mathcal{M})$ and π_s to denote the ultrafilter extension of $\mathcal{M}$ and the principle ultrafilter generated by s, respectively. We refer the reader to [1] for the definitions about these notions, as well as m-saturation. We first state (but not prove) a standard result in modal logic (cf. e.g., [1]).

Proposition 4.1 *For any $\mathcal{M}$ and s in $\mathcal{M}$, $ue(\mathcal{M})$ is an m-saturated model and $(\mathcal{M}, s) \equiv_{\Box} (ue(\mathcal{M}), \pi_s)$.*

Since **NCL** is a fragment of **ML**, any set of **NCL**-formulas can be viewed as a set of **ML**-formulas, thus we have:

Lemma 4.2 *Let $(\mathcal{M}, s)$ be a pointed model. Then $ue(\mathcal{M})$ is an **NCL**-saturated model and $(\mathcal{M}, s) \equiv_{\Delta} (ue(\mathcal{M}), \pi_s)$.*

Lemma 4.3 *Let $(\mathcal{M}, s)$ and $(\mathcal{N}, t)$ be pointed models. Then*

$$(\mathcal{M}, s) \equiv_{\Delta} (\mathcal{N}, t) \text{ implies } (ue(\mathcal{M}), \pi_s) \leftrightarroweq_{\Delta} (ue(\mathcal{N}), \pi_t).$$

Proof By Lemma 4.2 and Proposition 3.9. $\square$

We are now ready to prove the two characterization results: non-contingency logic is the Δ-bisimulation-invariant fragment of modal logic and of first-order logic.

Let φ be an **ML**-formula and α be a first-order formula. Call φ (resp. α) *invariant under Δ-bisimulation*, if for any models $(\mathcal{M}, s), (\mathcal{N}, t)$ such that $(\mathcal{M}, s) \leftrightarroweq_{\Delta} (\mathcal{N}, t)$, we have $\mathcal{M}, s \vDash \varphi$ iff $\mathcal{N}, t \vDash \varphi$ (resp. $\mathcal{M}, s \vDash \alpha$ iff $\mathcal{N}, t \vDash \alpha$).

Theorem 4.4 *An **ML**-formula is equivalent to an **NCL**-formula iff it is invariant under Δ-bisimulation.*

Proof It is clear for the direction 'only if' from Proposition 3.5.

For the converse direction, suppose that an **ML**-formula φ is invariant under Δ-bisimulation. Define

$$MOC(\varphi) = \{t(\psi) \mid \psi \text{ is an } \mathbf{NCL}\text{-formula and } \varphi \vDash t(\psi)\}^{7}$$

If we can show $MOC(\varphi) \vDash \varphi$, then by the compactness theorem for modal logic, for some finite subset T of $MOC(\varphi)$ such that $T \vDash \varphi$, i.e., $\vDash \bigwedge T \to \varphi$. In the meantime, the definition of $MOC(\varphi)$ implies that $\varphi \vDash \bigwedge T$, i.e., $\vDash \varphi \to \bigwedge T$.

---

[7] Here t is a translation function which recursively translates every **NCL** formula into the corresponding equivalent **ML** formula. In particular, for every **NCL** formula φ of the form $\Delta\psi$, $t(\varphi) = \Box t(\psi) \vee \Box t(\neg\psi)$.

Thus $\vDash \varphi \leftrightarrow \bigwedge T$. As each $\chi \in T$ is the translation of an **NCL**-formula, so is $\bigwedge T$. Then we are done.

So it remains to show $MOC(\varphi) \vDash \varphi$. Assume that $\mathcal{M}, s \vDash MOC(\varphi)$, we need to show $\mathcal{M}, s \vDash \varphi$. Let $X = \{t(\psi) \mid \mathcal{M}, s \vDash t(\psi), \psi \in \mathbf{NCL}\}$. We claim that $X \cup \{\varphi\}$ is satisfiable: if not, then by compactness theorem for modal logic again, for some finite $X' \subseteq X$ such that $\vDash \varphi \to \neg \bigwedge X'$, viz. $\varphi \vDash \neg \bigwedge X'$, then $\neg \bigwedge X' \in MOC(\varphi)$, and thus $\mathcal{M}, s \vDash \neg \bigwedge X'$, contradicting $\mathcal{M}, s \vDash X$ and $X' \subseteq X$.

Since $X \cup \{\varphi\}$ is satisfiable, it may as well assume that there is a pointed model $(\mathcal{N}, t)$ with $\mathcal{N}, t \vDash X \cup \{\varphi\}$. We can show that $(\mathcal{M}, s) \equiv_\Delta (\mathcal{N}, t)$: given any **NCL**-formula ψ, if $\mathcal{M}, s \vDash \psi$, then $\mathcal{M}, s \vDash t(\psi)$, and then $t(\psi) \in X$, thus $\mathcal{N}, t \vDash t(\psi)$, therefore $\mathcal{N}, t \vDash \psi$; if $\mathcal{M}, s \nvDash \psi$, then $\mathcal{M}, s \vDash \neg\psi$, and then $\mathcal{M}, s \vDash t(\neg\psi)$, thus $t(\neg\psi) \in X$, hence $\mathcal{N}, t \vDash t(\neg\psi)$, and therefore $\mathcal{N}, t \vDash \neg\psi$, i.e., $\mathcal{N}, t \nvDash \psi$.

Now construct the ultrafilter extensions of $\mathcal{M}$ and $\mathcal{N}$, denoted by $ue(\mathcal{M})$ and $ue(\mathcal{N})$, respectively. According to the fact that $(\mathcal{M}, s) \equiv_\Delta (\mathcal{N}, t)$ and Lemma 4.3, we have $(ue(\mathcal{M}), \pi_s) \leftrightarroweq_\Delta (ue(\mathcal{N}), \pi_t)$. From $\mathcal{N}, t \vDash \varphi$ and Proposition 4.1, it follows that $ue(\mathcal{N}), \pi_t \vDash \varphi$. By supposition and $(ue(\mathcal{M}), \pi_s) \leftrightarroweq_\Delta (ue(\mathcal{N}), \pi_t)$, we get $ue(\mathcal{M}), \pi_s \vDash \varphi$. By Proposition 4.1 again, one may conclude that $\mathcal{M}, s \vDash \varphi$, as desired. $\Box$

By proposition 3.4, if a formulas is invariant under Δ-bisimulation then it is invariant under $\Box$-bisimulation. The theorem below follows from Theorem 4.4 and Van Benthem Characterization Theorem (cf. e.g., [1]).

Theorem 4.5 *A first-order formula is equivalent to an **NCL**-formula iff it is invariant under Δ-bisimulation.*

5 Axiomatization of NCL over symmetric frames

In this section, we propose an axiomatization for non-contingency logic over the symmetric frames, a result so far not obtained in the extensive literature on the topic of non-contingency.

As mentioned at the end of Section 2, the schema AD can guide us to search for proper axioms in **NCL** for certain frame properties. For example, AD can help us in finding T-like axiom in **NCL** in a precise way.

$$\begin{aligned} & \nabla\neg\psi \to (\Box\neg\varphi \to \neg\varphi) && (1)\\ \Leftrightarrow\ & \nabla\neg\psi \wedge \Box\neg\varphi \to \neg\varphi && (2)\\ \Leftrightarrow\ & \nabla\neg\psi \wedge \Delta\neg\varphi \wedge \Delta(\neg\psi \to \neg\varphi) \to \neg\varphi && (3)\\ \Leftrightarrow\ & \Delta\varphi \wedge \Delta(\varphi \to \psi) \wedge \varphi \to \Delta\psi && (4) \end{aligned}$$

We write $\nabla\neg\psi \to (\Box\neg\varphi \to \neg\varphi)$ rather than $\Box\neg\varphi \to \neg\varphi$, since $\Box$ is definable in terms of Δ under the condition $\nabla\neg\psi$ for some $\neg\psi$. The above transition from (2) to (3) follows from Proposition 2.5. By using `TAUT`, Δ`Equ` and `RE`Δ below and Def∇, we then get the desired axiom (4), which was used in [10] to axiomatize **NCL** over the reflexive frames.

Similar idea guides us to the right direction of finding the B-like axiom (axiom $\Delta\mathtt{B}$ below) in **NCL**, but further fine-tunings are needed.

Definition 5.1 (Proof system $\mathbb{NCLB}$) *The proof system $\mathbb{NCLB}$ consists of the following axiom schemas and inference rules.*

TAUT	*all instances of tautologies*
Δ*Con*	$\Delta(\chi \to \varphi) \wedge \Delta(\neg\chi \to \varphi) \to \Delta\varphi$
Δ*Dis*	$\Delta\varphi \to \Delta(\varphi \to \psi) \vee \Delta(\neg\varphi \to \chi)$
Δ*Equ*	$\Delta\varphi \leftrightarrow \Delta\neg\varphi$
Δ*B*	$\varphi \to \Delta((\Delta\varphi \wedge \Delta(\varphi \to \psi) \wedge \neg\Delta\psi) \to \chi)$
MP	*From* φ *and* $\varphi \to \psi$ *infer* ψ
REΔ	*From* $\varphi \leftrightarrow \psi$ *infer* $\Delta\varphi \leftrightarrow \Delta\psi$

In the following, we write $\vdash \varphi$ if there is a proof of φ in $\mathbb{NCLB}$.

Proposition 5.2 *The rule (NECΔ): $\dfrac{\varphi}{\Delta\varphi}$ is derivable in $\mathbb{NCLB}$:*

Proof First, we show that $\vdash \Delta\top$:

(i)	$\top \to \Delta((\Delta\top \wedge \Delta(\top \to \psi) \wedge \neg\Delta\psi) \to \chi)$	$\Delta\mathtt{B}$
(ii)	$\Delta((\Delta\top \wedge \Delta(\top \to \psi) \wedge \neg\Delta\psi) \to \chi)$	$\mathtt{TAUT}(i)$
(iii)	$\Delta((\Delta\top \wedge \Delta\psi \wedge \neg\Delta\psi) \to \chi)$	$\mathtt{RE}\Delta(ii)$
(iv)	$\Delta(\bot \to \chi)$	$\mathtt{RE}\Delta(iii)$
(v)	$\Delta\top$	$\mathtt{RE}\Delta(iv)$

Next, suppose that $\vdash \varphi$, to show that $\vdash \Delta\varphi$. From the supposition and $\mathtt{TAUT}$ it follows that $\vdash \varphi \leftrightarrow \top$. Then using $\mathtt{RE}\Delta$ we get $\vdash \Delta\varphi \leftrightarrow \Delta\top$. Since we have shown $\vdash \Delta\top$, we conclude that $\vdash \Delta\varphi$. $\Box$

When we drop the axiom $\Delta\mathtt{B}$ and add the rule $\mathtt{NEC}\Delta$, we get the logical system $\mathbb{PLKW}$, which is shown to be sound and complete with respect to the class of arbitrary frames in [5, Section 4] (by taking Kw there as Δ). But note that $\mathtt{NEC}\Delta$ is indispensable in $\mathbb{PLKW}$, since $\mathtt{NEC}\Delta$ is *not* admissible in $\mathbb{PLKW} - \mathtt{NEC}\Delta$ (equivalently, $\mathbb{NCLB} - \Delta\mathtt{B}$). To see this, we can show that $\Delta\top$ is not provable in $\mathbb{PLKW} - \mathtt{NEC}\Delta$: define an auxiliary semantics $\Vdash$, which is the same as $\vDash$ except that wherein each $\Delta\varphi$ is interpreted always *false*, then we can show that $\mathbb{PLKW} - \mathtt{NEC}\Delta$ is sound with respect to $\Vdash$, but $\nVdash \Delta\top$, thus $\Delta\top$ is not provable in $\mathbb{PLKW} - \mathtt{NEC}\Delta$, therefore $\mathtt{NEC}\Delta$ cannot be admissible in $\mathbb{PLKW} - \mathtt{NEC}\Delta$.

Proposition 5.3 $\mathbb{NCLB}$ *is sound with respect to the class of symmetric frames.*

Proof Since $\mathbb{PLKW}$ is sound, we only need to show the validity of Axiom $\Delta\mathtt{B}$.

Given any symmetric model $\mathcal{M} = \langle S, R, V\rangle$ and $s \in S$, suppose that $\mathcal{M}, s \vDash \varphi$. Let t be an arbitrary world with sRt. By the symmetry of R, we have tRs. We show that $\mathcal{M}, t \vDash (\Delta\varphi \wedge \Delta(\varphi \to \psi) \wedge \neg\Delta\psi) \to \chi$. If $\mathcal{M}, t \vDash \Delta\varphi \wedge \Delta(\varphi \to \psi) \wedge \neg\Delta\psi$, then there exist t_1, t_2 such that tRt_1 and tRt_2 and $t_1 \vDash \psi$ and

$t_2 \vDash \neg\psi$. From $t \vDash \Delta\varphi$, tRs and the supposition, it follows that $t_1 \vDash \varphi$ and $t_2 \vDash \varphi$. Thus $t_1 \vDash \varphi \to \psi$ and $t_2 \vDash \neg(\varphi \to \psi)$, contrary to the fact that $t \vDash \Delta(\varphi \to \psi)$ and tRt_1 and tRt_2. Therefore $\mathcal{M}, t \nvDash \Delta\varphi \wedge \Delta(\varphi \to \psi) \wedge \neg\Delta\psi$, which implies that $\mathcal{M}, t \vDash (\Delta\varphi \wedge \Delta(\varphi \to \psi) \wedge \neg\Delta\psi) \to \chi$, as desired.

□

We now proceed with the completeness of $\mathbb{NCLB}$. First, some preparations.

Lemma 5.4

$$\vdash (\Delta(\varphi \to (\chi \to \psi)) \wedge \Delta(\neg\varphi \to \psi) \wedge \neg\Delta\psi \wedge \Delta\varphi) \to \Delta(\chi \to \psi)$$

Proof

(i)	$\Delta(\neg\varphi \to \psi) \wedge \neg\Delta\psi \to \neg\Delta(\varphi \to \psi)$	$\Delta\mathtt{Con}$
(ii)	$\Delta\varphi \to \Delta(\varphi \to \psi) \vee \Delta(\neg\varphi \to (\chi \to \psi))$	$\Delta\mathtt{Dis}$
(iii)	$\Delta(\varphi \to (\chi \to \psi)) \wedge \Delta(\neg\varphi \to (\chi \to \psi)) \to \Delta(\chi \to \psi)$	$\Delta\mathtt{Con}$
(iv)	$\Delta(\neg\varphi \to \psi) \wedge \neg\Delta\psi \wedge \Delta\varphi \to \Delta(\neg\varphi \to (\chi \to \psi))$	$\mathtt{TAUT}(i)(ii)$
(v)	$(\Delta(\varphi \to (\chi \to \psi)) \wedge \Delta(\neg\varphi \to \psi) \wedge \neg\Delta\psi \wedge \Delta\varphi) \to \Delta(\chi \to \psi)$	$\mathtt{TAUT}(iii)(iv)$

□

Proposition 5.5 *For all $k \geq 1$:*

$$\vdash \Delta(\bigwedge_{j=1}^{k} \varphi_j \to \neg\psi) \wedge \bigwedge_{j=1}^{k} \Delta\varphi_j \wedge \bigwedge_{j=1}^{k} \Delta(\psi \to \varphi_j) \to \Delta\psi$$

Proof By induction on k.

- $k = 1$. We need to show that $\vdash \Delta(\varphi_1 \to \neg\psi) \wedge \Delta\varphi_1 \wedge \Delta(\psi \to \varphi_1) \to \Delta\psi$. This is clear from $\mathtt{TAUT}$, $\mathtt{RE}\Delta$, $\Delta\mathtt{Con}$ and $\Delta\mathtt{Equ}$.
- Inductive step. Assume by induction hypothesis (IH) that the proposition holds for $k = n$. We now need to show that:

$$\vdash \Delta(\bigwedge_{j=1}^{n+1} \varphi_j \to \neg\psi) \wedge \bigwedge_{j=1}^{n+1} \Delta\varphi_j \wedge \bigwedge_{j=1}^{n+1} \Delta(\psi \to \varphi_j) \to \Delta\psi$$

The proof is as follows.

(i)	$\Delta(\bigwedge_{j=1}^{n} \varphi_j \to \neg\psi) \wedge \bigwedge_{j=1}^{n} \Delta\varphi_j$ $\wedge \bigwedge_{j=1}^{n} \Delta(\psi \to \varphi_j) \to \Delta\psi$	IH
(ii)	$(\Delta(\varphi_{n+1} \to (\bigwedge_{j=1}^{n} \varphi_j \to \neg\psi)) \wedge \Delta(\neg\varphi_{n+1} \to \neg\psi)$ $\wedge\neg\Delta\neg\psi \wedge \Delta\varphi_{n+1}) \to \Delta(\bigwedge_{j=1}^{n} \varphi_j \to \neg\psi)$	Lemma 5.4
(iii)	$(\Delta(\bigwedge_{j=1}^{n+1} \varphi_j \to \neg\psi)) \wedge \Delta(\psi \to \varphi_{n+1}) \wedge \neg\Delta\psi$ $\wedge\Delta\varphi_{n+1}) \to \Delta(\bigwedge_{j=1}^{n} \varphi_j \to \neg\psi)$	$\mathtt{TAUT}, \mathtt{RE}\Delta, \Delta\mathtt{Equ}, (ii)$
(iv)	$(\Delta(\bigwedge_{j=1}^{n+1} \varphi_j \to \neg\psi)) \wedge \bigwedge_{j=1}^{n+1} \Delta\varphi_j \wedge \neg\Delta\psi$ $\wedge \bigwedge_{j=1}^{n+1} \Delta(\psi \to \varphi_j) \to \Delta\psi$	$\mathtt{TAUT}(i)(iii)$
(v)	$\Delta(\bigwedge_{j=1}^{n+1} \varphi_j \to \neg\psi) \wedge \bigwedge_{j=1}^{n+1} \Delta\varphi_j$ $\wedge \bigwedge_{j=1}^{n+1} \Delta(\psi \to \varphi_j) \to \Delta\psi$	$\mathtt{TAUT}(iv)$

□

Next, we turn to the canonical model. The model defined below will be used to construct the desired canonical model for $\mathbb{NCLB}$, though it is not suitable for the system in question.

Definition 5.6 (Pseudo-canonical model) *Define* $\mathcal{M}^c = \langle S^c, R^c, V^c \rangle$ *as follows:*

- $S^c = \{s \mid s$ *is a maximal consistent set of* $\mathbb{NCLB}\}$
- *For all* $s, t \in S^c$, sR^ct *iff there exists* χ *such that:*
  · $\neg\Delta\chi \in s$, *and*
  · *for all* φ, $\Delta\varphi \wedge \Delta(\chi \to \varphi) \in s$ *implies* $\varphi \in t$.
- $V^c(p) = \{s \in S^c \mid p \in s\}$.

This definition of R^c can be viewed as a simplification of the canonical relation in [5, Definition 20] based on the almost-definability (Proposition 2.5). In the construction of canonical model for standard modal logic, the canonical relation R^c is usually defined by sR^ct holds iff for all φ, $\Box\varphi \in s$ implies $\varphi \in t$. According to the almost-definability, $\Box\varphi \in s$ can be replaced by $\Delta\varphi \wedge \Delta(\chi \to \varphi) \in s$ provided that $\neg\Delta\chi \in s$.

Analogous to the proof of Truth Lemma of $\mathbb{PLKW}$ ([5, Lemma 21]), we have

Lemma 5.7 *For all* $\varphi \in$ ***NCL*** *and* $s \in S^c$, $\mathcal{M}^c, s \vDash \varphi$ *iff* $\varphi \in s$.

Proof By induction on φ. The only non-trivial case is when $\varphi = \Delta\psi$.

'If': Assume that $\Delta\psi \in s$, we need to show $\mathcal{M}^c, s \vDash \Delta\psi$. Suppose not, then there exist $t_1, t_2 \in S^c$ such that sR^ct_1, sR^ct_2 and $t_1 \vDash \psi$ and $t_2 \nvDash \psi$. From $t_1 \vDash \psi$ and $t_2 \nvDash \psi$, and induction hypothesis, we have that $\psi \in t_1$ and $\psi \notin t_2$, respectively. From sR^ct_1 we infer that there is a χ_1 such that $\neg\Delta\chi_1 \in s$ and $(*)$: for all φ, $\Delta\varphi \wedge \Delta(\chi_1 \to \varphi) \in s$ implies $\varphi \in t_1$. Since $\Delta\psi \in s$ and $\psi \in t_1$, $\Delta\neg\psi \in s$ and $\neg\psi \notin t_1$. Now from $(*)$, it follows that $\neg\Delta(\chi_1 \to \neg\psi) \in s$, thus $\neg\Delta(\psi \to \neg\chi_1) \in s$ by $\mathtt{RE}\Delta$. Similarly, from sR^ct_2 we derive that there exists χ_2 such that $\neg\Delta(\chi_2 \to \psi) \in s$, i.e., $\neg\Delta(\neg\psi \to \neg\chi_2) \in s$. By the axiom $\Delta\mathtt{Dis}$, we obtain that $\neg\Delta\psi \in s$, contradiction.

'Only if': Suppose that $\Delta\psi \notin s$. Then $\neg\Delta\psi \in s$ and $\neg\Delta\neg\psi \in s$. We need to construct two points $t_1, t_2 \in S^c$ such that sR^ct_1 and sR^ct_2 and $\psi \in t_1$ and $\neg\psi \in t_2$. First, we have to show

(i) $\{\varphi \mid \Delta\varphi \wedge \Delta(\psi \to \varphi) \in s\} \cup \{\psi\}$ is consistent.

(ii) $\{\varphi \mid \Delta\varphi \wedge \Delta(\neg\psi \to \varphi) \in s\} \cup \{\neg\psi\}$ is consistent.

We prove item (i). Suppose the set is inconsistent. Then there exist $\varphi_1, \cdots, \varphi_n$ such that $\vdash \varphi_1 \wedge \cdots \wedge \varphi_n \to \neg\psi$ and $\Delta\varphi_k \wedge \Delta(\psi \to \varphi_k) \in s$ for all $k \in [1, n]$. From $\mathtt{NEC}\Delta$ follows that $\Delta(\varphi_1 \wedge \cdots \wedge \varphi_n \to \neg\psi) \in s$. Now from Proposition 5.5, we infer that $\Delta\psi \in s$, contradiction.

From item (i), the definition of R^c, and the observation that every consistent set can be extended to a maximal consistent set (Lindenbaum Lemma), we conclude that there is a t_1 such that $sR^c t_1$ and $\psi \in t_1$.

The proof of item (ii) is similar to item (i), and similarly, from item (ii), we conclude that there is a t_2 such that $sR^c t_2$ and $\neg\psi \in t_2$. □

We show that the relation R^c is *almost* symmetric.

Proposition 5.8 *For any $s, t \in S^c$, if $sR^c t$ and there exists a χ such that $\neg\Delta\chi \in t$ then $tR^c s$.*

Proof Assume that $sR^c t$ and $\neg\Delta\chi \in t$, we need to show $tR^c s$. Suppose not, then there exists φ such that $\Delta\varphi \wedge \Delta(\chi \to \varphi) \in t$ but $\varphi \notin s$ (thus $\neg\varphi \in s$). Since $sR^c t$, by definition, there is a ψ such that $\neg\Delta\psi \in s$ and ($\star$): for all θ : $\Delta\theta \wedge \Delta(\psi \to \theta) \in s$ implies $\theta \in t$. Thanks to $\Delta\mathtt{B}$, since $\neg\varphi \in s$, we have $\Delta((\Delta\neg\varphi \wedge \Delta(\neg\varphi \to \neg\chi) \wedge \neg\Delta\neg\chi) \to \neg\psi) \in s$ and $\Delta(\Delta\varphi \wedge \Delta(\neg\varphi \to \neg\chi) \wedge \neg\Delta\neg\chi) \in s$. By $\Delta\mathtt{Equ}$ and $\mathtt{RE}\Delta$, $\Delta(\Delta\varphi \wedge \Delta(\chi \to \varphi) \wedge \neg\Delta\chi) \wedge \Delta((\Delta\varphi \wedge \Delta(\chi \to \varphi) \wedge \neg\Delta\chi) \to \neg\psi) \in s$, finally we have $\Delta\neg(\Delta\varphi \wedge \Delta(\chi \to \varphi) \wedge \neg\Delta\chi) \wedge \Delta(\psi \to \neg(\Delta\varphi \wedge \Delta(\chi \to \varphi) \wedge \neg\Delta\chi)) \in s$. By ($\star$), we have $\neg(\Delta\varphi \wedge \Delta(\chi \to \varphi) \wedge \neg\Delta\chi) \in t$, contradiction. □

The above proposition tells us that R^c is *almost* symmetric: $sR^c t$ implies $tR^c s$, given that there exists a χ such that $\neg\Delta\chi \in t$. However, there are some states in $\mathcal{M}^c$ which include $\Delta\chi$ for any χ. In the sequel, we call them 'dead ends' due to the fact that those states cannot have outgoing transitions according to the definition of R^c. To turn $\mathcal{M}^c$ into a symmetric model, we handle these special points based on the following two crucial observations.

(i) if $sR^c t$ and t includes no $\neg\Delta\chi$ formula, then t has no outgoing R^c in $\mathcal{M}^c$ by definition. Now if we add a *unique* transition from t back to s to make it symmetric, then it does not change the truth values of formulas on both s and t.

(ii) However, there might be two or more transitions to a dead end t, e.g., $sR^c t$ and $uR^c t$, then adding both back arrows will change the truth values of formulas on t, since now t has two different successors and some $\Delta\varphi$ may not hold any more. We can fix this by replacing those dead ends with some new copies of themselves such that each copy has only one incoming transition. Essentially, we just split those dead ends.

Based on the above observations, we build the canonical model of $\mathbb{NCLB}$ formally. First let $D = \{t \mid t \in S^c, \Delta\chi \in t \text{ for all } \chi, \text{ and there exists an } s \in S^c \text{ such that } sR^c t\ \}$, where S^c and R^c are defined as in Definition 5.6. Let $\overline{D} = S^c \backslash D$.

Definition 5.9 (Canonical Model of $\mathbb{NCLB}$) *The canonical model $\mathcal{M}^+$ of $\mathbb{NCLB}$ is a tuple $\langle S^+, R^+, f, V^+ \rangle$ where:*

- $S^+ = \overline{D} \cup \{(s,t) \mid t \in D, sR^c t\}$
- *$sR^+ t$ iff one of the following cases holds:*

(i) $s, t \in \overline{D}$ *and* $sR^c t$,

(ii) $s \in \overline{D}$ *and* $t = (s, s') \in S^+$,

(iii) $t \in \overline{D}$ *and* $s = (t, t') \in S^+$.

- *f is a function assigning each state in S^+ to a maximal consistent set in S^c such that $f(s) = s$ for $s \in \overline{D}$, and $f((s,t)) = t$ for $(s,t) \in S^+$.*
- $V^+(p) = \{s \in S^+ \mid p \in f(s)\}$

The function f is introduced to label the maximal consistent sets in $\mathcal{M}^+$, since there can be multiple states in $\mathcal{M}^+$ sharing the same maximal consistent set.

Proposition 5.10

(i) *f is surjective.*

(ii) *if $s \in \overline{D}$ then sR^+t implies $f(s)R^c f(t)$.*

(iii) *if $f(s)R^c t$ then there exists $u \in S^+$ such that $f(u) = t$ and sR^+u.*

Proof For item (i), we need to show that for every $t \in S^c$, there exists a $u \in S^+$ such that $f(u) = t$. Given any $t \in S^c$, there are two cases to consider:

- $t \notin D$, i.e., $t \in \overline{D}$. By the definition of S^+, we have $t \in S^+$; by the definition of f, we have $f(t) = t$.
- $t \in D$. By the definition of D, there exists an $s \in S^c$ such that $sR^c t$. By the definition of S^+, we have $(s,t) \in S^+$. Then by the definition of f, we have $f((s,t)) = t$.

Either case implies that there exists a $u \in S^+$ such that $f(u) = t$.

For item (ii), suppose $s \in \overline{D}$ (thus $f(s) = s$) and sR^+t. If $t \in \overline{D}$ then $sR^c t$ and $f(t) = t$ by definitions, thus $f(s)R^c f(t)$; if $t \notin \overline{D}$ then $t = (s,s') \in S^+$ and $sR^c s'$, thus $f(t) = f((s,s')) = s'$, and hence $f(s)R^c f(t)$.

For item (iii), suppose $f(s)R^c t$. By the definition of D, we have $f(s) \notin D$. Since $f(s) \in S^c$, it follows that $f(s) \in \overline{D}$, thus $f(s) = s$: otherwise, $f(s) = t'$ and $s = (t,t') \in S^+$, then $t' \in \overline{D}$ and $t' \in D$, which is impossible. If $t \in \overline{D}$ then $f(t) = t$ and sR^+t. If $t \in D$ then $(s,t) \in S^+$ and $f((s,t)) = t$, and $sR^+(s,t)$.□

Notice that the condition $s \in \overline{D}$ in (ii) above is indispensable. For instance, suppose that $s = (t,t') \in S^+$ and $t \in \overline{D}$. By definition, sR^+t. By the definition of f, we have $f(s) = f((t,t')) = t'$ and $f(t) = t$. Since $t' \in D$, we do not have $t'R^c t$, i.e., it is not the case that $f(s)R^c f(t)$.

Lemma 5.11 *$\mathcal{M}^+$ is symmetric.*

Proof Suppose for any $s,t \in S^+$ that sR^+t. We need to show that tR^+s. According to the definition of R^+, we have three cases:

(i) $s,t \in \overline{D}$ and $sR^c t$. Then $t \notin D$. From the definition of D, we can see that there exists a χ such that $\neg\Delta\chi \in t$. According to Proposition 5.8 we have $tR^c s$ thus tR^+s.

(ii) $s \in \overline{D}$ and $t = (s,s')$. By the third condition of the definition of R^+ we have tR^+s.

(iii) $t \in \overline{D}$ and $s = (t,t')$. By the second condition of the definition of R^+ we have tR^+s.

□

We show that $\mathcal{M}^+$ preserves the truth values of formulas w.r.t. f:

Proposition 5.12 *For any $s \in S^+$ and any $\varphi \in$ **NCL**, we have*

$$\mathcal{M}^+, s \vDash \varphi \iff \mathcal{M}^c, f(s) \vDash \varphi.$$ [8]

Proof By induction on φ.

- $\varphi = p \in \mathbf{P}$. For any $s \in S^+$, $s \in V^+(p)$ iff $p \in f(s)$ iff $f(s) \in V^c(p)$.
- Boolean cases are immediate.
- $\varphi = \Delta\psi$. We show that $\mathcal{M}^+, s \nvDash \Delta\psi \iff \mathcal{M}^c, f(s) \nvDash \Delta\psi$.
  $\Rightarrow$: Suppose $\mathcal{M}^+, s \nvDash \Delta\psi$ then there are two points t and t' such that sR^+t, sR^+t', $\mathcal{M}^+, t \vDash \psi$ and $\mathcal{M}^+, t' \nvDash \psi$. First note that $s \in \overline{D}$, for otherwise s cannot have two different successors according to the definition of R^+. Now due to item (ii) of Proposition 5.10, $f(s)R^c f(t)$ and $f(s)R^c f(t')$. By IH, $\mathcal{M}^c, f(t) \vDash \psi$ and $\mathcal{M}^c, f(t') \nvDash \psi$. Thus $\mathcal{M}^c, f(s) \nvDash \Delta\psi$.
  $\Leftarrow$: Suppose $\mathcal{M}^c, f(s) \nvDash \Delta\psi$ then there are two points t and t' such that $f(s)R^c t$, $f(s)R^c t'$, $\mathcal{M}^c, t \vDash \psi$ and $\mathcal{M}^c, t' \nvDash \psi$. Now according to item (iii) of Proposition 5.10, there are $u, u' \in S^+$ such that $t = f(u)$, $t' = f(u')$, sR^+u and sR^+u'. By IH, we have $\mathcal{M}^+, u \vDash \psi$ and $\mathcal{M}^+, u' \nvDash \psi$, therefore $\mathcal{M}^+, s \nvDash \Delta\psi$.

□

From Lemma 5.7 and Proposition 5.12 we have:

Lemma 5.13 *For any $\varphi \in$ **NCL** and any $s \in S^+$, $\mathcal{M}^+, s \vDash \varphi$ iff $\varphi \in f(s)$.*

Now due to item (i) of Proposition 5.10, every $s \in S^c$ is an image of some u in $\mathcal{M}^+$ under f, thus each maximal consistent set is satisfiable in $\mathcal{M}^+$, which gives us the completeness theorem based on Lemma 5.11.

Theorem 5.14 (Soundness and Completeness of $\mathbb{NCLB}$) $\mathbb{NCLB}$ *is sound and strongly complete with respect to the class of symmetric frames.*

6 Conclusions and future work

We showed that necessity is almost definable in terms of non-contingency, which is demonstrated by the valid principle $\nabla\psi \to (\Box\varphi \leftrightarrow \Delta\varphi \wedge \Delta(\psi \to \varphi))$ (AD). We proposed notions of Δ-bisimulation and of Δ-bisimulation contraction. We also characterized non-contingency logic as the Δ-bisimulation invariant fragment of modal logic and of first-order logic. Inspired again by almost-definability, we axiomatized non-contingency logic over symmetric frames. This completes the spectrum of complete systems in the literature for non-contingency logic over the usual frame classes.

[8] We can also prove this proposition by showing that the function f is the graph of a Δ-bisimulation.

As mentioned in the introduction, in an epistemic setting, $\Delta\varphi$ is read as "the agent knows whether φ is true". There, it is natural to consider multi-agent scenarios which require multiple Δ-operators. For future work, we conjecture that the multi-agent version of the proof system $\mathbb{NCLB}$ also axiomatizes multi-agent non-contingency logic, where instead of unlabelled Δ-operators we employ labeled Δ_a-operators.

Other future work involves the dynamics of non-contingency logic. In [5] non-contingency logic with public announcements is axiomatized. This can be straightforwardly generalized to action models with non-contingency operators. A reduction axiom for knowing whether (non-contingency) consequences after update is $[\mathbf{M},\mathbf{s}]\Delta\psi \leftrightarrow (\mathbf{pre}(\mathbf{s}) \rightarrow \bigwedge_{\mathbf{sRt}}(\Delta[\mathbf{M},\mathbf{t}]\psi \vee \Delta[\mathbf{M},\mathbf{t}]\neg\psi))$. Such dynamics can also be added to the axiomatizations for non-contingency logic over various other frames, such as the underlying one on symmetric frames.

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References

[1] Blackburn, P., M. de Rijke and Y. Venema, "Modal Logic," Cambridge University Press, 2002.

[2] Brogan, A., *Aristotle's logic of statements about contingency*, Mind **76** (1967), pp. 49–61.

[3] Cresswell, M., *Necessity and contingency*, Studia Logica **47** (1988), pp. 145–149.

[4] Demri, S., *A completeness proof for a logic with an alternative necessity operator*, Studia Logica **58** (1997), pp. 99–112.

[5] Fan, J., Y. Wang and H. van Ditmarsch, *Knowing whether*, CoRR **abs/1312.0144** (2013), under submission.

[6] Humberstone, L., *The logic of non-contingency*, Notre Dame Journal of Formal Logic **36** (1995), pp. 214–229.

[7] Humberstone, L., *The modal logic of agreement and noncontingency*, Notre Dame Journal of Formal Logic **43** (2002), pp. 95–127.

[8] Humberstone, L., *Zolin and Pizzi: Defining necessity from noncontingency*, Erkenntnis **78** (2013), pp. 1275–1302.

[9] Kuhn, S., *Minimal non-contingency logic*, Notre Dame Journal of Formal Logic **36** (1995), pp. 230–234.

[10] Montgomery, H. and R. Routley, *Contingency and non-contingency bases for normal modal logics*, Logique et Analyse **9** (1966), pp. 318–328.

[11] Pizzi, C., *Contingency logics and propositional quantifications*, Manuscrito **22** (1999), pp. 283–303.

[12] Pizzi, C., *Necessity and relative contingency*, Studia Logica **85** (2007), pp. 395–410.

[13] Segerberg, K., "Classical Propositional Operators," Oxford, Clarendon Press, 1982.

[14] Steinsvold, C., *A note on logics of ignorance and borders*, Notre Dame Journal of Formal Logic **49(4)** (2008), pp. 385–392.

[15] van der Hoek, W. and A. Lomuscio, *A logic for ignorance*, Electronic Notes in Theoretical Computer Science **85(2)** (2004), pp. 117–133.
[16] Zolin, E., *Completeness and definability in the logic of noncontingency*, Notre Dame Journal of Formal Logic **40** (1999), pp. 533–547.
[17] Zolin, E., *Infinitary expressibility of necessity in terms of contingency*, Proceedings of the sixth ESSLLI student session (2001), pp. 325–334.