Fraïssé Limits, Hrushovski Property and Generic Automorphisms

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Abstract

This paper is a survey on the Fraïssé limits of the classes of finite graphs, finite rational metric spaces and finite K_n-free graphs. Hrushovski showed a property concerning extending partial isomorphisms for finite graphs, and this property turns out to yield results for the Fraïssé limit of the class of finite graphs. Similar properties were proven for the other two classes, which yielded analogue results concerning the generic automorphisms of their Fraïssé limits.

1 Introduction

The beginning of the story we're going to tell dated back to the 1950s. Roland Fraïssé published an important paper [3], where he established a method proven to be fruitful in constructing countably infinite homogeneous structures. Following Hodges's suggestion in [7], we now call this method Fraïssé construction. Among all the structures we obtain from Fraïssé construction, the first one that might draw most mathematician's attention was the Fraïssé limit of the class of finite graphs. In 1963, Erdős and Rényi [2] proved that, if a countable graph is chosen at random by selecting edges independently with probability 1/2, then the resulting graph is isomorphic to the Fraïssé limit of the class of finite graphs with probability 1. This is why we call this graph the *random graph.* (The graph is also known as Rado's graph, since it was Rado who gave the first concrete construction of it.)

In 1992, Hrushovski published his famous paper [9], where he proved that, every finite graph can be extended to a finite supergraph, such that all the partial isomorphisms of the original graph extend to automorphisms of the supergraph. It is shown that, this result, together with other results about generic automorphisms, yields several properties concerning the automorphism group of the random graph, for example, the small index property as proven by Hodges, Hodkinson, Lascar and Shelah in [8]. Kechris and Rosendal showed later in [10] that, their proof also applies to other Fraïssé limits, when certain conditions are met for the Fraïssé classes. Interesting enough, in many cases the conditions Kechris and Rosendal came up with can be implied from results analogue to Hrushovski's theorem for finite graphs. This leads us to the concern whether it's feasible to extend partial isomorphisms within other Fraïssé classes. Herwig and Lascar established a method in [6] which would help. All these results have brought about the manufacturing of properties in a large number of seemingly different structures. To summarize their work, we will consider in this article several most basic Fraïssé classes and prove parallel results for each of them.

This paper mainly consists of three parts. In the first part, we introduce Fraïssé construction and define the structures with which we concern in the following sections. In the second part, we give proofs of results that allow us to extend arbitrary partial isomorphism, namely the Hrushovski property, for the Fraïssé classes we mentioned in the first section. We also answer one of Hrushovski's questions he raised in [9]. In the last part, we focus on the automorphism groups of the Fraïssé limits and apply the Hrushovski property for each Fraïssé class to prove the existence of generic automorphisms. These proofs based on more recent results are much shorter than their original proofs. We also prove another result that shows the generic automorphisms are conjugate to their powers for these Fraïssé limits.

In this article we need some basic notions from first order logic, especially model theory. We recall that, a *language* $\mathcal{L}$ is a collection of function symbols, relational symbols and constant symbols. A language is *relational* if it contains no function symbols and constants. An $\mathcal{L}$*-structure* $\mathcal{M}$ is an underlying set M with *interpretation* of each symbol in $\mathcal{L}$. Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are *isomorphic* if there is a bijection $f\colon M \to N$ preserving the interpretation of all symbols in $\mathcal{L}$. A function g is a *partial isomorphism* between $\mathcal{M}$ and $\mathcal{N}$ if it is an isomorphism between substructures of $\mathcal{M}$ and $\mathcal{N}$. For an $\mathcal{L}$-structure $\mathcal{M}$ and an $\mathcal{L}$-formula ϕ, the notion that $\mathcal{M}$ *satisfies* ϕ (denoted by $\mathcal{M} \models \phi$) is defined by induction on the complexity of ϕ. For a more detailed introduction to these notions, the reader may refer to textbooks of model theory, such as Hodges [7] and Marker [12].

2 Fraïssé Limits

In his classic paper [3], Roland Fraïssé proved that, for a class $\mathcal{K}$ of structures with some certain properties, there exists a structure that contains isomorphic copies of every structure in $\mathcal{K}$. In order to state his theorem, we first introduce some definitions.

Definition 2.1. Let $\mathcal{K}$ be a class of structures.

1. $\mathcal{K}$ has the *hereditary property* (HP for short), if for all $A \in \mathcal{K}$ and all substructures B of A, we have $B \in \mathcal{K}$.

2. $\mathcal{K}$ has the *joint embedding property* (JEP for short), if for all $A, B \in \mathcal{K}$, there is $C \in \mathcal{K}$ such that both A and B can be embedded in C.

3. $\mathcal{K}$ has the *amalgamation property* (AP for short), if for all A, B and C in $\mathcal{K}$ with embeddings $f_1\colon A \to B$, $f_2\colon A \to C$, there is $D \in \mathcal{K}$ with embeddings $g_1\colon B \to D$, $g_2\colon C \to D$ such that $g_1 \circ f_1 = g_2 \circ f_2$.

Definition 2.2. Let $\mathcal{L}$ be a relational language and $\mathcal{M}$ be an $\mathcal{L}$-structure. The *age* of $\mathcal{M}$ is the class $\mathcal{K}$ of all finite structures that can be embedded in $\mathcal{M}$.

Definition 2.3. Let $\mathcal{L}$ be a relational language and $\mathcal{M}$ be an $\mathcal{L}$-structure. We call $\mathcal{M}$ *ultrahomogeneous* if every isomorphism between two finite substructures of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$.

With the definitions above, we can now state Fraïssé's theorem:

Theorem 2.4 (Fraïssé's Theorem, [7, Thm 7.1.2])**.** *Let $\mathcal{L}$ be a countable language and let $\mathcal{K}$ be a non-empty finite or countable class of finite $\mathcal{L}$-structures which has HP, JEP and AP. Then there is an $\mathcal{L}$-structure $\mathcal{M}$, unique up to isomorphism, such that $\mathcal{M}$ has cardinality at most ω, the class $\mathcal{K}$ is the age of $\mathcal{M}$ and $\mathcal{M}$ is ultrahomogeneous.* □

Following Hodges's terminology, a class which satisfies the conditions of the theorem is called a *Fraïssé class*, with the uniquely determined $\mathcal{L}$-structure $\mathcal{M}$ to be its *Fraïssé limit*. We now introduce the three main structures we're concerned with in this article.

Example 2.5 (The random graph)**.** Let $\mathcal{K}$ be the class of all finite graphs, where the language of graphs contains only one binary relation symbol E and no function symbols or constants. For every finite graph G, the universe of the structure is the set of all vertices in G; for every two vertices $v_1, v_2 \in G$, v_1 and v_2 are connected iff $G \models E(v_1, v_2) \wedge E(v_2, v_1)$. It's obvious that $\mathcal{K}$ has HP, JEP and AP, and thus the class $\mathcal{K}$ has a unique Fraïssé limit, which is called the *random graph* $\mathcal{R}$.

The word "random" here is due to Erdős and Rényi, as introduced before.

Example 2.6 (The Henson graph)**.** For each positive integer n, let $\mathcal{K}^n$ be the class of all finite graphs which do not include K_n, *i.e.*, the complete graph of size n, as an induced subgraph. The language we work with and its interpretation are the same as that in the previous example. It's obvious that HP and JEP hold for $\mathcal{K}^n$. To check AP, we take graph $A, B, C \in \mathcal{K}^n$ with embeddings $f_1 \colon A \to B$, $f_2 \colon A \to C$. Without loss of generality, we may assume $A \subseteq B$ and $A \subseteq C$. Let D be the set of $B \cup C$ and define two vertices to be adjacent in D iff they're adjacent in either B or C. Since neither B nor C contains K_n as an induced subgraph, D is K_n-free. Thus AP holds for the class $\mathcal{K}^n$. Since this Fraïssé class was first introduced and studied by Henson in [4], we call the Fraïssé limit of the class $\mathcal{K}^n$ the *Henson graph* H_n.

Example 2.7 (The rational Urysohn space)**.** Let $\mathcal{K}$ be the class of all finite rational metric spaces, *i.e.*, the class of finite metric spaces that take distances in positive rational numbers. The language $\mathcal{L}$ we work with contains a relation symbol R_r for each positive rational number r and no function symbols or constants. For every finite rational metric space X, the universe of the structure is the set of all points in X; for every two distinct points $a, b \in X$, we have $d(a, b) = r$ iff $X \models R_r(a, b) \wedge R_r(b, a)$.

Again, it's obvious that $\mathcal{K}$ has HP and JEP. To check AP, we take finite rational metric spaces A, B and C with embeddings $f_1 \colon A \to B$, $f_2 \colon A \to C$. Without loss of generality, we may assume $A \subseteq B$ and $A \subseteq C$. Let D be the set $B \cup C$. We define a metric on D by preserving the metric on B or C when $a, b \in B$ or $a, b \in C$ and setting $d_D(a, b)$ to be the minimum of $d_B(a, x) + d_C(x, b)$ for all $x \in A$ when $a \in B \setminus A$ and $b \in C \setminus A$. It's straight forward to check that (D, d_D) is a finite rational metric space, and therefore AP holds for the class $\mathcal{K}$. Due to Urysohn [17], we call the Fraïssé limit of $\mathcal{K}$ the *rational Urysohn space* $\mathbb{QU}$.

3 Hrushovski Property

In his famous work [9], Hrushovski proved the following result:

Theorem 3.1 (Hrushovski, [9])**.** *Given a finite graph X, there is a finite graph Z such that X is an induced subgraph of Z and every partial isomor-*

phism of X extends to an automorphism of Z.

Hrushovski's original proof was somewhat abstract. Given a finite graph X, let Y be the power set of X. Hrushovski considered the permutation group of Y and picked one of its subgroups G generated by some certain elements. Then he defined an equivalence relation $\sim$ on $G \times X$, and by endowing $G \times X/\sim$ with a graph structure, the desired graph Z is obtained.

The following shorter proof of the theorem was given by Cédric Milliet in [13]. For a finite graph X, we build the desired graph Z in two steps. For the first step, we extend X to a graph Y with uniform valency, *i.e.*, all vertices in Y have the same number of neighbours. For the second step, we embed Y into a finite graph Z. Finally we prove that every partial isomorphism of X now extends to an automorphism of Z.

Proof of Theorem 3.1. For the first step, let n be the maximum valency of X. We assume that n is odd, otherwise replace n with $n+1$. For each vertex x in X, we add some new vertices adjacent to and only to x so that the valency of x increases to n. In this new graph, each vertex would have valency either n or 1. Enumerate all the vertices of valency 1 by $x_1, x_2, ..., x_m$. We assume that $n \leq m$, otherwise we may add an isolated vertex to graph X before the construction begins. For each pair $1 \leq i < j \leq m$, we link x_i and x_j iff $j-i \leq (n-1)/2$ or $i+m-j \leq (n-1)/2$. In this way, each x_i would have valency $1 + 2(n-1)/2 = n$. We have now obtained a supergraph Y of X with uniform valency n.

For the second step, let E be the set of all edges in Y. We denote by $G(E,n)$ the set of all n-element subsets of E. Now we endow $G(E,n)$ with a graph structure such that two distinct vertices $x, y \in G(E,n)$ are adjacent iff $x \cap y \neq \emptyset$. Then, there is a natural embedding of Y into $Z = G(E,n)$ sending each vertex of Y to the n-set consisting of all edges related to this vertex in Y. Notice that, for every two vertices $x, y \in Y \subseteq Z$, we have $|x \cap y| \leq 1$.

At last, given a partial isomorphism σ of X, we extend it to an automorphism of $G(E,n)$. This can be done by building a permutation α of E such that it induces a natural permutation of $G(E,n)$ that extends σ.

Denote the domain of σ by $X_1 \subseteq G(E,n)$ and the range of σ by $X_2 \subseteq G(E,n)$. Let $e \in E$. Then one of the following holds:

(1) e belongs to two elements a and b in X_1. Then $\alpha(x)$ has to be the unique element of $\sigma(a) \cap \sigma(b)$;

(2) e belongs to only one element a in X_1. Notice that the cardinality of edges that belong to only one element must be the same for a and $\sigma(a)$. Let $\alpha|_a$ be one-to-one between these edges;

(3) e belongs to no elements in X_1. Since σ is an isomorphism, we have equally many edges in E that belongs to no elements in X_1 and X_2. Let α be one-to-one between these edges.

Clearly, α defined above is a permutation of E and the permutation of $G(E,n)$ it induces extends σ. □

On the size of Z

Hrushovski considered another question about extending partial isomorphisms in his original paper. Given a positive integer n, what is the minimum integer, denoted by $f(n)$, such that each graph of size at most n can be extended to a supergraph of size at most $f(n)$ satisfying the condition of the theorem? By Hrushovski's own proof of Theorem 3.1, when the size of X is n, the graph Z constructed would be of size at most $(2n2^n)!$. Hrushovski wondered whether it's possible to reduce the doubly-exponential bound; moreover, he conjectured that the bound should be 2^{cn^2}. In fact, Milliet's construction of Z showed us an even more precise estimation.

Proposition 3.2. *Let n be a positive integer large enough. Given a graph X with n vertices, there is a graph Z with at most n^{3n} vertices such that X embeds in Z and every partial isomorphism of X extends to an automorphism of Z.*

Proof. We check the construction in Milliet's proof.

In the first step, the maximum valency of X is at most n. Then we attach to each point at most $n+1$ new vertices. In the special case where there are not enough new vertices, we add a new vertex before the whole construction started. Thus, the graph Y we constructed is of size at most $(n+1)(n+2)$. Since Y is of uniform valency at most $n+1$, it has at most $(n+1)^2(n+2)/2$ edges.

Graph Z is constructed by taking all the n-subsets of the set of edges in Y. Therefore, the size of graph Z is at most

$$\binom{(n+1)^2(n+2)/2}{n} < \binom{n^3}{n} < n^{3n}$$

for n large enough. □

Remark. Note that $n^{3n} = 2^{3n\log_2 n} < 2^{cn^2}$ for n large enough. This estimation does answer Hrushovski's question.

Hrushovski property for metric spaces

Since finite graphs can be viewed as special cases of finite metric spaces, a natural question to ask here is whether the analogue property holds for finite metric space. We begin our exploration by adopting the same method used above. For the first step, we can prove by induction that, given a finite metric space with n elements, it can be extended to a finite metric space with 2^n elements such that for each element x in the latter space, the multiset $\{d(x,y)|\, y \neq x\}$ is the same. This result is not hard to obtain. What bothers us is the second step. Although it's easy to establish a graph structure on all the n-element sets of edges, defining a metric on all the n-element sets of edges that preserves the metric of the given space seems to be much more difficult. Indeed, such attempt is doomed to fail. The following example shows why it does not work:

Let $X = \{x_1, x_2, x_3, y_1\}$, with metric

$$\begin{aligned} d(x_i, x_j) &= 1 && \text{for } i \neq j; \\ d(x_i, y_1) &= 3 && \text{for } i = 1, 2, 3. \end{aligned}$$

The metric space we obtain from the first step would be $Y = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$, with metric

$$\begin{aligned} d(x_i, x_j) = d(y_i, y_j) &= 1 && \text{for } i \neq j; \\ d(x_i, y_j) &= 3 && \text{for } i, j = 1, 2, 3, 4. \end{aligned}$$

Now we'd like to define a metric on the set Z of all the 7-element sets of edges in Y, such that Y is embedded in Z. Following the spirit of our previous proof, if two sets in Z share only one element, we should define their distance to be the length of the element they share. But unfortunately, even a request as humble as such would lead to the disastrous outcome of violating the triangle inequality. Consider the images of x_1 and y_1 under the embedding, namely the sets

$$a = \{x_1x_2, x_1x_3, x_1x_4, x_1y_1, x_1y_2, x_1y_3, x_1y_4\}$$

and

$$b = \{y_1y_2, y_1y_3, y_1y_4, x_1y_1, x_2y_1, x_3y_1, x_4y_1\};$$

we have $a \cap b = \{x_1y_1\}$ and thus $d(a, b) = d(x_1, y_1) = 3$ as expected. However, when we take

$$c = \{x_2y_2, x_2y_3, x_3y_2, x_3y_3, x_1x_2, y_1y_2, x_3x_4\} \in Z,$$

we have $a \cap c = \{x_1x_2\}$ and $b \cap c = \{y_1y_2\}$, and thus $d(a, c) = d(b, c) = 1$, contradicting with $d(a, c) + d(b, c) \leq d(a, b)$.

The key issue here is that, while the structure of graph is kind of "tolerant" with whatever we attempt to define, the structure of metric space has more restrictions: it has to obey the triangle inequality. Therefore, constructing a finite graph is much easier than doing the same with finite metric space. That's why we can't generalize Milliet's proof of Hrushovski Property of graphs to that of finite metric spaces.

Now the question comes. Does the class of finite metric spaces in fact have Hroshovski Property? The positive answer to this question is revealed by Solecki in [15].

Solecki's proof is based the following profound lemma, which was established by Herwig and Lascar in [6].

Definition 3.3. Let $\mathcal{M}$ and $\mathcal{M}'$ be $\mathcal{L}$-structures. A *weak homomorphism* from $\mathcal{M}$ to $\mathcal{M}'$ is a map f from M to M' such that, for every n-ary relation symbol R of $\mathcal{L}$ and elements $a_1, a_2, ..., a_n$ in M such that $\mathcal{M} \models R(a_1, ..., a_n)$, we have $\mathcal{M}' \models R(f(a_1), ..., f(a_n))$.

Let $\mathcal{T}$ be a class of $\mathcal{L}$-structures. We say that $\mathcal{M}$ is *$\mathcal{T}$-free* if there is no weak homomorphism from every structure in $\mathcal{T}$ to $\mathcal{M}$.

Lemma 3.4 (Herwig and Lascar, [6, Thm 3.2])**.** *Let $\mathcal{L}$ be a finite relational language and $\mathcal{T}$ a finite class of finite $\mathcal{L}$-structures. If for every finite $\mathcal{T}$-free $\mathcal{L}$-structure C_1, there is a $\mathcal{T}$-free $\mathcal{L}$-structure C_2 such that $C_1 \subseteq C_2$ and every partial isomorphism of C_1 extends to an automorphism of C_2, then there exists a* finite *$\mathcal{T}$-free $\mathcal{L}$-structure C_3 such that $C_1 \subseteq C_3$ and every partial isomorphism of C_1 extends to an automorphism of C_3.* □

The notion of $\mathcal{T}$-free in the context of metric spaces is precisely the triangle inequality, which would be shown in the proof below. What the lemma states is the following: if we can find an infinite metric space such that all partial isometries extend to total isometries, such operation can also be done within finite metric spaces. With this powerful lemma as our new weapon, we now stand in the face of the proof.

Theorem 3.5 (Solecki, [15, Thm 2.1]). *Given a finite metric space X, there is a finite metric space Z such that $X \subseteq Z$ as metric spaces and every partial isometry of X extends to an isometry of Z.*

Proof. Given a finite metric space (X, d) where the distance takes value in $D = \{d(x_1, x_2) | \, x_1, x_2 \in X, x_1 \neq x_2\}$. Let $\mathcal{L}$ be the language with a binary relation symbol R_r for each $r \in D$ and no function symbols or constants.

For each positive integer n and sequence $\alpha = (r_0, r_1, ..., r_n) \in D^{n+1}$ such that $r_1 + ... + r_n \leq r_0$, let $\mathcal{M}_\alpha$ be the $\mathcal{L}$-structure with $n+1$ distinct elements $x_0, x_1, ..., x_n$ such that $\mathcal{M}_\alpha \models R_{r_i}(x_{i-1}, x_i) \wedge R_{r_i}(x_i, x_{i-1})$ for each $1 \leq i \leq n$ and $M_\alpha \models R_{r_0}(x_0, x_n) \wedge R_{r_0}(x_n, x_0)$ and no other relations holding between pairs of elements in $\mathcal{M}_\alpha$. Intuitively, $\mathcal{M}_\alpha$'s are the cases where the triangle inequality does not hold.

Let $\mathcal{T}$ be the class of all $\mathcal{M}_\alpha$'s. Notice that D is a finite set of positive numbers and thus there are only finitely many α's satisfying the requirements. Therefore, $\mathcal{T}$ is finite. Since X is a metric space, it is $\mathcal{T}$-free.

In order to use Lemma 3.4, we need to extend X to an infinite $\mathcal{T}$-free $\mathcal{L}$-structure $\mathcal{M}$ such that every partial isomorphism of X extends to an automorphism of $\mathcal{M}$. Let D' be the additive semigroup generated by D, and let $\mathcal{L}'$ be the language including the distances in D'. Applying the argument in Example 2.7, the class of all finite metric spaces whose distance takes value in D' has a Fraïssé limit $\mathcal{M}$. By the ultrahomogeneity of $\mathcal{M}$, every partial isometry of X now extends to a total isometry of $\mathcal{M}$. Therefore, when we view $\mathcal{M}$ as an $\mathcal{L}$-structure (forgetting all the distance relations that are not in $\mathcal{L}$), every partial isomorphism of X extends to an automorphism of $\mathcal{M}$. By Lemma 3.4, there is a finite $\mathcal{T}$-free $\mathcal{L}$-structure Y extending X such that every partial isomorphism of X extends to an automorphism of Y. Notice that this is not yet the result we desire, since Y is only a finite $\mathcal{T}$-free $\mathcal{L}$-structure, not a finite metric space.

We say that a sequence $y_0, y_1, ..., y_n$ of elements in Y is a *chain* between y_0 and y_n if for each $1 \leq i \leq n$ there is $r_i \in D$ such that $Y \models R_{r_i}(y_{i-1}, y_i) \wedge$

$R_{r_i}(y_i, y_{i-1})$, and we define $r_1 + ... + r_n$ to be the length of such a chain. Let Z be the substructure of Y consisting of all $y \in Y$ which is connected to some element in X by a chain. Clearly $X \subseteq Z$ as $\mathcal{L}$-structures.

Now we define a metric ρ on Z. If $a = b$, then $\rho(a, b) = 0$. Otherwise, we define $\rho(a, b)$ to be the length of the shortest chain between a and b. Since each element of Z is connected by a chain to some element in X and two distinct elements in X are always connected to each other, ρ is well-defined on Z. It's obvious that ρ satisfies the triangle inequality and thus is a metric on Z. It remains to show that ρ restricted to X coincides with d and that every partial isometry of X extends to a total isometry of Z.

For the first statement, we take $a, b \in X$ and check whether $d(a, b) = \rho(a, b)$. The inequality $\rho(a, b) \leq d(a, b)$ is clear since a, b is itself a chain. We assume towards contradiction that $\rho(a, b) < d(a, b)$. This indicates that there is a chain between a and b with distances $r_1, ...r_n \in D$ such that $r_1 + ... + r_n < d(a, b)$, which contradicts to the fact that Y is $\mathcal{T}$-free.

For the second statement, we take p to be a partial isometry of X. By the result of Lemma 3.4, p is extended to an automorphism p' of Y. Notice that the definition of chains and their length are preserved under automorphisms. Therefore, if two elements $a, b \in Y$ are connected to each other by a chain, $p'(a)$ and $p'(b)$ would also be connected by a chain of the same length. This not only ensures that $p'|_Z$ is an automorphism of Z but also shows that $p'|_Z$ preserves metric. Therefore, $p'|_Z$ is a total isometry of Z that extends p. □

Hrushovski property for K_n-free graphs

It is not hard to see that Lemma 3.4 and the existence of the Fraïssé limit play the most important roles in the proof of the theorem. In fact, the proof of the theorem can be generalized to obtain similar results for other classes of structures. We take the class of finite K_n-free graphs as an example. This theorem was first proven by Herwig in [5] before Lemma 3.4 was established:

Theorem 3.6 (Herwig, [5, Thm 2]). *Given a finite K_n-free graph X, there* *is a finite K_n-free graph Z such that X is an induced subgraph of Z and every partial isomorphism of X extends to an automorphism of Z.*

Proof. Let $\mathcal{L}$ be the language of graph with only one binary relation symbol E. Take $\mathcal{T}_n$ to be the class of all $\mathcal{L}$-structures $\mathcal{M}$ with exactly n elements such that, for each pair a, b of distinct elements in $\mathcal{M}$, we have $\mathcal{M} \models E(a, b) \vee E(b, a)$.

Given a finite K_n-free graph X, by Example 2.6, it can be embedded into the Henson graph H_n such that every partial isomorphism of X extends to an automorphism of H_n. By Lemma 3.4, there is a finite $\mathcal{T}_n$-free $\mathcal{L}$-structure Z such that every partial isomorphism of X extends to an automorphism of Z. Define a graph structure on Z such that two vertices a, b in Z are adjacent iff $Z \models E(a, b) \vee E(b, a)$. Clearly Z is now a K_n-free graph. Thus we have proved the theorem. □

4 Generic Automorphisms

In this section, we concern the automorphism groups of the Fraïssé limits we introduced in Section 2, namely the random graph $\mathcal{R}$, the rational Urysohn space $\mathbb{QU}$ and the Henson graph H_n. We denote by $\mathrm{Aut}(\mathcal{R})$, $\mathrm{Aut}(\mathbb{QU})$ and $\mathrm{Aut}(H_n)$ their automorphism groups and equip them with pointwise convergence topology as discrete sets. Thus, the basic open sets in each of the Polish groups are of the form:

$$\{h \in \mathrm{Aut}(\mathcal{R}) |\ h|_A = g|_A\} \text{ for some finite } A \subseteq \mathcal{R} \text{ and } g \in \mathrm{Aut}(\mathcal{R})$$
$$\{h \in \mathrm{Aut}(\mathbb{QU}) |\ h|_A = g|_A\} \text{ for some finite } A \subseteq \mathbb{QU} \text{ and } g \in \mathrm{Aut}(\mathbb{QU})$$
$$\{h \in \mathrm{Aut}(H_n) |\ h|_A = g|_A\} \text{ for some finite } A \subseteq H_n \text{ and } g \in \mathrm{Aut}(H_n)$$

Definition 4.1. We say that an automorphism g of a structure is *generic* if its conjugacy class is comeager under the pointwise convergence topology.

To see how this definition might work, we take a look at a simple example, which can be found in several different papers such as Truss [16] and Lascar [11].

Proposition 4.2 (Folklore)**.** *The countably infinite set* Ω *with empty language has a generic automorphism.*

Proof. Since the language is empty, an automorphism of Ω is precisely a permutation of Ω. We assert that a permutation σ of Ω is generic iff it contains no infinite cycles and infinitely many n-cycles for each n.

Firstly, every two permutations of Ω with the same cycle type are clearly conjugate. We only have to show that the class of all permutations of this cycle type is comeager.

For each positive integer n and k, let $A_{n,k}$ be the class of permutations with at least k disjoint n-cycles. Then $A_{n,k}$ is the union of $\{\sigma \in \mathrm{Aut}(\Omega)|\ \sigma|_{\Omega_0} = \tau\}$ for all $|\Omega_0| = nk$ and τ a permutation of Ω_0 with k disjoint n-cycles. Since each $\{\sigma \in \mathrm{Aut}(\Omega)|\ \sigma|_{\Omega_0} = \tau\}$ is a basic open set, $A_{n,k}$ is open. Now, given a basic open set D, the permutations in D are only determined within a finite set and may behave as whatever we want outside the set. Thus the intersection of $A_{n,k}$ and D is always non-empty, which implies that $A_{n,k}$ is dense.

Now enumerate the elements of Ω by $\omega_1, \omega_2, ...$ and let $B_{n,k} = \{\sigma \in \mathrm{Aut}(\Omega)|\ \sigma^k(\omega_n) = \omega_n\}$. By an argument similar to that for $A_{n,k}$, one can show that each $B_{n,k}$ is dense open.

Finally, let

$$A = \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} A_{n,k},\ B = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} B_{n,k}.$$

Then A is precisely the class of permutations with infinitely many n-cycles for each n, and B the class of permutations without infinite cycles. Since each $A_{n,k}$ and $B_{n,k}$ is dense open, $A \cap B$ is a dense G_δ, which completes the proof. □

What is special about the above example is that the conjugacy class of $\mathrm{Aut}(\Omega)$ is uniquely determined by its cycle type. For other structures where the language is not empty, it would be a lot harder to decide how a conjugacy class might look like. The existence of generic automorphisms for the random graph was established by Truss in [16], whose outline was similar to that in the above case, but with a much more complicated description of the comeager conjugacy class and a much longer proof. The result concerning rational Urysohn space was first proven by Solecki in [15], with a completely different approach that avoids showing explicitly what the comeager conjugacy class may look like. Both Truss and Solecki's proofs are closely related to the specific structure they concern, and it seems hard to apply their methods to other structures.

Fortunately, a necessary and sufficient condition for a Fraïssé limit to have generic automorphisms was later given by Kechris and Rosendal in [10], which allows us to obtain results on structures of different kinds with a single method. To state their result, we first introduce some notations:

Definition 4.3. For a class $\mathcal{K}$ of structures, we say that $\mathcal{K}$ has the *weak amalgamation property* (WAP for short), if for all $S \in \mathcal{K}$, there is $A \in \mathcal{K}$ with an embedding $e\colon S \to A$, such that for all $B, C \in \mathcal{K}$ with embeddings $f_1\colon A \to B$, $f_2\colon A \to C$, there are $D \in \mathcal{K}$ and embeddings $g_1\colon B \to D$, $g_2\colon C \to D$ such that $g_1 \circ f_1 \circ e = g_2 \circ f_2 \circ e$.

For each Fraïssé class $\mathcal{K}$, we associate it with the class $\mathcal{K}_p$ of all systems $S = \langle A, \psi \rangle$ such that $A \in \mathcal{K}$ and ψ is a partial isomorphism of A. We naturally say that $S = \langle A, \psi \rangle$ is embedded in $T = \langle B, \phi \rangle$ when A is embedded in B and ϕ extends ψ.

Theorem 4.4 (Kechris and Rosendal, [10, Thm 3.4])**.** *Let $\mathcal{K}$ be a Fraïssé class with its Fraïssé limit F. Then the following are equivalent:*

(1) F has a generic automorphism;

(2) $\mathcal{K}_p$ has JEP and WAP. □

Kechris and Rosendal's proof of the theorem in their paper involves the concept of "turbulence" and several properties related to it. We now skip the proof to some corollaries of the theorem concerning the Fraïssé classes mentioned before. Note that the way we prove these corollaries differ from their original proofs.

Corollary 4.5 (Truss, [16, Thm 3.2])**.** *The random graph $\mathcal{R}$ has a generic automorphism.*

Proof. Let $\mathcal{K}$ be the class of all finite graphs. Given $S = \langle A, \psi \rangle$ and $T = \langle B, \phi \rangle$ in $\mathcal{K}_p$, we take $U = \langle A \sqcup B, \psi \sqcup \phi \rangle$, where two vertices in $A \sqcup B$ are adjacent iff they're adjacent in either A or B. It's obvious that both S and T embed in U, and thus $\mathcal{K}_p$ has JEP.

Given $S = \langle A, \psi \rangle$ in $\mathcal{K}_p$, by Theorem 3.1 there is $T = \langle B, \phi \rangle$ in $\mathcal{K}_p$ such that A embeds in B and ϕ is an automorphism of B extending ψ. For every two systems $T_1 = \langle B_1, \phi_1 \rangle$ and $T_2 = \langle B_2, \phi_2 \rangle$ in $\mathcal{K}_p$ with embeddings $f_1 \colon B \to B_1$ and $f_2 \colon B \to B_2$ such that both ϕ_1 and ϕ_2 extend ϕ, take $C = B_1 \sqcup_B B_2$, where two vertices in C are adjacent iff they're adjacent in either B_1 or B_2. Since ϕ is an automorphism of B, ϕ_1 and ϕ_2 coincides on B when viewed as partial isomorphisms of C, and thus $\phi_1 \cup \phi_2$ is well-defined on C. It's obvious that $\phi_1 \cup \phi_2$ is a partial isomorphism of C that extends both ϕ_1 and ϕ_2. Thus WAP holds for $\mathcal{K}_p$.

By Theorem 4.4, the random graph $\mathcal{R}$ has a generic automorphism. □

The same proof also applies to the case of Henson graph.

Corollary 4.6. *The Henson graph H_n has a generic automorphism.*

Proof. Let $\mathcal{K}^n$ be the class of all finite K_n-free graphs. The rest of the proof is nearly identical to that of Corollary 4.5, except that we need to check each graph we construct is K_n-free.

Given $S = \langle A, \psi \rangle$ and $T = \langle B, \phi \rangle$ in $\mathcal{K}^n_p$, we take $U = \langle A \sqcup B, \psi \sqcup \phi \rangle$, where two vertices in $A \sqcup B$ are adjacent iff they're adjacent in either A or B. Obviously $A \sqcup B$ is K_n-free and both S and T embeds in U. Therefore $\mathcal{K}^n_p$ has JEP.

Given $S = \langle A, \psi \rangle$ in $\mathcal{K}_p^n$, by Theorem 3.6 there is $T = \langle B, \phi \rangle$ in $\mathcal{K}_p^n$ such that A embeds in B and ϕ is an automorphism of B extending ψ. For every two systems $T_1 = \langle B_1, \phi_1 \rangle$ and $T_2 = \langle B_2, \phi_2 \rangle$ in $\mathcal{K}_p^n$ with embeddings $f_1 \colon B \to B_1$ and $f_2 \colon B \to B_2$ such that both ϕ_1 and ϕ_2 extend ϕ, take $C = B_1 \sqcup_B B_2$, where two vertices in C are adjacent iff they're adjacent in either B_1 or B_2. Since ϕ is an automorphism of B, ϕ_1 and ϕ_2 coincides on B when viewed as partial isomorphisms of C, and thus $\phi_1 \cup \phi_2$ is well-defined on C. Since both B_1 and B_2 are K_n-free, so is C. It's straight forward to check that $\phi_1 \cup \phi_2$ is a partial isomorphism of C that extends both ϕ_1 and ϕ_2. Thus WAP holds for $\mathcal{K}_p^n$.

By Theorem 4.4, the Henson graph H_n has a generic automorphism. □

At last, we consider the case of rational Urysohn space.

Corollary 4.7 (Solecki, [15, Cor 4.1])**.** *The rational Urysohn space* $\mathbb{QU}$ *has a generic isometry.*

Proof. Let $\mathcal{K}$ be the class of all finite rational metric spaces.

To check JEP, given $S = \langle A, \psi \rangle$ and $T = \langle B, \phi \rangle$ in $\mathcal{K}_p$, we take $C = A \sqcup B$. The metric on C preserves that on both A and B, and for $a \in A$, $b \in B$, $d_C(a, b) = 2 \max\{d(A), d(B)\}$. It's clear that $\phi \sqcup \psi$ is now a partial isometry of C.

To check WAP, given $S = \langle A, \psi \rangle$ in $\mathcal{K}_p$, by Theorem 3.5 there is $T = \langle B, \phi \rangle$ in K_p such that A embeds in B and ϕ is a total isometry of B extending ψ. For every two systems $T_1 = \langle B_1, \phi_1 \rangle$ and $T_2 = \langle B_2, \phi_2 \rangle$ in $\mathcal{K}_p$ with embeddings $f_1 \colon B \to B_1$ and $f_2 \colon B \to B_2$ such that both ϕ_1 and ϕ_2 extend ϕ, take $C = B_1 \sqcup_B B_2$. Define a metric d_C on C such that it preserves the metric on both B_1 and B_2 and for $b_1 \in B_1 \setminus B$, $b_2 \in B_2 \setminus B$ we take $d_C(b_1, b_2)$ to be the minimum of $d_{B_1}(b_1, c) + d_{B_2}(b_2, c)$ for all $c \in B$. It's not hard to see that $\phi_1 \cup \phi_2$ is now a partial isometry of C extending both ϕ_1 and ϕ_2.

Therefore, by Theorem 4.4, the rational Urysohn space has a generic isometry. □

5 Powers of Generic Automorphism

Although Theorem 4.4 and its corollaries showed us the existence of generic automorphisms of many structures, we don't yet know much about how those generic automorphisms may behave. Rosendal showed in [14] that the conjugacy class of generic isometries of the rational Urysohn space is closed under powers. His theorem depends upon a lemma.

Lemma 5.1 (Rosendal, [14, Prop 11])**.** *Let $X \subseteq Y$ be finite rational metric spaces, with an isometry f of X and an isometry g of Y, such that $f^n = g|_X$. Then, there is a finite rational metric space Z extending Y and an isometry h of Z such that h extends f and $h^n|_Y = g$.* □

With the lemma above, Rosendal showed that:

Theorem 5.2 (Rosendal, [14, Prop 12])**.** *Let n be a positive integer. The generic isometry of the rational Urysohn space $\mathbb{QU}$ is conjugate with its nth power.* □

Now, we're going to mimic Rosendal's proof to obtain similar results for the random graph and the Henson graphs. We begin with two analogue lemmas.

Lemma 5.3. *Let $X \subseteq Y$ be finite graphs, with an automorphism f of X and an automorphism g of Y, such that $f^n = g|_X$. Then, there is a finite graph Z extending Y and an automorphism h of Z such that h extends f and $h^n|_Y = g$.*

Proof. We follow Rosendal's outline in the proof of Lemma 5.1.

Let $Y_1, ..., Y_n$ be n exact copies of Y and embed X into each Y_i by $l_i(x) = f^{-i}(x)$. For $x \in Y \setminus X$, we denote the copy of x in Y_i by x^i. We take $Z = \bigsqcup_X Y_i$ and endow it with a graph structure. For every two vertices a, b of Z, $E_Z(a, b)$ iff $E_{Y_i}(a, b)$ for some i.

We define a permutation h of Z as follows:

1. $h(x) = f(x)$ for $x \in X$;
2. $h(x^i) = x^{i+1}$ for $x \in Y \setminus X$ and $1 \leq i < n$;
3. $h(x^n) = (gx)^1$ for $x \in Y \setminus X$.

Now we check that h is an automorphism of Z. First, suppose $x, y \in X$:

$$\begin{aligned} E_Z(hx, hy) &\Leftrightarrow E_Z(fx, fy) \Leftrightarrow E_Y(l_i fx, l_i fy) \Leftrightarrow E_Y(f^{1-i}x, f^{1-i}y) \\ &\Leftrightarrow E_Y(f^{-i}x, f^{-i}y) \Leftrightarrow E_Z(x, y) \end{aligned}$$

Second, suppose $x \in X$ and $y \in Y \setminus X$. For $1 \leq i < n$:

$$\begin{aligned} E_Z(hx, hy^i) &\Leftrightarrow E_Z(fx, y^{i+1}) \Leftrightarrow E_{Y_{i+1}}(l_{i+1}fx, y) \Leftrightarrow E_{Y_{i+1}}(f^{-i}x, y) \\ &\Leftrightarrow E_{Y_i}(l_i x, y) \Leftrightarrow E_Z(x, y^i) \\ E_Z(hx, hy^n) &\Leftrightarrow E_Z(fx, (gy)^1) \Leftrightarrow E_{Y_1}(l_1 fx, gy) \Leftrightarrow E_{Y_1}(x, gy) \\ &\Leftrightarrow E_{Y_1}(g^{-1}x, y) \Leftrightarrow E_{Y_1}(f^{-n}x, y) \Leftrightarrow E_{Y_n}(l_n x, y) \Leftrightarrow E_Z(x, y^n) \end{aligned}$$

Finally, for $x, y \in Y \setminus X$ and $1 \leq i < j \leq n$:

$$\begin{aligned} &E_Z(hx^i, hy^i) \Leftrightarrow E_Z(x^{i+1}, y^{i+1}) \Leftrightarrow E_{Y_{i+1}}(x, y) \Leftrightarrow E_{Y_i}(x, y) \Leftrightarrow E_Z(x^i, y^i) \\ &E_Z(hx^i, hy^j) \Leftrightarrow E_Z(x^{i+1}, y^{j+1}) \Leftrightarrow \perp \Leftrightarrow E_Z(x^i, y^j) \end{aligned}$$

In conclusion, we've shown that h is an automorphism of Z.

Now we view g and f as automorphisms of the first copy Y_1 of Y. Obviously h extends f. It remains to check $h^n|_{Y_1} = g$. Take $x \in X$ and $y \in Y \setminus X$. Then we have $h^n x = h^{n-1} fx = ... = f^n x = gx$ and $h^n(y^1) = h^{n-1}(y^2) = ... = h(y^n) = (gy)^1 = g(y^1)$. This completes the proof. □

The same proof also applies to K_n-free graphs.

Lemma 5.4. *Let $X \subseteq Y$ be finite K_n-free graphs, with an automorphism f of X and an automorphism g of Y, such that $f^n = g|_X$. Then, there is a finite K_n-free graph Z extending Y and an automorphism h of Z such that* *h extends f and* $h^n|_Y = g$.

Proof. This is just a special case of the above lemma. There is only one more thing we have to check: when the given graphs X and Y are K_n-free, Z as constructed above is also K_n-free. Since $Z = \bigsqcup_X Y_i$, where each Y_i is an exact copy of Y, the result follows immediately. □

Now we come to the theorems:

Theorem 5.5. *Let n be a positive integer. The generic automorphism of the random graph $\mathcal{R}$ is conjugate with its nth power.*

Proof. We only need to show that there exist two generic automorphisms f and g such that $f = g^n$.

Recall that the basic open sets of $\mathrm{Aut}(\mathcal{R})$ are of the form:

$$U(g, A) = \{h \in \mathrm{Aut}(\mathcal{R}) |\ h|_A = g|_A\},$$

where $A \subseteq \mathcal{R}$ is finite and $g \in \mathrm{Aut}(\mathcal{R})$. Let $V = \cap_i V_i$ be the comeager conjugacy class of $\mathrm{Aut}(\mathcal{R})$ with each V_i dense open and enumerate the points in $\mathcal{R}$ by $a_1, a_2, \ldots$. We build two sequences of partial automorphisms f_i and g_i with finite domains $A_i \subseteq \mathcal{R}$ such that:

(1) $a_i \in A_i$, $A_i \subseteq A_{i+1}$;

(2) f_{i+1} extends f_i, g_{i+1} extends g_i;

(3) $U(f_{i+1}, A_{i+1}) \subseteq V_i$, $U(g_{i+1}, A_{i+1}) \subseteq V_i$

(4) $g_i^n = f_i$.

To start with, we set $A_0 = \emptyset$ with empty automorphisms $f_0 = g_0$. For the induction step, suppose A_i, f_i and g_i are given. Since $\mathcal{R}$ is universal and homogeneous, there is a finite $B \supseteq A_i \cup \{a_{i+1}\}$ and a partial isomorphism h with domain B such that h extends g_i. Since $U(h, B)$ is open and V_i is dense open, there is $U(k_0, C_0) \subseteq U(h, B) \cap V_i$. By Theorem 3.1, we can extend the partial isomorphism k_0 of $C_0 \cup k_0(C_0)$ to an automorphism k of some finite $C \subseteq \mathcal{R}$. Therefore, $U(k, C) \subseteq U(k_0, C_0) \subseteq V_i$. Repeat this construction for $U(k^n, C)$, we obtain an automorphism p of some finite $D \subseteq \mathcal{R}$ such that $U(p, D) \subseteq V_i$, $C \subseteq D$ and p extends k^n. By Lemma 5.3, there is an automorphism q of some finite $E \supseteq D$ such that $q|_C = k$ and $q^n|_D = p$. In conclusion, we have:

$$g_i \subseteq h \subseteq k \subseteq q,\ g_i^n \subseteq h^n \subseteq k^n \subseteq p \subseteq q^n;$$

$$A_i \cup \{a_{i+1}\} \subseteq B \subseteq C \subseteq D \subseteq E;$$

$$U(q, E) \subseteq U(k, C) \subseteq V_i,\ U(q^n, E) \subseteq U(p, D) \subseteq V_i.$$

Therefore, take $A_{i+1} = E$, $f_{i+1} = q^n$ and $g_{i+1} = q$, and the induction step is complete.

Finally, let $f = \cup_i f_i$ and $g = \cup_i g_i$. By (1) and (2), f and g are both automorphisms of $\mathcal{R}$. By (3), $f, g \in \cap_i V_i = V$ and thus are both generic automorphisms. By (4), $f = g^n$. The desired result follows. □

With nearly an identical proof, we also have:

Theorem 5.6. *Let m be a positive integer. The generic automorphism of the Henson graph H_m is conjugate with its nth power for each $n \geq 1$.*

Proof. Denote the basic open sets of $\mathrm{Aut}(H_m)$ by:

$$U(g, A) = \{h \in \mathrm{Aut}(H_m) |\ h|_A = g|_A\},$$

where $A \subseteq H_m$ is finite and $g \in \mathrm{Aut}(H_m)$. Let $V = \cap_i V_i$ be the comeager conjugacy class of $\mathrm{Aut}(H_m)$ with each V_i dense open and enumerate the points in $\mathcal{R}$ by $a_1, a_2, \ldots$. Build two sequences of partial automorphisms f_i and g_i with finite domains $A_i \subseteq H_m$ such that:

(1) $a_i \in A_i$, $A_i \subseteq A_{i+1}$;

(2) f_{i+1} extends f_i, g_{i+1} extends g_i;

(3) $U(f_{i+1}, A_{i+1}) \subseteq V_i$, $U(g_{i+1}, A_{i+1}) \subseteq V_i$

(4) $g_i^n = f_i$.

Set $A_0 = \emptyset$ with empty automorphisms $f_0 = g_0$. For the induction step, suppose A_i, f_i and g_i are given. By the universality and homogeneity of H_m, there is a finite $B \supseteq A_i \cup \{a_{i+1}\}$ and a partial isomorphism h with domain B such that h extends g_i. Since $U(h, B)$ is open and V_i is dense open, there is $U(k_0, C_0) \subseteq U(h, B) \cap V_i$. By Theorem 3.6, we can extend the partial isomorphism k_0 of $C_0 \cup k_0(C_0)$ to an automorphism k of some finite $C \subseteq H_m$. Therefore, $U(k, C) \subseteq U(k_0, C_0) \subseteq V_i$. Repeat this for $U(k^n, C)$ to obtain an automorphism p of some finite $D \subseteq H_m$ such that $U(p, D) \subseteq V_i$, $C \subseteq D$ and p extends k^n. By Lemma 5.4, there is an automorphism q of some finite $E \supseteq D$ such that $q|_C = k$ and $q^n|_D = p$. In conclusion, we have:

$$g_i \subseteq h \subseteq k \subseteq q, \ g_i^n \subseteq h^n \subseteq k^n \subseteq p \subseteq q^n;$$

$$A_i \cup \{a_{i+1}\} \subseteq B \subseteq C \subseteq D \subseteq E;$$

$$U(q, E) \subseteq U(k, C) \subseteq V_i, \ U(q^n, E) \subseteq U(p, D) \subseteq V_i.$$

Therefore, take $A_{i+1} = E$, $f_{i+1} = q^n$ and $g_{i+1} = q$, and the induction step is complete.

At last, take $f = \cup_i f_i$ and $g = \cup_i g_i$. By (1) and (2), f and g are both automorphisms of H_m. By (3), $f, g \in \cap_i V_i = V$ and thus are both generic automorphisms. By (4), $f = g^n$. This completes the proof. □

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