Gödel's First Incompleteness Theorems for Non-Recursively Enumerable Theories

Conden Chao

With some further work on "S. Salehi and P. Seraji 2015"

北京大学哲学系
PEKING UNIVERSITY PHILOSOPHY DEPARTMENT

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To Commemorate the
110 Anniversary of Gödel's Birth
1906—2016

Outline

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1. Introduction: Gödel's First Incompleteness Theorem
2. Generalized Meta-theoretical Properties
3. Generalizing to Non-Recursively Enumerable Theories
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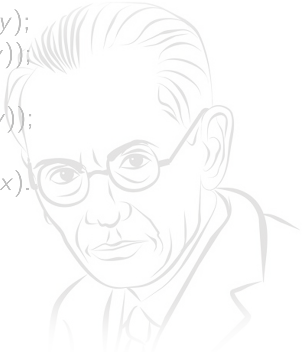
8. Conclusions: Diagrams for First Incompleteness

Introduction: Gödel's First Incompleteness Theorem

- Arithmetic language $\mathscr{L}_A$: non-logical symbols are $\bar{0}$, $\bar{S}$, $\bar{+}$, $\bar{\times}$ and equality symbol is $\equiv$.
- $\bar{n} = \bar{S}^n\bar{0} = \bar{S}\cdots\bar{S}\bar{0}$. ($n$ many $\bar{S}$)
- $x\bar{\leq}y$ is defined as $\exists z(z\bar{+}x\equiv y)$ and $x\bar{<}y$ is $x\bar{\leq}y \wedge x\not\equiv y$.
- The standard arithmetic model is $\mathcal{N} = (\mathbb{N}, 0, 1, \cdots, S, +, \times, \leq)$.
- Robinson arithmetic is the theory Q whose axioms are as follows

$$
\begin{aligned}
Q_1 &: \forall x \bar{S}x \not\equiv \bar{0};\\
Q_2 &: \forall x\forall y(\bar{S}x \equiv \bar{S}y \to x \equiv y);\\
Q_3 &: \forall x(x \not\equiv \bar{0} \to \exists y(x \equiv \bar{S}y));\\
Q_4 &: \forall x(x \bar{+} \bar{0} \equiv x);\\
Q_5 &: \forall x\forall y(x \bar{+} \bar{S}y \equiv \bar{S}(x \bar{+} y));\\
Q_6 &: \forall x(x \bar{\times} \bar{0} \equiv \bar{0});\\
Q_7 &: \forall x\forall y(x \bar{\times} \bar{S}y \equiv x \bar{\times} y \bar{+} x).
\end{aligned}
$$



Introduction: Gödel's First Incompleteness Theorem

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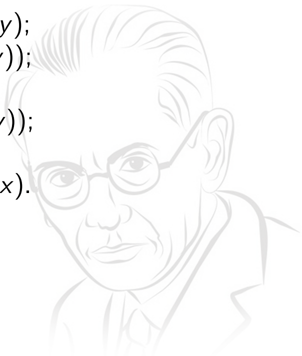
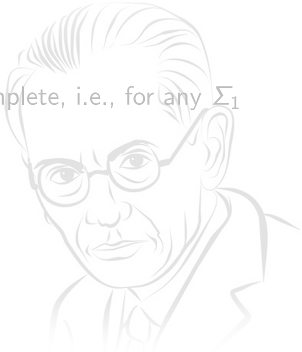


Introduction: Gödel's First Incompleteness Theorem

- (Σ_n, Π_n and Δ_n) Fix our arithmetic language $\mathscr{L}_A$ (notably that $\bar{n}$ ($n > 0$) and $\bar{\leq}$ are not non-logical symbols of it). The formulas $\Delta_0 = \Sigma_0 = \Pi_0$ is defined as follows:
  - all the atomic formulas such as $\tau \bar{\equiv} \sigma$, where τ, σ are terms, belong to Δ_0;
  - if $\phi, \psi \in \Delta_0$, then so $\neg\phi, \phi \wedge \psi, \phi \vee \psi \in \Delta_0$;
  - if τ is a term with $x \notin \mathrm{Vr}(\tau)$, and $\phi \in \Delta_0$, then so $\forall x \bar{\leq} \tau \phi, \exists x \bar{\leq} \tau \phi \in \Delta_0$.

  And recursively we can define Σ_n, Π_n and Δ_n sets of formulas:
  - $\phi \in \Sigma_n$ if $\phi = \exists \vec{x} \psi$ for some $\psi \in \Pi_{n-1}$;
  - $\phi \in \Pi_n$ if $\phi = \forall \vec{x} \psi$ for some $\psi \in \Sigma_{n-1}$;
  - $\phi \in \Delta_n$ if $\phi \in \Sigma_n \cap \Pi_n$.
- For all $n \in \mathbb{N}$, $\Sigma_n \cup \Pi_n \subseteq \Sigma_{n+1} \cap \Pi_{n+1}$.
- (Σ_1-completeness of Q) If $T \supseteq$ Q, then T is Σ_1-complete, i.e., for any Σ_1 sentence ϕ if $\mathcal{N} \vDash \phi$ then $T \vdash \phi$.



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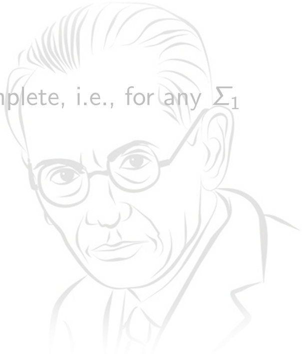

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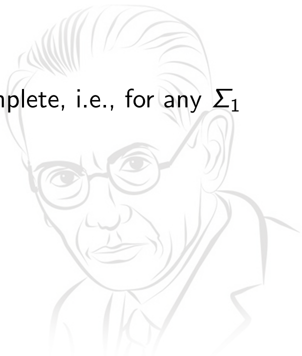

Introduction: Gödel's First Incompleteness Theorem

- A k-ary predicate $P \subseteq \mathbb{N}^k$ is representable in T if, there is a formula $\phi(\vec{x})$ such that for any $n_0, \cdots, n_{k-1} \in \mathbb{N}$

$$(n_0, \cdots, n_{k-1}) \in P \Longrightarrow T \vdash \phi(\overline{n_0}, \cdots, \overline{n_{k-1}}),$$
$$(n_0, \cdots, n_{k-1}) \notin P \Longrightarrow T \vdash \neg\phi(\overline{n_0}, \cdots, \overline{n_{k-1}}).$$

- A function $f : \mathbb{N}^k \to \mathbb{N}$ is representable in $T \supseteq \mathrm{Q}$ if, there is a formula $\phi(\vec{x}, y)$ such that for any $n_0, \cdots, n_{k-1} \in \mathbb{N}$

$$T \vdash \forall y[\phi(\overline{n_0}, \cdots, \overline{n_{k-1}}, y) \leftrightarrow y \equiv \overline{f(n_0, \cdots, n_{k-1})}].$$

- (Representability Theorem) Any recursive function (and hence every recursive predicate) is representable in $T \supseteq \mathrm{Q}$ and Δ_1.
- If T is recursively axiomatizable then proof and provability are arithmetized as a binary predicate $\mathrm{Be}_T(m, n)$ and a uary predicate $\mathrm{Beb}_T(n)$ respectively.
- And so if T is recursively axiomatizable, by Representability theorem proof and provability can be expressed by formulas

$$\mathrm{be}_T(x, y) \in \Delta_1,$$
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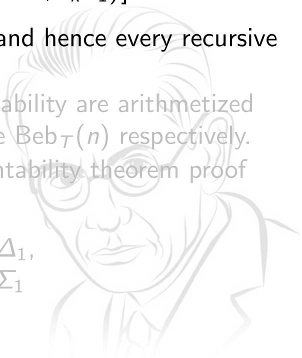

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Introduction: Gödel's First Incompleteness Theorem

- The natural number $\sharp\phi$ is the Gödel's code of ϕ and $\ulcorner\phi\urcorner = \overline{\sharp\phi} = \bar{S}^{\sharp\phi}\bar{0}$ is the term corresponding to the natural number $\sharp\phi$.
- (Fixed Point Lemma) Given any $\mathscr{L}_A$ formula $\phi(x)$ with $\mathrm{Fr}(\phi) = \{x\}$ and a theory $T \supseteq \mathrm{Q}$, we can effectively find a γ such that $T \vdash \gamma \leftrightarrow \phi(\ulcorner\gamma\urcorner)$.
  Proof. Suppose $x_0, x_1, y \neq x$ and $\psi(x_0, y, x_1)$ represents sub in T. For any $\delta(x)$ and any $n \in \mathbb{N}$, $T \vdash \psi(\ulcorner\delta\urcorner, y, \bar{n}) \leftrightarrow y \equiv \ulcorner\delta(\bar{n})\urcorner$. Setting $n = \sharp\delta$,

$$T \vdash \psi(\ulcorner\delta\urcorner, y, \ulcorner\delta\urcorner) \leftrightarrow y \equiv \ulcorner\delta(\ulcorner\delta\urcorner)\urcorner. \tag{1}$$

  Let $\theta(x) = \forall y(\psi(x, y, x) \to \phi(x; y))$. It's enough to show $\gamma = \theta(\ulcorner\theta\urcorner)$ is the desired fixed point of $\phi(x)$: in T we have

$$\begin{array}{lll} \gamma & = & \theta(\ulcorner\theta\urcorner) \\ & \leftrightarrow & \forall y(\psi(\ulcorner\theta\urcorner, y, \ulcorner\theta\urcorner) \to \phi(x; y)) \quad \text{substitute } \ulcorner\theta\urcorner \text{ for } x \text{ in } \theta(x) \\ & \leftrightarrow & \forall y(y \equiv \ulcorner\theta(\ulcorner\theta\urcorner)\urcorner \to \phi(x; y)) \quad \text{by (1) and } \delta = \theta(x) \\ & = & \forall y(y \equiv \ulcorner\gamma\urcorner \to \phi(x; y)) \quad \text{by } \gamma = \theta(\ulcorner\theta\urcorner) \\ & \leftrightarrow & \phi(\ulcorner\gamma\urcorner). \end{array}$$

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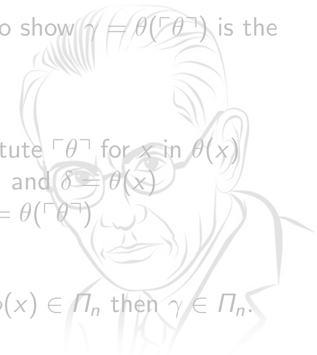

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Introduction: Gödel's First Incompleteness Theorem

Definition 1.1

T is ω-consistent if, there is no ϕ with $\phi = \exists x \psi(x)$ such that $T \vdash \exists x \psi(x)$ and $T \vdash \neg\psi(\overline{m})$ for all $m \in \mathbb{N}$.

Theorem 1.2 (Gödel's First Incompleteness)

Let $T \supseteq \mathrm{Q}$ be a recursively axiomatizable theory. If T is ω-consistent, then there is a Π_1 sentence γ such that $T \nvdash \gamma$ and $T \nvdash \neg\gamma$.

Proof.

Let γ be the fixed point of $\neg\mathrm{beb}(y)$. Then

$$T \vdash \gamma \leftrightarrow \neg\mathrm{beb}(\ulcorner\gamma\urcorner). \tag{2}$$

γ is as desired: (a) If $T \vdash \gamma$, then $\mathcal{N} \vDash \mathrm{beb}(\ulcorner\gamma\urcorner)$, and $T \vdash \mathrm{beb}(\ulcorner\gamma\urcorner)$ by Σ_1-completeness. But by (2) we have $T \vdash \neg\mathrm{beb}(\ulcorner\gamma\urcorner)$, a contradiction. So $T \nvdash \gamma$; (b) If $T \vdash \neg\gamma$, then by (2) we have $T \vdash \mathrm{beb}(\ulcorner\gamma\urcorner)$. Since $T \nvdash \gamma$, then for any $n \in \mathbb{N}$ we have $\neg\,\mathrm{Be}(n, \sharp\gamma)$, and by representability $T \vdash \neg\mathrm{be}(\bar{n}, \ulcorner\gamma\urcorner)$ for any $n \in \mathbb{N}$. By the ω-consistency of T, $T \nvdash \exists x \mathrm{be}(x, \ulcorner\gamma\urcorner)$, i.e., $T \nvdash \mathrm{beb}(\ulcorner\gamma\urcorner)$, a contradiction. So $T \nvdash \neg\gamma$. ❒

Introduction: Gödel's First Incompleteness Theorem

- The condition that T is recursively axiomatizable allows us to use a Σ_1 formula to express T in $\mathscr{L}_A$, and so we may write it as $\mathrm{Axiom}_T \in \Sigma_1$.
- ω-consistency was weakened by G. Kreisel as 1-consistency: there is no $\phi \in \Sigma_1$ with $\phi = \exists x \psi(x)$ for some $\psi(x) \in \Pi_0$ such that $T \vdash \exists x \psi(x)$ and $T \vdash \neg\psi(\overline{m})$ for all $m \in \mathbb{N}$.
- The conclusion could be written as T isn't Π_1-deciding (T is Π_1-deciding if for any $\phi \in \Pi_1$ either $T \vdash \phi$ or $T \vdash \neg\phi$).

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Corollary 1.3

1. *If* $\mathrm{Q} \subseteq T$ *and* $\mathrm{Axiom}_T \in \Sigma_1$ *and* T *is* 1*-consistent, then* T *isn't* Π_1*-deciding.*
2. *If* $\mathrm{Q} \subseteq T$ *and* $\mathrm{Axiom}_T \in \Sigma_1$ *and* T *is* Σ_1*-sound, then* T *isn't* Π_1*-deciding.*

Proof.

(2) Σ_1-soundness (T is Σ_1-sound if, for any $\phi \in \Sigma_1$ with $T \vdash \phi$ we have $\mathcal{N} \vDash \phi$) is stronger than 1-consistency. ❐

Introduction: Gödel's First Incompleteness Theorem

Theorem 1.4 (Gödel-Rosser's First Incompleteness)

Let $T \supseteq \mathsf{Q}$ be a recursively axiomatizable theory. If T is consistent, then there is a Π_1 sentence γ such that $T \nvdash \gamma$ and $T \nvdash \neg\gamma$.

Introduction: Gödel's First Incompleteness Theorem

Theorem 1.4 (Gödel-Rosser's First Incompleteness)

Let $T \supseteq \mathrm{Q}$ be a recursively axiomatizable theory. If T is consistent, then there is a Π_1 sentence γ such that $T \nvdash \gamma$ and $T \nvdash \neg\gamma$.

Similarly, since Σ_0-soundness is equivalent to consistency whence $\mathrm{Q} \subseteq T$, Gödel-Rosser's First Incompleteness theorem could be written as

Theorem 1.5

If $\mathrm{Q} \subseteq T$ and $\mathrm{Axiom}_T \in \Sigma_1$ and T is Σ_0-sound, then T isn't Π_1-deciding.

Corollary 1.6

1. *If $\mathrm{Q} \subseteq T$ and $\mathrm{Axiom}_T \in \Sigma_1$ and T is Π_1-sound, then T isn't Π_1-deciding.*
2. *If $\mathrm{Q} \subseteq T$ and $\mathrm{Axiom}_T \in \Sigma_1$ and $T \subseteq \mathrm{Th}(\mathcal{N})$, then T isn't Π_1-deciding.*

Outline

Generalized Meta-theoretical Properties

Notation 2.1

Let T be a theory and Γ a set of sentences, then

$$\Gamma(\mathcal{N}) = \{\phi \in \Gamma \mid \mathcal{N} \models \phi\}$$

where $\mathcal{N}$ is the standard arithmetic model. And so $\Sigma_n(\mathcal{N})$ and $\Pi_n(\mathcal{N})$ denotes Σ_n sentences and Π_n sentences respectively true in $\mathcal{N}$.

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Lemma 2.2 (Cf. Corollary 1.76 of [6])

1. *$\Sigma_n(\mathcal{N})$ is defined by a Σ_n formula Σ_n-$True(x)$.*
2. *$\Pi_n(\mathcal{N})$ is defined by a Π_n formula Π_n-$True(x)$.*

Remark 2.3

$\phi \in \Sigma_n(\mathcal{N})$ iff $\mathcal{N} \vDash \Sigma_n$-True$(\ulcorner\phi\urcorner)$ and $\phi \in \Pi_n(\mathcal{N})$ iff $\mathcal{N} \vDash \Pi_n$-True$(\ulcorner\phi\urcorner)$.

Generalized Meta-theoretical Properties

Definition 2.4 (Γ-consistency)

Let T be a theory and Γ a set of sentences, then T is Γ-consistent with if $T + \Gamma$ is consistent.

Generalized Meta-theoretical Properties

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Let T be a theory and Γ a set of sentences, then T is Γ-consistent with if $T + \Gamma$ is consistent.

- We will survey the relations between $\Sigma_n(\mathcal{N})$- and $\Pi_n(\mathcal{N})$-consistency later.

Generalized Meta-theoretical Properties

Definition 2.5 (Γ-deciding)

Let T be a theory and Γ a set of sentences, then T is Γ-deciding if, for any $\phi \in \Gamma$ either $T \vdash \phi$ or $T \vdash \neg\phi$; otherwise T isn't Γ-deciding.

Generalized Meta-theoretical Properties

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Lemma 2.6

1. *Σ_{n+1}-deciding implies Π_n-deciding and Σ_n-deciding;*
2. *Π_{n+1}-deciding implies Σ_n-deciding and Π_n-deciding;*
3. *Σ_n-deciding is equivalent to Π_n-deciding*
4. *Syntactic completeness implies Σ_n-deciding and Π_n-deciding.*

Generalized Meta-theoretical Properties

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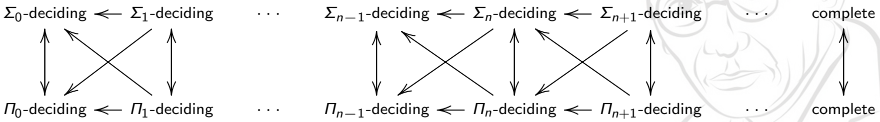

Generalized Meta-theoretical Properties

Definition 2.7 (n-consistency)

Let T be a theory and Γ a set of sentences.

- T is ω-cosistent if, there is no ϕ with $\phi = \exists x \psi(x)$ such that $T \vdash \exists x \psi(x)$ and $T \vdash \neg\psi(\overline{m})$ for all $m \in \mathbb{N}$; otherwise T is ω-inconsistent.
- T is n-cosistent if, there is no $\phi \in \Sigma_n$ with $\phi = \exists x \psi(x)$ for some $\psi(x) \in \Pi_{n-1}$ such that $T \vdash \exists x \psi(x)$ and $T \vdash \neg\psi(\overline{m})$ for all $m \in \mathbb{N}$; otherwise T is n-inconsistent.

Generalized Meta-theoretical Properties

Definition 2.7 (*n*-consistency)

Let T be a theory and Γ a set of sentences.

- ☛ T is ω-cosistent if, there is no ϕ with $\phi = \exists x\psi(x)$ such that $T \vdash \exists x\psi(x)$ and $T \vdash \neg\psi(\overline{m})$ for all $m \in \mathbb{N}$; otherwise T is ω-inconsistent.
- ☛ T is n-cosistent if, there is no $\phi \in \Sigma_n$ with $\phi = \exists x\psi(x)$ for some $\psi(x) \in \Pi_{n-1}$ such that $T \vdash \exists x\psi(x)$ and $T \vdash \neg\psi(\overline{m})$ for all $m \in \mathbb{N}$; otherwise T is n-inconsistent.

Lemma 2.8

1. *n-consistency implies consistency;*
2. $(n+1)$*-consistency implies n-consistency.*
3. ω*-consistency implies n-consistency and consistency.*

Generalized Meta-theoretical Properties

Definition 2.9 (Γ-soundness with respect to $\mathcal{N}$)

Let T be a theory and Γ a set of sentences.

- T is sound (with respect to $\mathcal{N}$) if, for any ϕ with $T \vdash \phi$ we have $\mathcal{N} \vDash \phi$; otherwise T isn't sound.
- T is Γ-sound (with respect to $\mathcal{N}$) if, for any $\phi \in \Gamma$ with $T \vdash \phi$ we have $\mathcal{N} \vDash \phi$; otherwise T isn't Γ-sound.

Generalized Meta-theoretical Properties

Lemma 2.10

1. *Σ_{n+1}-soundness implies Σ_n-soundness and Π_n-soundness;*
2. *Π_{n+1}-soundness implies Π_n-soundness and Σ_n-soundness;*
3. *Σ_n-soundness implies Π_{n+1}-soundness, and hence Π_n-soundness;*
4. *Soundness implies Σ_n-soundness and Π_n-soundness.*

Generalized Meta-theoretical Properties

Lemma 2.10

1. *Σ_{n+1}-soundness implies Σ_n-soundness and Π_n-soundness;*
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Proof.

(3). Let $\phi \in \Pi_{n+1}$ be such that $\phi = \forall x \theta(x)$ for some $\theta \in \Sigma_n$ and $T \vdash \forall x \theta(x)$. Then $T \vdash \theta(\overline{m})$ for all $m \in \mathbb{N}$, and by Σ_n-soundness $\mathcal{N} \models \theta(\overline{m})$ for all $m \in \mathbb{N}$. Hence $\mathcal{N} \models \forall x \theta(x)$. ❒

Generalized Meta-theoretical Properties

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1. *Σ_{n+1}-soundness implies Σ_n-soundness and Π_n-soundness;*
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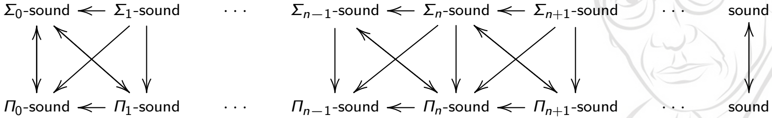

Generalized Meta-theoretical Properties

Definition 2.11 (completeness, and Γ-completeness with respect to $\mathcal{N}$)

Let T be a theory and Γ a set of sentences.

- ☛ T is (syntactically) complete if, for any ϕ either $T \vdash \phi$ or $T \vdash \neg\phi$; otherwise T isn't complete.
- ☛ T is (semantically) complete (with respect to $\mathcal{N}$) if, for any ϕ with $\mathcal{N} \vDash \phi$ we have $T \vdash \phi$; otherwise T isn't complete.
- ☛ T is (semantically) Γ-complete (with respect to $\mathcal{N}$) if, for any $\phi \in \Gamma$ with $\mathcal{N} \vDash \phi$ we have $T \vdash \phi$; otherwise T isn't Γ-complete.

Generalized Meta-theoretical Properties

Lemma 2.12

1. *Σ_{n+1}-completeness implies Σ_n-completeness and Π_n-completeness;*
2. *Π_{n+1}-completeness implies Π_n-completeness and Σ_n-completeness;*
3. *Σ_n-completeness doesn't imply Π_n-completeness;*
4. *Π_n-completeness implies Σ_{n+1}-completeness, and hence Σ_n-completeness;*
5. *Semantical completeness implies Σ_n-completeness and Π_n-completeness.*

Generalized Meta-theoretical Properties

Lemma 2.12

1. *Σ_{n+1}-completeness implies Σ_n-completeness and Π_n-completeness;*
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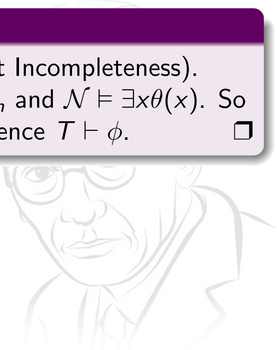
3. *Σ_n-completeness doesn't imply Π_n-completeness;*
4. *Π_n-completeness implies Σ_{n+1}-completeness, and hence Σ_n-completeness;*
5. *Semantical completeness implies Σ_n-completeness and Π_n-completeness.*

Proof.

(3) Q is Σ_1-complete but not Π_1-complete (by Gödel's First Incompleteness).
(4) Let $\phi \in \Sigma_{n+1}$ be such that $\phi = \exists x \theta(x)$ for some $\theta \in \Pi_n$ and $\mathcal{N} \vDash \exists x \theta(x)$. So $\mathcal{N} \vDash \theta(\overline{m})$ for some $m \in \mathbb{N}$. By Π_n-complete $T \vdash \theta(\overline{m})$. Hence $T \vdash \phi$. ❐



Generalized Meta-theoretical Properties

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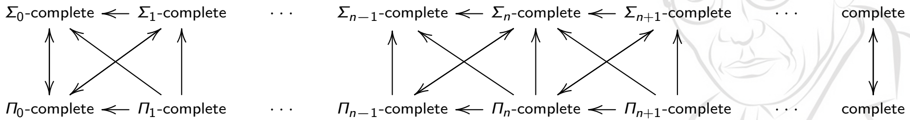


Generalized Meta-theoretical Properties

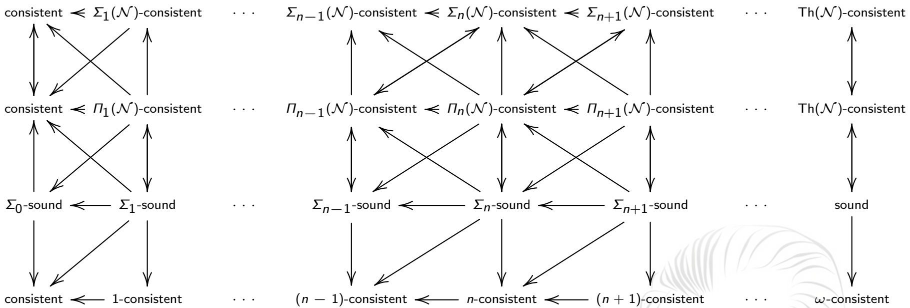
Lemma 2.13

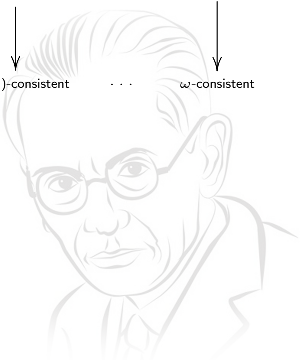
1. *Soundness is equivalent to* $\mathrm{Th}(\mathcal{N})$*-consistency, and* $\mathcal{N} \vDash T$*;*
2. Π_n*-soundness is equivalent to* $\Sigma_n(\mathcal{N})$*-consistency for all* $n \in \mathbb{N}$*;*
3. Σ_n*-soundness is equivalent to* $\Pi_n(\mathcal{N})$*-consistency for all* $n \in \mathbb{N}$*;*
4. Σ_n*-soundness implies n-consistency for all* $n \in \mathbb{N}$*;*
5. *n-consistency doesn't imply* Σ_n*-soundness for all* $n \geq 3$*;*
6. *n-consistency and* Σ_{n-1}*-completeness imply* Σ_n*-soundness for all* $n \in \mathbb{N}$*.*

And if $\mathrm{Q} \subseteq T$*, then*

7. Σ_2*-soundness is equivalent to 2-consistency;*
8. Σ_1*-soundness is equivalent to 1-consistency;*
9. Σ_0*-soundness is equivalent to consistency.*

Generalized Meta-theoretical Properties





Generalized Meta-theoretical Properties

Definition 2.14 (Γ-definable theories)

Let T be a theory and Γ a set of formulas.

- T is definable if there is some Ω of sentences axiomatizing T and some formula $\mathrm{Axiom}_T(x)$ such that

$$\Omega = \{\phi \mid \mathcal{N} \vDash \mathrm{Axiom}_T(\ulcorner \phi \urcorner) \text{ and } \phi \text{ is a sentence}\}.$$

- T is Γ-definable if there is some Ω of sentences axiomatizing T and some formula $\mathrm{Axiom}_T(x) \in \Gamma$ such that

$$\Omega = \{\phi \mid \mathcal{N} \vDash \mathrm{Axiom}_T(\ulcorner \phi \urcorner) \text{ and } \phi \text{ is a sentence}\}.$$

Generalized Meta-theoretical Properties

Lemma 2.15

1. *Σ_n-definability implies Σ_{n+1}- and Π_{n+1}-definability;*
2. *Π_n-definability implies Π_{n+1}- and Σ_{n+1}-definability;*
3. *Σ_{n+1}-definability implies Π_n-definability;*
4. *T is recursively enumerable iff T is Σ_0-definable iff T is Σ_1-definable.*

Generalized Meta-theoretical Properties

Lemma 2.15

1. *Σ_n-definability implies Σ_{n+1}- and Π_{n+1}-definability;*
2. *Π_n-definability implies Π_{n+1}- and Σ_{n+1}-definability;*
3. *Σ_{n+1}-definability implies Π_n-definability;*
4. *T is recursively enumerable iff T is Σ_0-definable iff T is Σ_1-definable.*

Proof I.

(3) Suppose T is axiomatized by Ω and $\text{Axiom}_T(x) = \exists x_1 \cdots \exists x_m \psi(x, x_1, \cdots, x_m)$ with $\psi \in \Pi_n$ defines $\sharp\Omega$. Then $\text{Axiom}_T(x)$ is equivalent to $\exists y \delta(x, y)$ with $\delta(x, y) = \exists x_1 \bar{\leq} y \cdots \exists x_m \bar{\leq} y \psi(x, x_1, \cdots, x_m) \in \Pi_n$. So

$$\Omega' = \{\phi \wedge (\bar{k} \bar{\equiv} \bar{k}) \mid \mathcal{N} \vDash \delta(\ulcorner \phi \urcorner, \bar{k}) \text{ and } \phi \in \Omega\}$$

also axiomatizes T. And it's easy to see that Ω' is defined by the Π_n formula

$$\text{Axiom}'_T(x) = \exists y \bar{\leq} x \exists z \bar{\leq} x [\delta(x, y) \wedge (x \bar{\equiv} \ulcorner \gamma_y \wedge (\gamma_z \bar{\equiv} \gamma_z) \urcorner)],$$

where γ_y is the formula by encoding y and γ_z is the term by encoding z.

Generalized Meta-theoretical Properties

Proof II.

(4) Clearly 'Σ_1-definability $\Longrightarrow$ Σ_0-definability' follows from (3) and 'Σ_0-definability $\Longrightarrow$ Recursive enumerability' is trivial. While 'Recursive enumerability $\Longrightarrow$ Σ_1-definability' follows from the following claim.

If T is recuresively enumerable then T is axiomatized by a recursive set.

Suppose T is axiomatized by a recursively enumerable set Ω. Then there is some effective algorithm enumerating Ω as $\phi_1, \phi_2, \cdots$. For any n, let

$$\psi_n =_{df} \underbrace{\phi_n \wedge (\phi_n \wedge \cdots))}_{n \text{ many}}.$$

and Ω' be the set of such ψ_n. Clearly T is axiomatized by the recursive Ω'. ❒

Proof II.

(4) Clearly 'Σ_1-definability $\Longrightarrow$ Σ_0-definability' follows from (3) and 'Σ_0-definability $\Longrightarrow$ Recursive enumerability' is trivial. While 'Recursive enumerability $\Longrightarrow$ Σ_1-definability' follows from the following claim.

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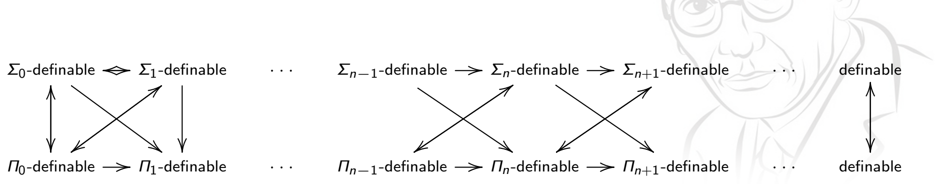

Outline

1. Introduction: Gödel's First Incompleteness Theorem
2. Generalized Meta-theoretical Properties
3. Generalizing to Non-Recursively Enumerable Theories
4. Σ_n-soundness is sufficient
5. Π_n-soundness is also sufficient
6. n-consistency is also sufficient
7. Consistency isn't sufficient
8. Conclusions: Diagrams for First Incompleteness

Generalizing to Non-Recursively Enumerable Theories

Notation 3.1

Suppose the set of axioms for T is defined by $\mathrm{Axiom}_T(x)$ and $\mathrm{Q} \subseteq T$. We define $\mathrm{be}(x, y)_T$ and $\mathrm{beb}_T(y)$ corresponding to concepts 'proof in T' and 'provable in T' respectively as:

$$\begin{aligned} \mathrm{be}_T(x, y) &=_{df} \quad \exists x_1 \cdots \exists x_k(\mathrm{Axiom}_T(x_1) \wedge \cdots \wedge \mathrm{Axiom}_T(x_k) \wedge \mathrm{be}_{\mathrm{Q}}(x, \ulcorner \chi_{x_1} \wedge \cdots \wedge \chi_{x_k} \rightarrow \chi_y \urcorner)), \\ \mathrm{beb}_T(y) &=_{df} \quad \exists x \exists x_1 \cdots \exists x_k(\mathrm{Axiom}_T(x_1) \wedge \cdots \wedge \mathrm{Axiom}_T(x_k) \wedge \mathrm{be}_{\mathrm{Q}}(x, \ulcorner \chi_{x_1} \wedge \cdots \wedge \chi_{x_k} \rightarrow \chi_y \urcorner)). \end{aligned}$$

where χ_x is the formula by encoding x.

Generalizing to Non-Recursively Enumerable Theories

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Suppose the set of axioms for T is defined by $\text{Axiom}_T(x)$ and $\mathsf{Q} \subseteq T$. We define $\text{be}(x, y)_T$ and $\text{beb}_T(y)$ corresponding to concepts 'proof in T' and 'provable in T' respectively as:

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where χ_x is the formula by encoding x.

Remark 3.2

- ☛ If $\text{Axiom}_T(x) \in \Sigma_n$, then $\text{beb}_T(y) \in \Sigma_n$ and $\neg\text{beb}_T(y) \in \Pi_n$.
- ☛ If $\text{Axiom}_T(x) \in \Pi_n$, then $\text{beb}_T(y) \in \Sigma_{n+1}$ and $\neg\text{beb}_T(y) \in \Pi_{n+1}$.

Generalizing to Non-Recursively Enumerable Theories

We generalize Corollary 1.6 (2) i.e., 'If $\mathsf{Q} \subseteq T$ and $\mathrm{Axiom}_T \in \Sigma_1$ and $T \subseteq \mathrm{Th}(\mathcal{N})$, then T isn't Π_1-deciding' to non-recursively enumerable (non-r.e.) theories:

Theorem 3.3

If $\mathsf{Q} \subseteq T$ *and* $\mathrm{Axiom}_T \in \Sigma_n$ *and* $T \subseteq \mathrm{Th}(\mathcal{N})$*, then* T *isn't* Π_n*-deciding.*

Generalizing to Non-Recursively Enumerable Theories

We generalize Corollary 1.6 (2) i.e., 'If $\mathsf{Q} \subseteq T$ and $\mathsf{Axiom}_T \in \Sigma_1$ and $T \subseteq \mathsf{Th}(\mathcal{N})$, then T isn't Π_1-deciding' to non-recursively enumerable (non-r.e.) theories:

Theorem 3.3

If $\mathsf{Q} \subseteq T$ and $\mathsf{Axiom}_T \in \Sigma_n$ and $T \subseteq \mathsf{Th}(\mathcal{N})$, then T isn't Π_n-deciding.

Proof.

Let γ be the fixed point of $\neg\mathsf{beb}_T(y)$

$$T \vdash \gamma \leftrightarrow \neg\mathsf{beb}_T(\ulcorner\gamma\urcorner). \tag{3}$$

Clearly γ could be Π_n, and it suffices to show γ is independent of T:

- $T \nvdash \gamma$. If $T \vdash \gamma$. Then $\mathcal{N} \vDash \mathsf{beb}(\ulcorner\gamma\urcorner)$ and $\mathcal{N} \vDash \gamma$. And since $\mathcal{N} \vDash \gamma \leftrightarrow \neg\mathsf{beb}_T$, then $\mathcal{N} \vDash \neg\mathsf{beb}(\ulcorner\gamma\urcorner)$, a contradiction.
- $T \nvdash \neg\gamma$. If $T \vdash \neg\gamma$. Then $\mathcal{N} \vDash \neg\gamma$. And since $\mathcal{N} \vDash \gamma \leftrightarrow \neg\mathsf{beb}_T$, then $\mathcal{N} \vDash \mathsf{beb}_T(\ulcorner\gamma\urcorner)$, and hence $T \vdash \gamma$, a contradiction to to $T \nvdash \gamma$.

We can also show that $\mathcal{N} \vDash \gamma$. ❒

Generalizing to Non-Recursively Enumerable Theories

Corollary 3.4

If Q $\subseteq T$ *and* $\text{Axiom}_T \in \Pi_n$ *and* $T \subseteq \text{Th}(\mathcal{N})$, *then* T *isn't* Π_{n+1}*-deciding.*

Proof.

This is because $\text{Axiom}_T \in \Pi_n \subseteq \Sigma_{n+1}$. ❒

Outline

Σ_n-soundness is sufficient

Theorem 4.1

If $\mathsf{Q} \subseteq T$ *and* $\mathsf{Axiom}_T \in \Pi_n$ *and* T *is* Σ_n*-sound, then* T *isn't* Π_{n+1}*-deciding.*

Σ_n-soundness is sufficient

Theorem 4.1

If $\mathrm{Q} \subseteq T$ *and* $\mathrm{Axiom}_T \in \Pi_n$ *and* T *is* Σ_n*-sound, then* T *isn't* Π_{n+1}*-deciding.*

Proof I.

Define

$$\mathrm{pro}_T(y) =_{df} \exists x[\mathrm{be}_T(x, y) \wedge \forall z \bar{<} x \neg \mathrm{be}_T(z, \overline{\neg}(y))].$$

Set $T^* = T + \Pi_n(\mathcal{N})$. Then T^* is Π_n-complete and Σ_{n+1}-complete, and consistent by Σ_n-soundness. One claim is needed.

Σ_n-soundness is sufficient

Theorem 4.1

If $\mathrm{Q} \subseteq T$ *and* $\mathrm{Axiom}_T \in \Pi_n$ *and* T *is* Σ_n*-sound, then* T *isn't* Π_{n+1}*-deciding.*

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Define

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Set $T^* = T + \Pi_n(\mathcal{N})$. Then T^* is Π_n-complete and Σ_{n+1}-complete, and consistent by Σ_n-soundness. One claim is needed.

Lemma 4.2

For all $n \in \mathbb{N}$, $\mathrm{Q} \vdash \forall x(x \,\bar{\leq}\, \bar{n} \leftrightarrow \bigvee_{q \leq n} x \,\bar{=}\, \bar{q})$ *and* $\mathrm{Q} \vdash \forall x(x \,\bar{\leq}\, \bar{n} \vee \bar{n} \,\bar{\leq}\, x)$.

Claim

1. *If* $T \vdash \delta$*, then* $T^* \vdash \mathrm{pro}_T(\ulcorner \delta \urcorner)$.
2. *If* $T \vdash \neg\delta$*, then* $T^* \vdash \neg\mathrm{pro}_T(\ulcorner \delta \urcorner)$.

Σ_n-soundness is sufficient

Proof II.

Let's turn to the theorem, and let γ be the fixed point of $\neg\text{pro}_T(y)$. Then

$$T \vdash \gamma \leftrightarrow \neg\text{pro}_T(\ulcorner\gamma\urcorner). \tag{4}$$

Clearly γ could be Π_{n+1}. It suffices to show that γ is independent of T: if $T \vdash \gamma$, then by the Claim 1 we have $T^* \vdash \text{pro}_T(\ulcorner\gamma\urcorner)$, but (4) gives us $T^* \vdash \neg\text{pro}_T(\ulcorner\gamma\urcorner)$, a contradiction to consistency of T^*, and so $T \nvdash \gamma$; if $T \vdash \neg\gamma$, then by the Claim 2 we have $T^* \vdash \neg\text{pro}_T(\ulcorner\gamma\urcorner)$, but (4) gives us $T^* \vdash \text{pro}_T(\ulcorner\gamma\urcorner)$, also a contradiction to consistency of T^*, and so $T \nvdash \neg\gamma$. ❐

Σ_n-soundness is sufficient

Corollary 4.3

1. *if* $\mathrm{Q} \subseteq T$ *and* $\mathrm{Axiom}_T \in \Sigma_n$ *and* T *is* Σ_{n-1}*-sound, then* T *isn't* Π_n*-deciding.*
2. *if* $\mathrm{Q} \subseteq T$ *and* $\mathrm{Axiom}_T \in \Sigma_n$ *and* T *is* Σ_n*-sound, then* T *isn't* Π_n*-deciding.*

Proof.

(1) By Lemma 2.15 (3), Axiom_T could also be Π_{n-1}, and then by Theorem 4.1 T isn't Π_n-deciding.
(2) By (1) and Σ_n-soundness implies Σ_{n-1}-soundness. ❐

Outline

Π_n-soundness is sufficient

Theorem 5.1

If Q $\subseteq T$ *and* $\text{Axiom}_T \in \Pi_n$ *and* T *is* Π_{n+1}*-sound, then* T *isn't* Π_{n+1}*-deciding.*

Proof.

This is because Π_{n+1}-soundness is equivalent to Σ_n-soundness. ❐

Π_n-soundness is sufficient

Theorem 5.1

If Q $\subseteq T$ *and* Axiom$_T \in \Pi_n$ *and* T *is* Π_{n+1}*-sound, then* T *isn't* Π_{n+1}*-deciding.*

Proof.

This is because Π_{n+1}-soundness is equivalent to Σ_n-soundness. ❐

Corollary 5.2

If Q $\subseteq T$ *and* Axiom$_T \in \Sigma_n$ *and* T *is* Π_n*-sound, then* T *isn't* Π_n*-deciding.*

Proof.

Since Axiom$_T \in \Sigma_n$ then Axiom$_T \in \Pi_{n-1}$, and then the conclusion suffices from Theorem 5.1. ❐

Outline

n-consistency is sufficient

Lemma 6.1

Th($\mathcal{N}$) *is the only complete and* ω*-consistent extension of* PA *(indeed* Q*).*

Lemma 6.2

If $\mathsf{Q} \subseteq T$ *and* T *is* Π_n*-deciding and* T *is* n*-consistent, then* T *is* Π_n*-complete.*

n-consistency is sufficient

Lemma 6.1

Th($\mathcal{N}$) *is the only complete and* ω*-consistent extension of* PA *(indeed* Q*).*

Lemma 6.2

If $\mathrm{Q} \subseteq T$ *and* T *is* Π_n*-deciding and* T *is n-consistent, then* T *is* Π_n*-complete.*

Proof I.

Suppose T isn't Π_n-complete, then there is some $\phi \in \Pi_n$ such that $\mathcal{N} \models \phi$ and $T \nvdash \phi$; by Π_n-decidability of T we have $T \vdash \neg\phi$, and so

$$\mathcal{N} \models \phi \text{ and } T \vdash \neg\phi \text{ and } \phi \in \Pi_n. \tag{5}$$

We may write $\phi = \forall x \exists y \psi(x, y)$ for some $\psi \in \Pi_{n-2}$. By $T \vdash \exists x \neg \exists y \psi(x, y)$ and the n-consistency of T we have $T \nvdash \exists y \psi(\bar{k}, y)$ for some $k \in \mathbb{N}$. Since T is Π_n-deciding then $T \vdash \forall y \neg \psi(\bar{k}, y)$. Since $\mathcal{N} \models \forall x \exists y \psi(x, y)$, then $\mathcal{N} \models \psi(\bar{k}, \bar{l})$ for some $l \in \mathbb{N}$, and clearly $T \vdash \neg\psi(\bar{k}, \bar{l})$. So for $\chi = \psi(\bar{k}, \bar{l})$ we have

$$\mathcal{N} \models \chi \text{ and } T \vdash \neg\chi \text{ and } \chi \in \Pi_{n-2}. \tag{6}$$

n-consistency is sufficient

Proof II.

Proceeding in this way (from n to $n-2$) we can show that there is some δ such that

$$\mathcal{N} \vDash \delta \text{ and } T \vdash \neg\delta \text{ and either } \delta \in \Pi_1 (n \text{ is odd}) \text{ or } \delta \in \Pi_0 (n \text{ is even}). \quad (7)$$

If $\delta \in \Pi_1$ then write $\delta = \forall x \theta(x)$ for some $\theta \in \Pi_0$. By $T \vdash \exists x \neg\theta(x)$ and the 1-consistency of T we have $T \nvdash \theta(\overline{m})$ for some $m \in \mathbb{N}$. Since T is Π_0-deciding then $T \vdash \neg\theta(\overline{m})$. And also we have $\mathcal{N} \vDash \forall x \theta(x)$, then $\mathcal{N} \vDash \theta(\overline{m})$. So for there is some γ (either δ in (7) or $\theta(\overline{m})$) such that

$$\mathcal{N} \vDash \gamma \text{ and } T \vdash \neg\gamma \text{ and } \gamma \in \Pi_0. \quad (8)$$

By Σ_1-completeness of $T \supseteq \mathrm{Q}$ and $\mathcal{N} \vDash \gamma$ we have $T \vdash \gamma$. Also we have $T \vdash \neg\gamma$, a contradiction to the consistency of T following from its n-consistency. ❒

n-consistency is sufficient

Theorem 6.3

If $\mathrm{Q} \subseteq T$ and $\mathrm{Axiom}_T \in \Pi_n$ and T is n-consistent, then T isn't Π_{n+1}-deciding.

Proof.

Let T satisfy the conditions in the theorem. If T isn't Π_n-deciding, then T isn't Π_{n+1}-deciding. So we suppose T is Π_n-deciding, then T is Π_n-complete by Lemma 6.2, and so $\Pi_n(\mathcal{N}) \subseteq T$, and so T is Σ_n-sound. Then T isn't Π_{n+1}-deciding by Theorem 4.1. ❐

n-consistency is sufficient

Theorem 6.3

If $\mathrm{Q} \subseteq T$ *and* $\mathrm{Axiom}_T \in \Pi_n$ *and* T *is n-consistent, then* T *isn't* Π_{n+1}*-deciding.*

Proof.

Let T satisfy the conditions in the theorem. If T isn't Π_n-deciding, then T isn't Π_{n+1}-deciding. So we suppose T is Π_n-deciding, then T is Π_n-complete by Lemma 6.2, and so $\Pi_n(\mathcal{N}) \subseteq T$, and so T is Σ_n-sound. Then T isn't Π_{n+1}-deciding by Theorem 4.1. ❐

- ☛ It is interesting to note that for $n > 3$ all the incompleteness proofs (presented as above) with the assumption of $\Sigma_n(\Pi_{n-1})$-soundness are constructive, while all the incompleteness proofs with the assumption of n-consistency are all non-constructive (i.e., the independent sentence is not constructed explicitly, and only its mere existence is proved).

n-consistency is sufficient

Corollary 6.4

1. *If* $Q \subseteq T$ *and* $\text{Axiom}_T \in \Sigma_n$ *and* T *is* $(n-1)$*-consistent, then* T *isn't* Π_n*-deciding.*
2. *If* $Q \subseteq T$ *and* $\text{Axiom}_T \in \Sigma_n$ *and* T *is* n*-consistent, then* T *isn't* Π_n*-deciding.*
3. *If* $Q \subseteq T$ *and* $\text{Axiom}_T \in \Sigma_n$ *and* T *is* ω*-consistent, then* T *isn't* Π_n*-deciding.*
4. *If* $Q \subseteq T$ *and* $\text{Axiom}_T \in \Pi_{n-1}$ *and* T *is* ω*-consistent, then* T *isn't* Π_n*-deciding.*

Proof.

(1) By Theorem 6.3 and Σ_n-definability is equivalent to Π_{n-1}-definability.
(2) By (1) and n-consistency implies $(n-1)$-consistency.
(3) By (2) and ω-consistency implies n-consistency.
(3) By (3) and Σ_n-definability is equivalent to Π_{n-1}-definability. ❐

Outline

1. Introduction: Gödel's First Incompleteness Theorem
2. Generalized Meta-theoretical Properties
3. Generalizing to Non-Recursively Enumerable Theories
4. Σ_n-soundness is sufficient
5. Π_n-soundness is also sufficient
6. n-consistency is also sufficient
7. **Consistency isn't sufficient**
8. Conclusions: Diagrams for First Incompleteness

Consistency isn't sufficient

Lemma 7.1

There is a complete (and consistent) theory T such that $\mathsf{Q} \subseteq T$ and T is Σ_{n+2}-definable and T is Σ_n-sound.

Consistency isn't sufficient

Lemma 7.1

There is a complete (and consistent) theory T such that $\mathrm{Q} \subseteq T$ and T is Σ_{n+2}-definable and T is Σ_n-sound.

Proof I.

Let $S = \mathrm{Q} + \Pi_n(\mathcal{N})$(clearly $S = \mathrm{Q} = \mathrm{Q} + \Pi_0(\mathcal{N})$ when $n = 0$). We get the completion of S in Lindenbaum's way: enumerate all the sentences as $\phi_0, \phi_1, \cdots$ and define

$$\begin{aligned} T_0 &= S; \\ T_{n+1} &= \begin{cases} T_n \cup \{\phi_n\} & T_n \cup \{\phi_n\} \text{ is consistent,} \\ T_n \cup \{\neg\phi_n\} & \text{otherwise;} \end{cases} \\ T &= \bigcup_{n\in\mathbb{N}} T_n. \end{aligned}$$

Clearly $\mathrm{Q} \subseteq T$, and T is Σ_n-sound since $\Pi_n(\mathcal{N}) \subseteq S \subseteq T$. It suffices to show that T is Σ_{n+2}-definable.

Consistency isn't sufficient

Proof II.

Now define $\mathrm{Axiom}_T(x)$ as

$$\exists y \Big[\mathrm{finseq}(y) \wedge y_{\ell\mathrm{en}(y)-1} \dot{\equiv} x \wedge$$
$$\forall k \dot{<} \ell\mathrm{en}(y) \Big[\mathrm{Sent}(y_k) \wedge \forall z \dot{\leq} y \big[\mathrm{Senth}(z, k) \wedge$$
$$[\mathrm{con}'(S + y \restriction k + z) \to y_k \dot{\equiv} z \ \vee\ \neg \mathrm{con}'(S + y \restriction k + z) \to y_k \dot{\equiv} \dot{\neg}(z)]\big]\Big]\Big].$$

And

$$\mathrm{con}'(S + y \restriction k + z) = \forall v \forall w (\Pi_n\text{-true}(v) \to \neg \mathrm{beb}_{\mathrm{Q}}(w, \ulcorner \delta_v \wedge \delta_{y_0} \wedge \cdots \wedge \delta_{y_{k-1}} \wedge \delta_z \to \bot \urcorner)).$$

It's easy to check that $\mathrm{Axiom}_T(x) \in \Sigma_{n+2}$ and T is defined by it. ❒

Consistency isn't sufficient

Theorem 7.2 (Optimal Gödel-Rosser's First Incompleteness)

If $\mathsf{Q} \subseteq T$ and T is Σ_{n+2}-definable and T is consistent, then T may be complete.

Proof.

This is the case for $n = 0$ in Σ_n-sound since Σ_0-soundness is equivalent to consistency under $\mathsf{Q} \subseteq T$. ❒

Consistency isn't sufficient

Corollary 7.3

1. *If* Q $\subseteq T$ *and* $\text{Axiom}_T \in \Sigma_n$ *and* T *is* Σ_{n-2}*-sound, then* T *may be* Π_n*-deciding.*
2. *If* Q $\subseteq T$ *and* $\text{Axiom}_T \in \Sigma_n$ *and* T *is* Π_{n-1}*-sound, then* T *may be* Π_n*-deciding.*
3. *If* Q $\subseteq T$ *and* $\text{Axiom}_T \in \Sigma_n$ *and* T *is* $(n-2)$*-consistent, then* T *may be* Π_n*-deciding.*
4. *If* Q $\subseteq T$ *and* $\text{Axiom}_T \in \Pi_{n-1}$ *and* T *is* Σ_{n-2}*-sound, then* T *may be* Π_n*-deciding.*
5. *If* Q $\subseteq T$ *and* $\text{Axiom}_T \in \Pi_{n-1}$ *and* T *is* Π_{n-1}*-sound, then* T *may be* Π_n*-deciding.*
6. *If* Q $\subseteq T$ *and* $\text{Axiom}_T \in \Pi_{n-1}$ *and* T *is* $(n-2)$*-consistent, then* T *may be* Π_n*-deciding.*

Proof.

(1) Suppose for sake of a contradiction that none of such T is Π_n-deciding, then none of such T is complete, a contradiction to Lemma 7.1.
(2) By (1) and Σ_{n-2}-soundness is equivalent to Π_{n-1}-soundness.
(3) By (1) and Σ_{n-2}-soundness implies $(n-2)$-consistency.
(4) By (1) and Σ_n-definability is equivalent to Π_{n-1}-definability.
(5) By (2) and Σ_n-definability is equivalent to Π_{n-1}-definability.
(6) By (3) and Σ_n-definability is equivalent to Π_{n-1}-definability. ❒

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Conclusions: Diagrams for First Incompleteness

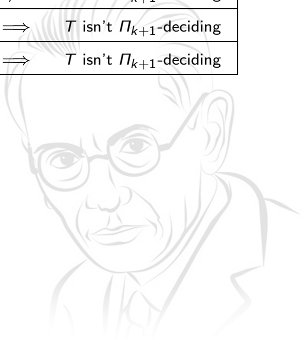
First Incompleteness Theorems for Σ_n-definable Theories($n > 1$)

Gödel-Rosser's 1st 1.5	$Q \subseteq T \land \mathrm{Axiom}_T \in \Sigma_1 \land T$ is Σ_0-sound	$\Longrightarrow$	T isn't Π_1-deciding
Corollary 1.3 (2)	$Q \subseteq T \land \mathrm{Axiom}_T \in \Sigma_1 \land T$ is Σ_1-sound	$\Longrightarrow$	T isn't Π_1-deciding
Corollary 7.3 (1)	$Q \subseteq T \land \mathrm{Axiom}_T \in \Sigma_n \land T$ is Σ_{n-2}-sound	$\not\Longrightarrow$	T isn't Π_n-deciding
Corollary 4.3 (1)	$Q \subseteq T \land \mathrm{Axiom}_T \in \Sigma_n \land T$ is Σ_{n-1}-sound	$\Longrightarrow$	T isn't Π_n-deciding
Corollary 4.3 (2)	$Q \subseteq T \land \mathrm{Axiom}_T \in \Sigma_n \land T$ is Σ_n-sound	$\Longrightarrow$	T isn't Π_n-deciding
Theorem 3.3	$Q \subseteq T \land \mathrm{Axiom}_T \in \Sigma_n \land T$ is sound	$\Longrightarrow$	T isn't Π_n-deciding
Gödel-Rosser's 1st 1.5	$Q \subseteq T \land \mathrm{Axiom}_T \in \Sigma_1 \land T$ is Π_0-sound	$\Longrightarrow$	T isn't Π_1-deciding
Corollary 1.6 (1)	$Q \subseteq T \land \mathrm{Axiom}_T \in \Sigma_1 \land T$ is Π_1-sound	$\Longrightarrow$	T isn't Π_1-deciding
Corollary 7.3 (2)	$Q \subseteq T \land \mathrm{Axiom}_T \in \Sigma_n \land T$ is Π_{n-1}-sound	$\not\Longrightarrow$	T isn't Π_n-deciding
Corollary 5.2	$Q \subseteq T \land \mathrm{Axiom}_T \in \Sigma_n \land T$ is Π_n-sound	$\Longrightarrow$	T isn't Π_n-deciding
Theorem 3.3	$Q \subseteq T \land \mathrm{Axiom}_T \in \Sigma_n \land T$ is sound	$\Longrightarrow$	T isn't Π_n-deciding
Gödel-Rosser's 1st	$Q \subseteq T \land \mathrm{Axiom}_T \in \Sigma_1 \land T$ is consistent	$\Longrightarrow$	T isn't Π_1-deciding
Gödel's 1st 1.3 (1)	$Q \subseteq T \land \mathrm{Axiom}_T \in \Sigma_1 \land T$ is 1-consistent	$\Longrightarrow$	T isn't Π_1-deciding
Corollary 7.3 (3)	$Q \subseteq T \land \mathrm{Axiom}_T \in \Sigma_n \land T$ is $(n-2)$-consistent	$\not\Longrightarrow$	T isn't Π_n-deciding
Corollary 6.4 (1)	$Q \subseteq T \land \mathrm{Axiom}_T \in \Sigma_n \land T$ is $(n-1)$-consistent	$\Longrightarrow$	T isn't Π_n-deciding
Corollary 6.4 (2)	$Q \subseteq T \land \mathrm{Axiom}_T \in \Sigma_n \land T$ is n-consistent	$\Longrightarrow$	T isn't Π_n-deciding
Corollary 6.4 (3)	$Q \subseteq T \land \mathrm{Axiom}_T \in \Sigma_n \land T$ is ω-consistent	$\Longrightarrow$	T isn't Π_n-deciding

Conclusions: Diagrams for First Incompleteness

First Incompleteness Theorems for Π_k-definable Theories($k > 0$)

Corollary 7.3 (4)	$Q \subseteq T \wedge Axiom_T \in \Pi_k \wedge T$ is	Σ_{k-1}-sound	$\not\Longrightarrow$	T isn't Π_{k+1}-deciding
Theorem 4.1	$Q \subseteq T \wedge Axiom_T \in \Pi_k \wedge T$ is	Σ_k-sound	$\Longrightarrow$	T isn't Π_{k+1}-deciding
Corollary 3.4	$Q \subseteq T \wedge Axiom_T \in \Pi_k \wedge T$ is	sound	$\Longrightarrow$	T isn't Π_{k+1}-deciding
Corollary 7.3 (5)	$Q \subseteq T \wedge Axiom_T \in \Pi_k \wedge T$ is	Π_k-sound	$\not\Longrightarrow$	T isn't Π_{k+1}-deciding
Theorem 5.1	$Q \subseteq T \wedge Axiom_T \in \Pi_k \wedge T$ is	Π_{k+1}-sound	$\Longrightarrow$	T isn't Π_{k+1}-deciding
Theorem 3.3	$Q \subseteq T \wedge Axiom_T \in \Pi_k \wedge T$ is	sound	$\Longrightarrow$	T isn't Π_{k+1}-deciding
Corollary 7.3 (6)	$Q \subseteq T \wedge Axiom_T \in \Pi_k \wedge T$ is	$(k-1)$-consistent	$\not\Longrightarrow$	T isn't Π_{k+1}-deciding
Theorem 6.3	$Q \subseteq T \wedge Axiom_T \in \Pi_k \wedge T$ is	k-consistent	$\Longrightarrow$	T isn't Π_{k+1}-deciding
Corollary 6.4 (4)	$Q \subseteq T \wedge Axiom_T \in \Pi_k \wedge T$ is	ω-consistent	$\Longrightarrow$	T isn't Π_{k+1}-deciding



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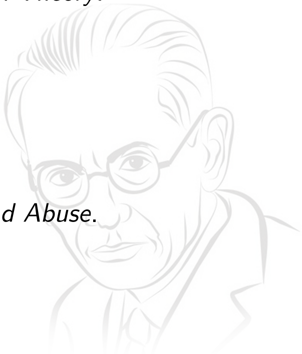

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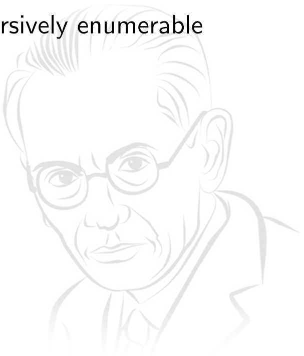

THANKS!