Frame Definability of First-Order Modal Logic

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Outline

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Language of FOML

The FOML-Language τ contains the following symbols:

1. A countably infinite set of variables.
2. $\neg$, $\wedge$.
3. $\exists$.
4. $\equiv$.
5. A countably infinite set of constants.
6. A countably infinite set of *n*-ary predicates for each n $\geq$ 1.
7. Modal operator $\Diamond$

Formula

Definition (FOML-Formula)

$$\phi = P(t_1, \cdots, t_n) \mid t_1 = t_2 \mid \neg\phi \mid \phi \wedge \phi \mid \exists x\phi \mid \Diamond\phi$$

where P is an n-ary predicate and $t_1, \cdots, t_n$ are τ-terms.

- τ-terms and τ-atomic formulas are defined as in FOL.
- Bound and free variables of a formula are as usual.
- A τ- sentence is a formula without any free variable.

Constant Domain Model

Definition

A constant domain Kripke model is a quadruple $\mathfrak{M} = \langle W, R, D, I \rangle$, where

- W is a non-empty set of possible worlds.
- R is a binary relation on W.
- D is a non-empty set.
-
  1. $I(w, P) \subseteq D^n$ for $w \in W$ and each n-ary predicate P.
  2. $I(w, c) = I(w', c) \in D$, for each $w, w' \in W$ and each constant c.

Varying Domain Kripke Model

Definition

A varying domain Kripke model is a tuple $\mathfrak{M} = \langle W, R, D, I, \{D(w)\}_{w \in W} \rangle$, where

- $\langle W, R, D, I \rangle$ is a constant domain model.
- For each $w \in W$, $D(w) \neq \emptyset$ is a domain of w and $D = \bigcup_{w \in W} D(w)$.

We write $\mathfrak{M} = \langle W, R, D, I \rangle$ for short.
$\mathfrak{F} = \langle W, R \rangle$ and $\mathfrak{S} = \langle W, R, D \rangle$ are called *frame* and *skeleton* respectively.
$(\mathfrak{M}, w)$ is called a *pointed model*.

An **assignment** is a function σ, assigning to each variable v an element $\sigma(v)$ in D.
$\sigma(a/x)$ is an assignment which maps x to a and agrees with σ on all variables distinct from x.
$t^{\mathfrak{M},\sigma}$ is the interpretation of t in $\mathfrak{M}$ under the assignment σ.

Definition

Let $\mathfrak{M}$ be a **varying domain model** and σ, for every $w \in W$ and every assignment σ, a τ-formula $\phi(x_1, \cdots, x_n)$ is satisfied at w with respect to σ is defined inductively as follows

$$
\begin{array}{lcl}
\mathfrak{M}, w \vDash_\sigma t_1 = t_2 & \iff & t_1^{\mathfrak{M},\sigma} = t_2^{\mathfrak{M},\sigma} \\
\mathfrak{M}, w \vDash_\sigma P(t_1, \cdots, t_n) & \iff & (t_1^{\mathfrak{M},\sigma}, \cdots, t_n^{\mathfrak{M},\sigma}) \in I(w, P) \\
\mathfrak{M}, w \vDash_\sigma \neg\phi & \iff & \mathfrak{M}, w \nvDash_\sigma \neg\phi \\
\mathfrak{M}, w \vDash_\sigma \phi \wedge \psi & \iff & \mathfrak{M}, w \vDash_\sigma \phi \text{ and } \mathfrak{M}, w \vDash_\sigma \psi \\
\mathfrak{M}, w \vDash_\sigma \exists x\phi(x) & \iff & \text{for some } a \in D(w), \mathfrak{M}, w \vDash_{\sigma(a/x)} \phi(x) \\
\mathfrak{M}, w \vDash_\sigma \Diamond\phi & \iff & \text{for some } w' \in W \text{ with } wRw', \mathfrak{M}, w' \vDash_\sigma \phi
\end{array}
$$

- A formula ϕ is valid in a model $\mathfrak{M}$, denoted by $\mathfrak{M} \vDash \phi$, if for every $w \in W$ and every σ, $\mathfrak{M}, w \vDash_\sigma \phi$.
- A formula ϕ is valid in a frame $\mathfrak{F}$ (or a skeleton $\mathfrak{S}$) iff ϕ is valid in every model based on $\mathfrak{F}$ (or $\mathfrak{S}$).

Two Remarks

1. If the satisfaction of predicates in any world is restricted to its domain, [2]p201[4] i.e. $P(a_1, \cdots, a_n)$ is true in w if and only if $a_i \in D(w)$ for $1 \leq i \leq n$ and $(a_1, \cdots, a_n) \in I(w, P)$, then the sentence $\forall_x \Diamond(P(x) \vee \neg P(x))$ is not valid. a_i might not exist in $D(w)$, it does exist under alternative circumstances we are willing to consider, and consequently talk about a_i is meaningful [3]p102. We follow this and have such sentences valid in our semantics.
2. In a varying domain model $\mathfrak{M}$, $c^{\mathfrak{M}}$ might not necessarily belong to $D(w)$, for $w \in W$. Then the sentence $\exists_x(x = c)$ is not valid in varying domian models.

   We need to assume that in each w each constant c has the same interpretation, otherwise for example the sentence $c = c' \to \Box(c = c')$ is not valid.

Definition (Submodel, Elementary Submodel)

For two first-order Kripke models, we call $\mathfrak{M}$ a **submodel** of $\mathfrak{N}$, denoted by $\mathfrak{M} \subseteq \mathfrak{N}$, if

- $\mathfrak{F}_{\mathfrak{M}}$ is a subframe of $\mathfrak{F}_{\mathfrak{N}}$.
- $D_{\mathfrak{M}}(w) \subseteq D_{\mathfrak{N}}(w)$, for each $w \in W_{\mathfrak{M}}$.
- $I_{\mathfrak{M}}(w, P) = I_{\mathfrak{N}}(w, P) \cap D^n_{\mathfrak{M}}$, for each $w \in W_{\mathfrak{M}}$ and each n-ary predicate P.
- $I_{\mathfrak{M}}(c) = I_{\mathfrak{N}}(c)$, for each constant symbol c.

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- $I_{\mathfrak{M}}(c) = I_{\mathfrak{N}}(c)$, for each constant symbol c.

The model $\mathfrak{M}$ is an **elementary submodel** of $\mathfrak{N}$, denoted by $\mathfrak{M} \preceq \mathfrak{N}$, if $\mathfrak{M} \subseteq \mathfrak{N}$ and for each $w \in W_{\mathfrak{M}}$, $a_1, \cdots, a_n \in D_{\mathfrak{M}}$ and each formula $\phi(x_1, \cdots, x_n)$,

$$\mathfrak{M}, w \models \phi(x_1, \cdots, x_n) \iff \mathfrak{N}, w \models \phi(x_1, \cdots, x_n)$$

Outline

Bisimulation

In this section we focus on **varying domain models**.

Let $D^\star$ be the set of all finite sequences over D. We use the abbreviations $w\bar{a}$ for $(w, a_1 \cdots a_n)$ and $\phi(\bar{x})$ for $\phi(x_1, \cdots, x_n)$.

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Definition

Let $\mathfrak{M}$ and $\mathfrak{N}$ be two Kripke models. The relation $Z \subseteq (W_{\mathfrak{M}} \times D^\star_{\mathfrak{M}}) \times (W_{\mathfrak{N}} \times D^\star_{\mathfrak{N}})$ is a **bisimulation** if for each $((w, \bar{a}), (v, \bar{b})) \in Z$ with $|\bar{a}| = |\bar{b}|$ the following three conditions hold:

1. for each atomic formulas $\phi(\bar{x})$, we have $\mathfrak{M}, w \vDash \phi(\bar{a})$ if and only if $\mathfrak{N}, v \vDash \phi(\bar{b})$.
2. ($\Diamond$-forth) If $wR_{\mathfrak{M}}w'$, then there is $v' \in W_{\mathfrak{N}}$ such that$vR_{\mathfrak{N}}v'$ and $(w', \bar{a})Z(v', \bar{b})$.
   ($\Diamond$-back) If $vR_{\mathfrak{N}}v'$, then there is $w' \in W_{\mathfrak{M}}$ such that $wR_{\mathfrak{M}}w'$ and $(w', \bar{a})Z(v', \bar{b})$.
3. ($\exists$-forth) If $c \in D_{\mathfrak{M}}(w)$, then there is $d \in D_{\mathfrak{N}}(v)$ with $(w, \bar{a}c)Z(v, \bar{b}d)$.
   ($\exists$-back) If $d \in D_{\mathfrak{N}}(v)$, then there is $c \in D_{\mathfrak{M}}(w)$ with $(w, \bar{a}c)Z(v, \bar{b}d)$.

$w\bar{a}$ and $v\bar{b}$ **bisimilar**, denoted by $(\mathfrak{M}, w\bar{a}) \rightleftarrows (\mathfrak{N}, v\bar{b})$, if there is a bisimulation Z between $\mathfrak{M}$ and $\mathfrak{N}$ such that $(w, \bar{a})Z(v, \bar{b})$.
$(w, v) \in Z$ means $(w\lambda, v\lambda) \in Z$ where λ is the empty sequence in $D^\star_{\mathfrak{M}}$ and $D^\star_{\mathfrak{N}}$.

Proposition

Suppose $(\mathfrak{M}, w\bar{a}) \rightleftarrows (\mathfrak{N}, v\bar{b})$*, then for any formula* $\phi(\bar{x})$ *with* $|\bar{x}| = |\bar{a}| = |\bar{b}|$*,*
$\mathfrak{M}, w \vDash \phi(\bar{a}) \iff \mathfrak{N}, v \vDash \phi(\bar{b})$*.*

Proposition

Suppose $(\mathfrak{M}, w\bar{a}) \rightleftarrows (\mathfrak{N}, v\bar{b})$, *then for any formula* $\phi(\bar{x})$ *with* $|\bar{x}| = |\bar{a}| = |\bar{b}|$, $\mathfrak{M}, w \vDash \phi(\bar{a}) \iff \mathfrak{N}, v \vDash \phi(\bar{b})$.

Proof.

By induction on $\phi(\bar{x})$.

- The case $\phi(\bar{x})$ is atomic formula is imediate.
- $\phi(\bar{x}) = \neg\psi(\bar{x})$. $\mathfrak{M}, w \vDash \neg\psi(\bar{a}) \iff \mathfrak{M}, w \nvDash \psi(\bar{a}) \overset{IH}{\iff} \mathfrak{N}, v \nvDash \psi(\bar{b}) \iff \mathfrak{N}, v \vDash \neg\psi(\bar{b})$.
- $\phi(\bar{x}) = (\phi_1 \wedge \phi_2)(\bar{x})$. $\mathfrak{M}, w \vDash (\phi_1 \wedge \phi_2)(\bar{a}) \iff \mathfrak{M}, w \vDash \phi_1(\bar{a}) \wedge \phi_2(\bar{a}) \iff \mathfrak{M}, w \vDash \phi_1(\bar{a})$ and $\mathfrak{M}, w \vDash \phi_2(\bar{a}) \overset{IH}{\iff} \mathfrak{N}, v \vDash \phi_1(\bar{b})$ and $\mathfrak{N}, v \vDash \phi_2(\bar{b}) \iff \mathfrak{N}, v \vDash \phi_1(\bar{b}) \wedge \phi_2(\bar{b}) \iff \mathfrak{N}, v \vDash (\phi_1 \wedge \phi_2)(\bar{b})$.
- $\phi(\bar{x}) = \exists_y\psi(\bar{x})(y)$.
  $\mathfrak{M}, w \vDash \exists_y\psi(\bar{a})(y) \implies$ there is a $c \in D_{\mathfrak{M}}(w)$, such that $\mathfrak{M}, w \vDash \psi(\bar{a}c) \overset{\exists\text{-forth}}{\implies}$ there is $d \in D_{\mathfrak{N}}(v)$ and $(w, \bar{a}c)Z(v, \bar{b}d) \overset{IH}{\implies} \mathfrak{N}, v \vDash \psi(\bar{b}d) \implies$ there is a $d \in D_{\mathfrak{N}}(v)$, such that $\mathfrak{N}, v \vDash \psi(\bar{b}d) \implies \mathfrak{N}, v \vDash \exists_y\psi(\bar{b})(y)$.
  The other direction can be done using $\exists$-back condition and induction hypothesis.
- $\phi(\bar{x}) = \Diamond\psi(\bar{x})$. $\mathfrak{M}, w \vDash \Diamond\psi(\bar{a}) \implies$ for some $w' \in W_{\mathfrak{M}}$ with wRw', $\mathfrak{M}, w' \vDash \psi(\bar{a}) \overset{\Diamond\text{-forth}}{\implies}$ there is $v' \in W_{\mathfrak{N}}$ such that $vRv' \overset{IH}{\implies} \mathfrak{N}, v' \vDash_\sigma \psi(\bar{b}) \implies \mathfrak{N}, v \vDash \Diamond\psi(\bar{b})$. The other direction can be done using $\exists$-back condition and induction hypothesis.

The notation $(\mathfrak{M}, w\bar{a}) \equiv (\mathfrak{N}, v\bar{b})$ means that for every formula $\phi(\bar{x})$ with at most n free variables among $\bar{x}$, we have $\mathfrak{M}, w \vDash \phi(\bar{a})$ if and only if $\mathfrak{N}, v \vDash \phi(\bar{b})$.

Definition

Two pointed Kripke models $(\mathfrak{M}, w)$ and $(\mathfrak{N}, v)$ are **elementary equivalent**, denoted by $(\mathfrak{M}, w) \equiv (\mathfrak{N}, v)$, if for every **sentence** ϕ, we have $\mathfrak{M}, w \vDash \phi$ if and only if $\mathfrak{N}, v \vDash \phi$.

Translation

First-order modal logic can be viewed as a **fragment** of two-sorted first order logic. For a FOML language τ, the corresponding two-sorted language τ^{cor}'s sorts are s_W and s_O. s_W corresponds to the set of **possible worlds** and s_O the set of **objects**.

There are two extra binary predicates $R(u, u')$ and $E(u, x)$. $R(u, u')$ means u is related to u' via the accessibility relation R and $E(u, x)$ means the object x is in the domain $D(u)$.

τ^{cor} contains all constant symbols of τ and for any n-ary predicate $p(x_1, \cdots, x_n) \in \tau$, τ^{cor} includes an $(n+1)$-ary predicate $P(u, x_1, \cdots, x_n)$.

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Definition

A first-order modal τ-formula can be inductively translated to a τ^{cor}-formula as follows:

- $ST_u(t_1 = t_2) = t_1 = t_2$.
- $ST_u(p(x_1, \cdots, x_n)) = P(u, x_1, \cdots, x_n)$.
- $ST_u(\neg\phi) = \neg ST_u(\phi)$.
- $ST_u(\phi \wedge \psi) = ST_u(\phi) \wedge ST_u(\psi)$.
- $ST_u(\Diamond\phi) = \exists u'(R(u, u') \wedge ST_{u'}(\phi))$.
- $ST_u(\exists x\phi(x)) = \exists x(E(u, x) \wedge ST_u(\phi))$.

Let $\mathfrak{M}^*$ be a τ^{cor}-structure of a first-order Kripke model $\mathfrak{M}$.

Proposition

Let $\phi(x_1, \cdots, x_n)$ be a first-order modal formula. Then for all Kripke models $\mathfrak{M}$ with $w \in W$ and $a_1 \cdots a_n \in D$,

$$\mathfrak{M}, w \vDash \phi(a_1, \cdots, a_n) \iff \mathfrak{M}^* \vDash ST_u(\phi)(w, a_1, \cdots, a_n).$$

Proof.

By induction on the complexity of $\phi(x_1, \cdots, x_n)$.

- Atomic formula, negation and conjunction cases are imediate.
- $\phi(x_1, \cdots, x_n) = \exists_y \psi(y)(x_1, \cdots, x_n)$. $\mathfrak{M}, w \vDash \exists_y \psi(y)(a_1, \cdots, a_n) \iff$ there is a $b \in D(w)$ and $\mathfrak{M}, w \vDash \psi(b)(a_1, \cdots, a_n) \overset{IH}{\iff}$ there is a $b \in D(w)$ and $\mathfrak{M}^* \vDash ST_u(\psi(b))(w, a_1, \cdots, a_n) \iff \mathfrak{M}^* \vDash \exists_y (E(u, y) \wedge ST_u(\psi(b)))(w, a_1, \cdots, a_n) \iff \mathfrak{M}^* \vDash ST_u(\exists_y \psi(y))(w, a_1, \cdots, a_n)$.
- $\phi(x_1, \cdots, x_n) = \Diamond \psi(x_1, \cdots, x_n)$. $\mathfrak{M}, w \vDash \Diamond \psi(a_1, \cdots, a_n) \iff$ there is a $w' \in W_{\mathfrak{M}}$, such that wRw' and $\mathfrak{M}, w' \vDash \psi(a_1, \cdots, a_n) \overset{IH}{\iff}$ there is a $w' \in W_{\mathfrak{M}}$, such that wRw' and $\mathfrak{M}^* \vDash ST_u(\psi)(w', a_1, \cdots, a_n) \iff \mathfrak{M}^* \vDash \exists u'(R(u, u') \wedge ST_{u'}(\psi))(w, a_1, \cdots, a_n) \iff \mathfrak{M}^* \vDash ST_u(\Diamond \psi)(w, a_1, \cdots, a_n)$

□

Modally-saturated model

For a varying domain Kripke model $\mathfrak{M} = (W, R, D, I)$ and a finite subset $A \subseteq D$, the language τ_A is an expansion of τ by adding a new constant c_a for every $a \in A$. The τ_A-Kripke model $\mathfrak{M}_A$ expands $\mathfrak{M}$ naturally by interpreting any constant c_a as a itself.

Definition

Let $\Gamma(\bar{x})$ be a set of τ_A-formula whose free variable are among $\bar{x} = x_1, \cdots, x_n$. The set of formulas $\Gamma(\bar{x})$ is an $\exists$-**type** if $(\mathfrak{M}_A, w)$ if for all finite subsets $\Gamma_0(\bar{x})$ of $\Gamma(\bar{x})$, we have $\mathfrak{M}_A, w \vDash \exists \bar{x} \bigwedge \Gamma_0(\bar{x})$. Similarly, $\Gamma(\bar{x})$ is a $\Diamond$-**type** of $(\mathfrak{M}_A, w)$ with respect to some $\bar{a} = a_1, \cdots, a_n \in D$, if $\mathfrak{M}_A, w \vDash \Diamond \bigwedge \Gamma_0(\bar{a})$ for all finite $\Gamma_0 \subseteq \Gamma$.
A type of $(\mathfrak{M}, w)$ is either a $\Diamond$-*type* or an $\exists$-*type* of $(\mathfrak{M}_A, w)$ for some finite subset $A \subseteq D$.

Modally-saturated model

Definition

An $\exists$-*type* $\Gamma(\bar{x})$ of $(\mathfrak{M}_A, w)$ is **realized** in $(\mathfrak{M}, w)$ if there are $a_1, \cdots, a_n \in D(w)$ such that $\mathfrak{M}_A, w \vDash \Gamma(\bar{a})$. Likewise, a $\Diamond$-*type* $\Gamma(\bar{x})$ with respect to a $\bar{a}$ is realized in $(\mathfrak{M}, w)$, if there is an element $w' \in W$ such that wRw' and $\mathfrak{M}_A, w \vDash \Gamma(\bar{a})$.

A model $\mathfrak{M}$ is **modally-saturated**, or m-saturated for short, if for every $w \in W$ and each finite subset A of D, every type of $(\mathfrak{M}_A, w)$ is realized in $(\mathfrak{M}, w)$.

Analogy of the Hennessy-Milner Theorem

Theorem

Let $\mathfrak{M}$ and $\mathfrak{N}$ be two m-saturated Kripke models. Then $(\mathfrak{M}, w\bar{a}) \equiv (\mathfrak{N}, v\bar{b})$ if and only if $(\mathfrak{M}, w\bar{a}) \rightleftarrows (\mathfrak{N}, v\bar{b})$.

Proof.

The right to left direction has been shown in previous proposition. Atomic formula case is trivial.
Show that $Z = \{(w\bar{a}, v\bar{b}) \mid (\mathfrak{M}, w\bar{a}) \equiv (\mathfrak{N}, v\bar{b}_0)\}$ is a bisimulation between $\mathfrak{M}$ and $\mathfrak{N}$ with $w_0\bar{a}_0 Z v_0\bar{b}_0$.

- *$\Diamond$-forth . Assume that $(w, \bar{a}), (w', \bar{a}') \in (W_{\mathfrak{M}} \times D^{\star}_{\mathfrak{M}})$ such that $wR_{\mathfrak{M}}w'$ and $(\mathfrak{M}, w\bar{a}) \equiv (\mathfrak{N}, v\bar{b}_0)$. Let $\Gamma(\bar{x})$ be the set of formulas true at $(\mathfrak{M}, w'\bar{a})$.*

  $\mathfrak{M}, w'\bar{a} \vDash \Gamma(\bar{x}) \Longrightarrow$ *every finite subset* $\Gamma_0(\bar{x})$ *of* $\Gamma(\bar{x}), \mathfrak{M}, w' \vDash \wedge\Gamma_0(\bar{a}) \Longrightarrow \mathfrak{M}, w \vDash \Diamond \wedge \Gamma_0(\bar{a}) \overset{\equiv}{\Longrightarrow} \mathfrak{N}, v \vDash \Diamond \wedge \Gamma_0(\bar{b}) \overset{\text{Let } B=\{b_1,\cdots,b_n\}}{\Longrightarrow} \mathfrak{N}_B, v \vDash \Diamond \wedge \Gamma_0(\bar{c}_b) \Longrightarrow$
  $\Gamma(\bar{x})$ *is a* $\Diamond$*-type of* $(\mathfrak{N}_B, v)$ *w.r.t.* $\bar{c}_b \in D_{\mathfrak{N}}(v) \overset{\mathfrak{N} \text{ is m-saturated, thus } \Gamma(\bar{x}) \text{ is realized}}{\Longrightarrow}$
  there is a $v' \in W_{\mathfrak{N}}$ *s.t.* $\mathfrak{N}_B, v' \vDash \Gamma(\bar{c}_b) \Longrightarrow \mathfrak{N}, v' \vDash \Gamma(\bar{b}) \Longrightarrow \mathfrak{N}, v'\bar{b} \vDash \Gamma(\bar{x})$.
- *$\Diamond$-back case is similar.*

cont.

- $\exists$-forth . *Assume that there is* $c \in D_{\mathfrak{M}}(w)$. *Let* $\Gamma(\bar{x}y)$ *be the set of formulas true at* $(\mathfrak{M}, w\bar{a}c)$, *then*

  $\mathfrak{M}, w\bar{a}c \models \Gamma(\bar{x}y) \Longrightarrow$ *every finite subset* $\Gamma_0(\bar{x}y) \subseteq \Gamma(\bar{x}y)$, $\mathfrak{M}, w\bar{a}c \models \wedge\Gamma_0(\bar{x}y) \Longrightarrow \mathfrak{M}, w\bar{a} \models \wedge\Gamma_0(\bar{x}c) \Longrightarrow \mathfrak{M}, w\bar{a} \models \exists_y \wedge \Gamma_0(\bar{x}y) \Longrightarrow \mathfrak{M}, w \models \exists_y \wedge \Gamma_0(\bar{a}y) \stackrel{\equiv}{\Longrightarrow} \mathfrak{N}, v \models \exists_y \wedge \Gamma_0(\bar{b}y) \xrightarrow{\text{Let } B=\{b_1,\cdots,b_n\}} \mathfrak{N}_B, v \models \exists_y \wedge \Gamma_0(\bar{c}_b y) \Longrightarrow \Gamma(\bar{c}_b y)$ *is an* $\exists$-*type of* $\mathfrak{N}_B, v \xrightarrow{\mathfrak{N} \text{ is m-saturated, thus } \Gamma(\bar{c}_b y) \text{ is realized}}$ *there is a* $d \in D_{\mathfrak{N}}(v)$ *s.t.* $\mathfrak{N}_B, v \models \Gamma(\bar{c}_b d) \Longrightarrow \mathfrak{N}, v \models \Gamma(\bar{b}d) \Longrightarrow \mathfrak{N}, v\bar{b}d \models \Gamma(\bar{x}y)$

□

Outline

Theorem (Goldblatt-Thomason Theorem for PML)

Let K be an elementary class of frames. Then K is definable by a set of propositional modal formulas if and only if

- *(GT1) it is closed under taking bounded morphic images, generated subframes, disjoint unions.*
- *(GT2) reflects ultrafilter extensions.*

In this section we focus on **constant domain models**.

A τ-formula $\phi(x_1, \cdots, x_n)$ is **valid at a world** w **of the frame** $\mathfrak{F}$, denoted by $\mathfrak{F}, w \models \phi(x_1, \cdots, x_n)$, if for every constant domain τ-model $\mathfrak{M}$ based on $\mathfrak{F}$, we have $\mathfrak{M}, w \models \forall x_1 \cdots \forall x_n \phi(x_1, \cdots, x_n)$.

$\phi(x_1, \cdots, x_n)$ is **valid on the frame** $\mathfrak{F}$, denoted by $\mathfrak{F} \models \phi(x_1, \cdots, x_n)$, if it is valid at every world $w \in W$.

Definition (FOML-*definable*)

A class of Kripke frames K is FOML-definable if for some language τ there exists a set of first-order modal τ-**sentences** Λ such that for any frame $\mathfrak{F}$, $\mathfrak{F} \in K$ if and only if $\mathfrak{F} \vDash \Lambda$.

More generally, a set of first-order modal **formulas** Λ **with free variables** among $x_1, \cdots, x_n$ defines K if for any Kripke frame $\mathfrak{F}$, $\mathfrak{F} \in K$ if and only if $\mathfrak{F} \vDash \forall \bar{x} \phi(\bar{x})$ for every $\phi(\bar{x}) \in \Lambda$.

Let $\theta(p_1, \cdots, p_n)$ is a propositional modal formula and $P_1(x), \cdots, P_n(x)$ are atomic formulas where $1 \leq i \leq n$, is a unary predicate and x is a single variable. The **sbustitution** of $\theta(p_1, \cdots, p_n)$ is a first-order modal formula $\theta(P_1(x), \cdots, P_n(x))$ which is obtained by **uniformly replacing** each proposition p_i by an atomic formula $P_i(x)$ for some variable x.

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Proposition

Let K be any class of frames. If K is PML-definable then it is also FOML-definable.

Proof.

Suppose K is definable by a set of propositional modal formulas Λ_P. Let $\tau = \{P_1, P_2, \cdots, P_n, \cdots\}$ consist of countably many unary predicates and put Λ_F to be a set of all formulas of $\forall_x \theta(P_1(x), \cdots, P_n(x))$ for $\theta \in \Lambda_P$, then for every $\mathfrak{F}$,
$\mathfrak{F} \in K \iff \mathfrak{F} \vDash \theta(p_1, \cdots, p_n) \iff$ for every $\mathfrak{M}$ and $w, \mathfrak{M}, w \vDash \theta(p_1, \cdots, p_n) \iff$ for every $a \in D, \mathfrak{M}, w \vDash \theta(P_1(a), \cdots, P_n(a) \iff \mathfrak{F} \vDash \forall_x \theta(P_1(x), \cdots, P_n(x)$. □

An Example

Example

The class of frames in which every world has a reflexive accessible world ($\forall x \exists y(Rxy \wedge Ryy)$) is not definable by any set of propositional modal formulas [1], since it does not reflect ultrafilter extensions. However, this class is definable by the formula $\Diamond \forall x(\Box P(x) \rightarrow P(x))$.

This class of frames are not definable by any set of propositional modal formulas, since it **does not reflect ultrafilter extensions**.(Bluebook P142)[5]

Ultrafilter Extension

Definition (Filter)

A **filter** F over a set W is a subset of $\mathcal{P}(W)$ such that

- $W \in F$
- $X, Y \in F$ implies $X \cap Y \in F$
- $X \in F$ and $X \subseteq Y$ implies $Y \in F$

A **proper filter** is a filter such that $\emptyset \notin F$. An **ultrafilter** is a proper filter such that either $X \in F$ or $W \setminus X \in F$.

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Definition (Ultrafilter Extension)

For a Kripke frame $\mathfrak{F} = (W, R)$, the **ultrafilter extension** $\mathfrak{ue}\mathfrak{F}$ of $\mathfrak{F}$ is defined as the frame $(Uf(W), R^{ue})$. $Uf(W)$ is the set of ultrafilters over W and R^{ue} is a binary relation over $Uf(W)$. For all $u, u' \in Uf(W)$, $R^{ue}uu'$ if $m_\diamond(X) = \{w \in W \mid Rww' \text{ for some } w' \in X\} \in u$ for every $X \in u'$, where

The **principal ultrafilter** generated by w is the filter generated by the singleton set $\{w\}$: $\pi_w = \{X \subseteq W \mid w \in X\}$.
A class of frames K **reflects ultrafilter extensions** if $\mathfrak{ue}\mathfrak{F} \in K$ implies $\mathfrak{F} \in K$.

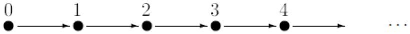


Figure: The Frame $\mathfrak{N} = (\mathbb{N}, <)$

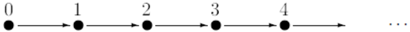
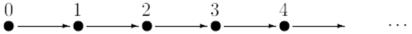


Figure: The Frame $\mathfrak{N} = (\mathbb{N}, <)$

There are two kinds of ultrafilters over an infinite set: the principal ultrafilters that are $1-1$ sorrespondence with the points of the set, and the non-principal ones that contain all co-finite sets, and only infinite sets.

For any pair of ultrafilters u, u', if u' is non-principal, then for any $X \in u'$, since X is infinite, then for any $n \in \mathbb{N}$, there is an $m \in X$ such that $n < m$. This shows that $m_\diamond(X) = \mathbb{N}$. Because $\mathbb{N}$ is an element of every ultrafilter. Therefore for any ultrafilter u, $m_\diamond(X) \in u$, this means $R^{ue}uu'$.



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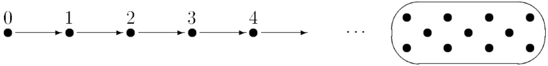

Figure: The Ultrafilter Extension of $\mathfrak{N}$

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$\Diamond \forall x(\Box P(x) \rightarrow P(x))$ is clearly valid in this class of frame.

For the converse, suppose that $\mathfrak{F}$ is a frame with some world w_0 that does not have any reflexive successor. Let $\mathfrak{M} = (\mathfrak{F}, D, I)$ be a **constant domain model** based on $\mathfrak{F}$ such that $|D| \geq |\{w' | Rw_0 w'\}|$. Moreover, for each w' there exists a distinct element $d_{w'} \in D$ such that $P(d_{w'})$ **is false in** w' **but it is true in all successors of** w'. Then for any w' with $Rw_0 w'$ we have $\mathfrak{M}, w' \not\models \forall_x(\Box P(x) \rightarrow P(x))$. So $\mathfrak{M}, w_0 \not\models \Diamond \forall_x(\Box P(x) \rightarrow P(x))$.

Bounded Morphic Images, Generated Subframes and Disjoint Unions

Definition (Bounded Morphic Images)

Let $\mathfrak{F}$ and $\mathfrak{G}$ be two Kripke frames. A function $f : W_{\mathfrak{F}} \to W_{\mathfrak{G}}$ is a bounded morphism from $\mathfrak{F}$ to $\mathfrak{G}$ if

- $R_{\mathfrak{F}} ww'$ implies $R_{\mathfrak{G}} f(w) f(w')$
- if $R_{\mathfrak{G}} f(w) v$, then there exists $w' \in W\mathfrak{F}$ such that $f(w') = v$ and $R_{\mathfrak{F}} ww'$.

If f is a *surjective*, then we say $\mathfrak{G}$ is a *bounded morphic image* of $\mathfrak{F}$, $\mathfrak{F} \twoheadrightarrow \mathfrak{G}$.

Definition (Generated Subframe)

A frame $\mathfrak{F}$ is a *generated subframe* of $\mathfrak{G}$, denoted by $\mathfrak{F} \rightarrowtail \mathfrak{G}$, if $W_{\mathfrak{F}} \subseteq W_{\mathfrak{G}}$ and $R_{\mathfrak{F}} = R_{\mathfrak{G}} \upharpoonright W_{\mathfrak{F}}$ and for every $w \in W_{\mathfrak{F}}$ if $R_{\mathfrak{G}} ww'$ then $w' \in W_{\mathfrak{F}}$.

Definition (Disjoint Union)

Suppose $\langle \mathfrak{F}_i : i \in I \rangle$ is a family of disjoint Kripke frames. The *disjoint union* of $\mathfrak{F}_i$'s is a frame $\mathfrak{F} = \biguplus_{i \in I} \mathfrak{F}_i$ in which $W = \bigcup_{i \in I} W_i$ and $R = \bigcup_{i \in I} R_i$.

Proposition

*The validity of **first-order modal sentences** is preserved under bounded morphic images, generated subframes, and disjoint unions.*

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*The validity of **first-order modal sentences** is preserved under bounded morphic images, generated subframes, and disjoint unions.*

Proof.

The proof is a generalization of the proof of the theorem for the propositional version.[1] Here, we take bounded morphic images case as an example.

- Assume that $\mathfrak{F} \twoheadrightarrow \mathfrak{F}'$. If $\mathfrak{F} \vDash \phi$, then $\mathfrak{F}' \vDash \phi$

Assume that f is a surjective bounded morphism form $\mathfrak{F}$ onto $\mathfrak{F}'$. For every first-order modal sentence ϕ, suppose that $\mathfrak{F}' \nvDash \phi$.Then there is a model $\mathfrak{M}' = (W', R', D', I')$ based on $\mathfrak{F}' = (W', R')$ and a $w' \in W'$ s.t. $\mathfrak{M}', w' \nvDash \phi$.
Let $\mathfrak{M} = (W, R, D, I)$ be a model based on $\mathfrak{F} = (W, R)$, where for every $w \in W$, $D(w) = D'(f(w))$, for every $w \in W$ and every constant c, $I(w, c) = I'(f(w), c)$, and for every n-ary predicate P, $I(w, P) = \{(a_1, \cdots, a_n) \in D^n \mid (a_1, \cdots, a_n) \in I'(f(w), P)\}$. Since f is surjective, then there is a $w \in W$ s.t. $f(w) = w'$. By induction on sentence ϕ, we have $\mathfrak{M}', w' \nvDash \phi \iff \mathfrak{M}, w \nvDash \phi$.

- $\phi = c_1 = c_2$. $\mathfrak{M}', w' \nvDash c_1 = c_2 \iff I'(w', c_1) \neq I'(w', c_2) \iff I'(f(w), c_1) \neq I'(f(w), c_2) \iff I(w, c_1) \neq I(w, c_2) \iff \mathfrak{M}, w \nvDash c_1 = c_2$.
- $\phi = P(c_1, \cdots, c_n)$. $\mathfrak{M}', w' \nvDash P(c_1, \cdots, c_n) \iff (I'(w', c_1), \cdots, I'(w', c_n)) \notin I'(w', P) \iff (I'(f(w), c_1), \cdots, I'(f(w), c_n)) \notin I'(f(w), P) \iff (I(w, c_1), \cdots, I(w, c_n)) \notin I(w, P) \iff \mathfrak{M}, w \nvDash P(c_1, \cdots, c_n)$.

□

cont.

- $\phi = \Diamond\psi$.
  $\mathfrak{M}, w \vDash \Diamond\psi \iff$ there is a $v \in W$, wRv and $\mathfrak{M}, v \vDash \psi \overset{\text{definition of bounded morphism}}{\iff}$ there is a $f(v) \in W'$, $f(w)R'f(v)$ and $\mathfrak{M}, v \vDash \psi \overset{IH}{\iff}$ there is a $f(v) \in W'$, $f(w)R'f(v)$ and $\mathfrak{M}', f(v) \vDash \psi \iff \mathfrak{M}', f(w) \vDash \Diamond\psi \iff \mathfrak{M}', w' \vDash \Diamond\psi$.
- $\phi = \exists_y \phi(y)$. $\mathfrak{M}, w \vDash \exists_y \phi(y) \iff$ there is a $a \in D(w)$ s.t. $\mathfrak{M}, w \vDash \phi(a) \overset{IH}{\iff}$ there is a $a \in D'(f(w))$ s.t. $\mathfrak{M}', w' \vDash \phi(a) \iff$ there is a $a \in D'(w')$ s.t. $\mathfrak{M}', w' \vDash \phi(a) \iff \mathfrak{M}', w' \vDash \exists_y \phi(y)$

Hence, $\mathfrak{F} \nvDash \phi$. □

Definition (Universal)

A first-order modal formula $\phi(x_1, \cdots, x_n)$ is universal if it is in the form $\forall y_1, \cdots, \forall y_m \psi(x_1, \cdots, x_n, y_1, \cdots, y_m)$ where ψ is a quantifier-free formula.

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Proposition

Let K be an elementary class of frames definable by a set of universal modal sentences in some language τ. Then K is definable by a set of propositional modal formulas.

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Let K be an elementary class of frames definable by a set of universal modal sentences in some language τ. Then K is definable by a set of propositional modal formulas.

Proof.

Suppose that the class K is defined by a set of universal modal sentences Λ ($\mathfrak{F} \in K \iff \mathfrak{F} \models \Lambda$for every $\mathfrak{F}$). By the previous proposition K has property $GT1$. We show K **reflects ultrafilter extensions**, and hence by Goldblatt-Thomason theorem we get the desired result.

Assume that $\mathfrak{ue}\mathfrak{F} \in K$ for some $\mathfrak{F}$. To show $\mathfrak{F} \models \Lambda$, suppose $\mathfrak{F} \not\models \Lambda$, then there is a universal sentence $\phi = \forall \bar{x} \psi(\bar{x}) \in \Lambda$ and a modal $\mathfrak{M}$ based on $\mathfrak{F}$ and a $w \in W_{\mathfrak{F}}$ such that $\mathfrak{M}, w \not\models \phi$. Let $\mathfrak{ue}\mathfrak{F}$ be a model based on $\mathfrak{ue}\mathfrak{F}$ **with $D_{\mathfrak{M}}$ as domain**. For each $u \in Uf(W)$ and each n-ary predicate P, $(a_1, \cdots, a_n) \in I^{\mathfrak{ue}}(u, P)$ **if and only if** $\{w \in W_{\mathfrak{F}} \mid (a_1, \cdots, a_n) \in I^{\mathfrak{M}}(w, P)\} \in u$. By induction on the quantifier-free formula ψ, we have $\mathfrak{M}, w \models \psi(a_1, \cdots, a_n) \iff \mathfrak{ue}(\mathfrak{M}), \pi_w \models \psi(a_1, \cdots, a_n)$.
Hence $\mathfrak{ue}(\mathfrak{M}), \pi_w \not\models \phi$, a contradiction. □

Goldblatt-Thomason Theorem for FOML

Theorem (Goldblatt-Thomason Theorem for FOML)

Let K be an elementary class of frames. Then K is definable by a set of first-order modal τ-sentences if and only if it is closed under bounded morphic images, generated subframes, and disjoint unions.[5]

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Let K be an elementary class of frames. Then K is definable by a set of first-order modal τ-sentences if and only if it is closed under bounded morphic images, generated subframes, and disjoint unions.[5]

Proof.

Let ϕ be first-order modal τ sentence and $Th(K) = \{\phi \mid K \vDash \phi\}$. We need to show that for any frame $\mathfrak{F} \vDash Th(K) \Longrightarrow \mathfrak{F} \in K$.
Assume that $\mathfrak{F} \vDash Th(K)$ where $\mathfrak{F} = (W_{\mathfrak{F}}, R_{\mathfrak{F}})$. By GT1, since any frame is a bounded morphic image of the disjoint union of its pointed-generated subframs, we assume that $\mathfrak{F}$ is point-generated by w_0.
Let τ_1 be a language consisting of a unary predicate P, a binary predicate R and new constants c_w, for each $w \in W_{\mathfrak{F}}$. Put

$$\tau' = \tau_1 \cup \{R_\theta \mid \theta(\bar{x}) \text{ is an } \mathcal{L}_R \text{ - } \textit{formula}\}$$

where for each first-order $\mathcal{L}_R$ - formula $\theta(x_1, \cdots, x_k)$ we add a k-ary predicate $R_\theta(x_1, \cdots, x_k)$.

cont.

Define a τ_1-model $\mathfrak{M} = (\mathfrak{F}, D, I)$ based on $\mathfrak{F}$ where $D = W\mathfrak{F}$, in each world w, $I(w, P) = \{w\}$ and $R(w, w')$ holds if and only if $(w, w') \in R_{\mathfrak{F}}$. Interpret each constant c_w as w. τ_1 model $\mathfrak{M}$ can be viewed as a τ' model by interpreting each predicate R_θ as

$$I(w, R_\theta) = \{(w_1, \cdots, w_k) \in D^k \mid \mathfrak{M}, w \vDash \theta(w_1, \cdots, w_k)\}$$

By induction on $\mathcal{L}_R$-formulas we have for each $\mathcal{L}_R$-formula $\theta(x_1, \cdots, x_n)$ and $w, w' \in W$, $I(w, R_\theta) = I(w', R_\theta)$. Moreover

$$I(w, R_\theta) = \{(w_1, \cdots, w_k) \in D^k \mid \mathfrak{M} \vDash \theta(w_1, \cdots, w_k)\}$$

Let Δ be the set of τ'-sentences true in $(\mathfrak{M}, w_0)$. Since $\mathfrak{F}$ is pointe-generated by w_0, for each $w \in W$ the length of the shortest path from w_0 to w,n_w is finite. For all $w, w' \in W$, $n \in \mathbb{N}$, and $\mathcal{L}_R$-formulas $\theta(\bar{x})$,$\theta_1(\bar{x})$,$\theta_2(\bar{x})$. Δ contains, in particular, the following sentences.

cont.

1. $\Diamond^{n_w} P(c_w)$.
2. $\Box^n c_w \neq c_{w'}$, if $w \neq w'$.
3. $\Box^n \exists_x (P(x) \wedge \forall_y (P(y) \rightarrow x = y))$.
4. $\Box^n R(c_w, w_{w'})$, if $(w, w') \in R_{\mathfrak{F}}$.
5. $\Box^n \neg R(c_w, c_{w'})$, if $(w, w') \notin R_{\mathfrak{F}}$.
6. $\Box^n \forall_x \forall_y (P(x) \wedge \Diamond P(y) \rightarrow R(x, y))$.
7. $\Box^n \forall_x \forall_y (P(x) \wedge R(x, y) \rightarrow \Diamond P(y))$.
8. $\forall_{x_1}, \cdots, x_k (\theta(x_1, \cdots, x_k) \rightarrow \Box^n \theta(x_1, \cdots, x_k))$
9. $\forall_{x_1}, \cdots, x_k (\Diamond^n \theta(x_1, \cdots, x_k) \rightarrow \theta(x_1, \cdots, x_k))$
10. $\Box^n \forall_x \forall_y (R_{R(x,y)}(x, y) \leftrightarrow R(x, y))$.
11. $\Box^n \forall_x, \cdots, x_k (R_{\neg\theta}(x_1, \cdots, x_k) \leftrightarrow \neg R_\theta(x_1, \cdots, x_k))$.
12. $\Box^n \forall_x, \cdots, x_k (R_{\theta_1 \wedge \theta_2}(x_1, \cdots, x_k) \leftrightarrow (R_{\theta_1}(x_1, \cdots, x_k) \wedge R_{\theta_2}(x_1, \cdots c, x_k)))$.
13. $\Box^n \forall_x, \cdots, x_k (R_{\exists_y \theta(y, x_1, \cdots, x_k)}(x_1, \cdots, x_k) \leftrightarrow \exists_y R_{\theta_{(y, x_1, \cdots, x_k)}}(y, x_1, \cdots, x_k))$.
14. $\Box^n \forall_x, \cdots, x_k (R_\theta(x_1, \cdots, x_k) \leftrightarrow \theta(x_1, \cdots, x_k))$.

1-7 describe the properties of predicates P and R in $\mathfrak{M}$. 8, equivalent to 9, expresses the validity of each $\mathcal{L}$-formula is equivalent to its satisfiability inside $\mathfrak{M}$. 10-13 state the inductive interpretation of the auxiliary predicates in $\mathfrak{M}$. 14 is the conjunction of 10-13.

cont.

Claim 1. *Δ is finitely satisfiable in K.*

For if not, there is a finite subset δ of Δ which is not satisfiable in any constant domain model based on the frames of K, then $\neg \bigwedge \sigma \in Th(K)$, hence $\mathfrak{F} \vDash \neg \bigwedge \sigma$ which contradicts $\mathfrak{M}, w_0 \vDash \bigwedge \sigma$.
Since K is an elementary class of frames, it is closed under ultraproducts. Hence , there is a frame $\mathfrak{G} \in K$, a constant domain model $\mathfrak{N} = (\mathfrak{G}, D_{\mathfrak{N}}, I_{\mathfrak{N}})$ based on $\mathfrak{G}$ and a point v_0 such that $\mathfrak{N}, v_0 \vDash \Delta$. Once again, w.l.o.g, we assume that $\mathfrak{G}$ is generated by v_0.
Let $\mathfrak{G}' = (W', R')$ where $W' = \{a \in D_{\mathfrak{N}} \mid \mathfrak{N}, v \vDash P(a)$, for some $v \in \mathfrak{G}\}$, and $(a, b) \in R'$ if and only if $\mathfrak{N} \vDash R(a, b)$.

Claim 2. $\mathfrak{G} \twoheadrightarrow \mathfrak{G}'$

Define $h : \mathfrak{G} \to \mathfrak{G}'$ by letting $h(v)$ be the unique element a_v of $D_{\mathfrak{N}}$ such that $P(a_{a_v})$ holds in $v \in \mathfrak{G}$. h is a well-defined surjective function since $\mathfrak{G}$ is generated by v_0 and it satisfies 3. By 6-9, h is a bounded morphism.

cont.

Claim 3. *$\mathfrak{F}$ is isomorphic to a first-order elementary $\mathcal{L}_{\mathcal{R}}$-substurcture of $\mathfrak{G}'$.*

Let $i : \mathfrak{F} \to \mathfrak{G}'$ be defined as $i(w) = c_w^{\mathfrak{N}}$, which is well defined by 1. By 2,4 and 5 the map i is an $\mathcal{L}_{\mathcal{R}}$-embedding, since for any $w, w' \in W_{\mathfrak{F}}$,
$\mathfrak{F} \vDash wR_{\mathfrak{F}}w' \iff \mathfrak{G}' \vDash i(w)Ri(w')$
*To show it is the desired elementary embedding, we **expand $\mathfrak{F}$ and $\mathfrak{G}'$ to first-order τ'-structures and show i is a τ'-embedding**.*
First, interpret each constant c_w as w in $\mathfrak{F}$ and as $a = c_w^{\mathfrak{N}}$ in $\mathfrak{G}'$. Interpret each $R_\theta \in \tau'$ in $\mathfrak{F}$ as

$$(w_1, \cdots, w_k) \in R_\theta^{\mathfrak{F}} \text{ if } \mathfrak{M} \vDash R_\theta(w_1, \cdots, w_k)$$

for $w_1, \cdots, w_k \in \mathfrak{F}$, and as

$$(a_1, \cdots, a_k) \in R_\theta^{\mathfrak{G}'} \text{ if } \mathfrak{N} \vDash R_\theta(a_1, \cdots, a_k)$$

, for $a_1, \cdots, a_k \in \mathfrak{G}'$.

For each $\mathcal{L}_{\mathcal{R}}$-formula $\theta_{(y,x_1,\cdots,x_k)}$ and constant symbols $c_{w_1}, \cdots, c_{w_k}$, both $\mathfrak{F}$ and $\mathfrak{G}'$ satisfy the formula $(R_\theta(c_{w_1}, \cdots, c_{w_k}) \leftrightarrow \theta(c_{w_1}, \cdots, c_{w_k}))$.

The argument goes by induction on $\theta_{(y,x_1,\cdots,x_k)}$. For sturcture $\mathfrak{G}'$, $\wedge$ and $\neg$ cases are easy. For $\theta(x_1, \cdots, x_k) = \exists_y \psi(y, x_1, \cdots, x_k)$.

cont.

We first show that $\mathfrak{G}' \models R_{\exists_y \psi}(c_{w_1}, \cdots, c_{w_k}) \iff \mathfrak{G}' \models \exists_y R_\psi(y, c_{w_1}, \cdots, c_{w_n})$.

$\Longrightarrow$: $\mathfrak{G}' \models R_{\exists y \psi}(c_{w_1}, \cdots, c_{w_k}) \overset{by\ definition}{\Longrightarrow} \mathfrak{N} \models R_{\exists_y \psi}(c_{w_1}, \cdots, c_{w_k}) \overset{13}{\Longrightarrow} \mathfrak{N} \models \exists_y R_\psi(y, c_{w_1}, \cdots, c_{w_k}) \overset{(\mathfrak{M}, w_0) \equiv (\mathfrak{N}, v_0)}{\Longrightarrow} \mathfrak{M} \models \exists_y R_\psi(y, c_{w_1}, \cdots, c_{w_k}) \Longrightarrow$ *there exists* $w \in \mathfrak{F}$, *s.t.* $\mathfrak{M} \models R_\psi(c_w, c_{w_1}, \cdots, c_{w_k}) \Longrightarrow \mathfrak{N} \models R_\psi(c_w, c_{w_1}, \cdots, c_{w_k}) \overset{by\ definition}{\Longrightarrow} \mathfrak{G}' \models R_\psi(c_w, c_{w_1}, \cdots, c_{w_k}) \Longrightarrow \mathfrak{G}' \models \exists_y R_\psi(y, c_{w_1}, \cdots, c_{w_k})$.

$\Longleftarrow$: $\mathfrak{G}' \models \exists_y R_\psi(y, c_{w_1}, \cdots, c_{w_k}) \Longrightarrow \mathfrak{G}' \models R_\psi(a, c_{w_1}, \cdots, c_{w_k})$ *for some* $a \in \mathfrak{G}' \Longrightarrow \mathfrak{N} \models R_\psi(a, c_{w_1}, \cdots, c_{w_k}) \Longrightarrow \mathfrak{N} \models \exists_y R_\psi(y, c_{w_1}, \cdots, c_{w_k}) \overset{13}{\Longrightarrow} \mathfrak{N} \models R_{\exists_y \psi}(c_{w_1}, \cdots, c_{w_k}) \Longrightarrow \mathfrak{G}' \models R_{\exists_y \psi}(c_{w_1}, \cdots, c_{w_k})$.

The case for structure $\mathfrak{F}$ *can be done analogously.*

cont.

The inductive definitions of R_θ's imply that both structures $\mathfrak{F}$ and $\mathfrak{G}'$ satisfy $(R_\theta(c_{w_1}, \cdots, c_{w_k}) \leftrightarrow \theta(c_{w_1}, \cdots, c_{w_k}))$. Since

$$\mathfrak{F} \models R_\theta(c_{w_1}, \cdots, c_{w_k}) \text{ if and only if } \mathfrak{M} \models R_\theta(c_{w_1}, \cdots, c_{w_k})$$

$$\mathfrak{G}' \models R_\theta(c_{w_1}, \cdots, c_{w_k}) \text{ if and only if } \mathfrak{N} \models R_\theta(c_{w_1}, \cdots, c_{w_k})$$

therefore

$$\mathfrak{F} \models R_\theta(c_{w_1}, \cdots, c_{w_k}) \text{ if and only if } \mathfrak{G}' \models R_\theta(c_{w_1}, \cdots, c_{w_k})$$

$$\mathfrak{F} \models \theta(c_{w_1}, \cdots, c_{w_k}) \text{ if and only if } \mathfrak{G}' \models \theta(c_{w_1}, \cdots, c_{w_k})$$

Hence, i is an elementary $\mathcal{L}_R$-embedding. By closure of K under bounded morphic images, $\mathfrak{G}' \in K$. Scine K is an elementary class of frames, Claim 3 implies that $\mathfrak{F} \in K$. □

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