A Brief Introduction to First-Order Modal Logic: Completeness Proofs

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Overview

Introduction

- At first glance, first-order modal logic (FOML) seems straightforward: nothing other than a combination of propositional modal logic (PML) and first-order logic (FOL).
- However, it cannot bring us a intuitively satisfying semantics. There are plenty of semantics based on different philosophical backgrounds or technical considerations.
- We will mainly talk about three different semantics according to domain settings.
- For simplicity, we will go along with the philosophical viewpoint that takes terms as *rigid designators*.

Language of FOML

An FOML-language $\mathcal{L}$ includes the following symbols:

1. A countably infinite set of variables.
2. A countably infinite set of n-place predicates for each $n \geq 1$.
3. Boolean connections $\neg, \vee$.
4. Identity predicate $\approx$.
5. Quantifier $\forall$.
6. Modal operator $\Box$.

Language of FOML

- Our FOML language does not include function and constant symbols. One reason is for simplicity. Another reason is that the expressivity will not be weaken by omitting those symbols.

Formula

Definition (FOML-Formula)

FOML-formulas are defined as:

$$\varphi ::= Px_1 \cdots x_n \mid x \approx y \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box x\varphi \mid \forall\varphi$$

where P is an n-place predicate symbol, $x_1, \cdots, x_n$ and x, y are variables.

We can also define the following formulas as:

$$\exists x\varphi := \neg\forall x\neg\varphi$$

$$\Diamond\varphi := \neg\Box\neg\varphi$$

Constant Domain Model

Definition

A constant domain model is a tuple $\mathcal{M} = \langle W, R, D, \{V_w\}_{w \in W} \rangle$ where

- W is a non-empty set;
- R is a binary relation on W;
- D is a non-empty set;
- for each $w \in W$, V_w is a function to each n-place predicate assigns a subset of D^n.

Given a constant domain model $\mathcal{M} = \langle W, R, D, \{V_w\}_{w \in W} \rangle$, an assignment σ is a function that to each variable assigns an element of D.

We call the pair $\mathcal{F} = \langle W, R \rangle$ *frame* and triple $\mathcal{S} = \langle W, R, D \rangle$ *skeleton*.

Constant Domain

Definition

Semantics Let $\mathcal{M} = \langle W, R, D, \{V_w\}_{w \in W} \rangle$ be a model, for each $w \in W$ and each assignment σ we define:

$$
\begin{aligned}
\mathcal{M}, w, \sigma \models Px_1, \cdots, x_n &\Leftrightarrow \langle \sigma_1), \cdots, \sigma(x_n)\rangle \in V_w(P) \\
\mathcal{M}, w, \sigma \models x \approx y &\Leftrightarrow \sigma(x) = \sigma(y) \\
\mathcal{M}, w, \sigma \models \neg\varphi &\Leftrightarrow \text{not } \mathcal{M}, w, \sigma \models \varphi \\
\mathcal{M}, w, \sigma \models \varphi \vee \psi &\Leftrightarrow \mathcal{M}, w, \sigma \models \varphi \text{ and } \mathcal{M}, w, \sigma \models \psi \\
\mathcal{M}, w, \sigma \models \Box\varphi &\Leftrightarrow \text{for any } v \in W \text{ such that } Rwv, \\
&\quad\;\; \mathcal{M}, w, \sigma \models \varphi \\
\mathcal{M}, w, \sigma \models \forall x\varphi &\Leftrightarrow \text{for any } a \in D,\ \mathcal{M}, w, \sigma(a/x) \models \varphi
\end{aligned}
$$

$\sigma(a/x)$ is an assignment which maps x to a and agrees with σ on all variables distinct from x.

Constant Domain

- A formula φ is said to be *true* at w if $\mathcal{M}, w, \sigma \vDash \varphi$; otherwise is said to be false.
- A formula φ is satisfiable iff φ is true at some w. A set Λ of formulas is satisfiable iff every $\varphi \in \Lambda$ is true at some w.
- A formula φ is valid in a model $\mathcal{M}$ iff for every $w \in W$ and every assignment σ, $\mathcal{M}, w, \sigma \vDash \varphi$.
- A formula φ is valid in a frame $\mathcal{F}$ (or a skeleton $\mathcal{S}$) iff φ is valid in every model based on $\mathcal{F}$ (or $\mathcal{S}$).

System **QS**

Given a normal propositional modal logic system **S**, we define the corresponding first-order modal system **QS** as :

Axiom Schemas

S'	All **FOML** substitution instances of a theorem of **S**.
I1	$x \approx x$
I2	$x \approx y \rightarrow (\varphi \rightarrow \psi)$(where φ and ψ differ only in that φ has free x where ψ has free y)
$\Box$NI	$x \not\approx y \rightarrow \Box(x \not\approx y)$
$\forall$1	$\forall x\varphi \rightarrow \varphi[y/x]$ (where $\varphi[y/x]$ is φ with free y replacing every free x)

Rules

MP	$\dfrac{\varphi \rightarrow \psi \quad \varphi}{\psi}$
N	$\dfrac{\varphi}{\Box\varphi}$
$\forall$2	$\dfrac{\varphi \rightarrow \psi}{\varphi \rightarrow \forall x\psi}$ (x is not free in φ)

System **QS**

We have another principle known as *Barcan formula* which is stated by the schema

$$\text{BF} \qquad \forall x \Box \varphi \to \Box \forall x \varphi$$

System **QS** + **BF** is **QS** with the addition of BF.

System **QS**

Theorem (BFc)

$\vdash_{QK} \Box\forall x\varphi \rightarrow \forall x\Box\varphi$

Proof.

1	$\forall x\varphi \rightarrow \varphi$	($\forall$1)
2	$\Box(\forall x\varphi \rightarrow \varphi)$	(N,1)
3	$\Box\forall x\varphi \rightarrow \Box\varphi$	(K,MP,2)
4	$\Box\forall x\varphi \rightarrow \forall x\Box\varphi$	($\forall$2,3)

□

System **QS**

Theorem (BF on QB)

$\vdash_{QB} \forall x \Box \varphi \rightarrow \Box \forall x \varphi$

Proof.

1	$\forall x \Box \varphi \rightarrow \Box \Diamond \forall x \Box \varphi$	(B)
2	$\forall x \Box \varphi \rightarrow \Box \varphi$	($\forall$1)
3	$\Diamond \forall x \Box \varphi \rightarrow \Diamond \Box \varphi$	(RK$\Diamond$)
6	$\Diamond \forall x \Box \varphi \rightarrow \forall x \varphi$	($\forall$2,5)
7	$\Box \Diamond \forall x \Box \varphi \rightarrow \Box \forall x \varphi$	(RK,6)
8	$\forall x \Box \varphi \rightarrow \Box \forall x \varphi$	(1,7)

□

As a result, **QB** + **BF** ≡ **QB** and **QS5** + **BF** ≡ **QS5**

System **QS**

Theorem ($\Box$I)

$\vdash_{\mathbf{QS}} x \approx y \rightarrow \Box(x \approx y)$

Proof.

1	$x \approx y \rightarrow (\Box x \approx x \rightarrow \Box x \approx y)$	(I2)
2	$\Box x \approx x \rightarrow (x \approx y \rightarrow \Box x \approx y)$	(PL,1)
3	$\Box x \approx x$	(I1,N)
4	$x \approx y \rightarrow \Box x \approx y$	(2,3)

$\Box$

System **QS**

In first-order modal logic,System **QS** soundness is with respect to skeleton-validity.

Theorem (Soundness)

If $\mathcal{F}$ is a frame for a normal propositional modal logic system **S**, *then $\mathcal{S} = \langle \mathcal{F}, D \rangle$ is a skeleton for* **QS** + **BF**.

Proof.

Let $\mathfrak{M}$ be the class of models based on $\mathcal{F}$. We need to prove that each instance of `S'` , $\forall$1, `I1`, `I2`,$\Box$`NI` and `BF` is valid in every model in $\mathfrak{M}$ and the rules `MP`,`N`,$\forall$2 preserve the validity in every such model.

We only prove the cases of `S'`, `BF` and $\Box$`NI`, for the rest is almost the same like the proves in propositional modal logic and first-order logic.

Soundness of System **QS+BF**

cont'd.

- S': Assume that α is a theorem of **S** and φ is a **FOML**-formula by substituting **FOML**-formulas $\psi_1, \cdots, \psi_n$ for propositional variables $p_1, \cdots, p_n$ in α.
  Suppose that φ is not valid on $\mathfrak{M}$, i.e. there is a model $\mathcal{M} = \langle \mathcal{F}, D, \{V_w\}_{w \in W} \rangle$ and an assignment σ such that for some $w \in W$, $\mathcal{M}, w, \sigma \not\vDash \varphi$. Let $\mathcal{M}' = \langle \mathcal{F}, V \rangle$ be a propositional logic model with the same frame $\mathcal{F}$ and for every $w \in W_{\mathcal{F}}$ and every $p_i (1 \leq i \leq n)$, $p_i \in V'(w)$ iff $\mathcal{M}, w, \sigma \vDash \psi_i$. By induction on α, one can easily show that $\langle \mathcal{F}, V' \rangle \not\vDash \alpha$. Since α is not valid in $\langle \mathcal{F}, V' \rangle$ and $\mathcal{F}$ is a frame for **S**, α is not a theorem of **S**. Therefore, if α is a theorem of **S**, then φ is valid in $\mathfrak{M}$.
  By assumption, α is valid in $\mathfrak{M}$.

Soundness of System **QS+BF**

cont'd.

- $\texttt{BF}$: Suppose that $\mathcal{M}, w, \sigma \vDash \forall x \Box \varphi$. By semantics, for any $a \in D$, $\mathcal{M}, w, \sigma(a/x) \vDash \Box\varphi$. Then for any $w' \in W$ such that wRw', $\mathcal{M}, w', \sigma(a/x) \vDash \varphi$. Since this holds for any $a \in D$, we have that $\mathcal{M}, w', \sigma \vDash \forall x \varphi$. Since this holds for any $w' \in W$ with wRw', we finally have $\mathcal{M}, w, \sigma \vDash \Box \forall x \varphi$. Therefore, every instance of $\texttt{BF}$ is valid in $\mathfrak{M}$.
- $\Box\texttt{NI}$: Suppose $\sigma(x) \neq \sigma(y)$. Then there will be no $w \in W$ such that $\mathcal{M}, w, \sigma \vDash x \approx y$. So $\mathcal{M}, w, \sigma \vDash \Box(x \not\approx y)$.

$\Box$

Completeness of System **QS+BF**

First, we recall the strategies of the proofs of Completeness Theorem in propositional modal logic and first order logic.

Completeness Theorem of Propositional Modal Logic

- A propositional system **S** is complete w.r.t. a structure $\mathcal{S}$ iff every **S**-consistent set of formulas Σ is satisfiable on $\mathcal{S}$.
- We need a canonical model $\mathcal{M}^c$ such that for any formula φ, $\mathcal{M}^c, w \vDash \varphi$ iff $\varphi \in w$. (Truth Lemma)
- Prove that every **S**-consistent set Σ can be extended to a maximal consistent set Σ^+. (Lindenbaum's Lemma)
- Use maximal consistent sets as states set W^c in $\mathcal{M}^c$ which ensures Σ belongs to some state in $\mathcal{M}^c$.
- Define R^c of $\mathcal{M}^c$ as $wR^c u$ if $\psi \in u$ implies $\Diamond\psi \in w$ and define V^c as $V^c(p) = \{w \in W^c \mid p \in w\}$.
- Prove that this canonical model $\mathcal{M}^C$ is possible. (Existence Lemma)

Completeness of System **QS+BF**

Completeness Theorem of First Order Logic

- Every **Q**-consistent set Γ can be extended to a maximal consistent set Λ.
- We also need a canonical model (interpretation) $\mathfrak{A}$ and a assignment σ such that $\mathfrak{A}, \sigma \models \Gamma$ iff $\Gamma \subseteq \Lambda$.
- Λ must be a Henkin set (if $\neg\forall x\varphi \in \Lambda$, then there is a y such that $\neg\varphi[y/x] \in \Lambda$ to ensure that $\forall x\varphi \in \Lambda$ iff $\mathfrak{A}, \sigma \models \forall x\varphi$.
- Prove that every **Q**-consistent set Γ can be extended to a maximal Henkin consistent set Λ. (Extend language with new symbols.)
- Let $A^{\mathfrak{A}}$ be the set of equivalence classes over $\approx$. For every n-place predicate P, $P^{\mathfrak{A}} = \{\langle x_1, \cdots, x_n\rangle \mid Px_1, \cdots, x_n \in \Lambda\}$.
- Every **Q**-consistent set Γ is satisfiable.

Completeness of System **QS+BF**

Now we try to sketch the proof strategy of completeness theorem of **QS+BF**.

- Maximal consistent sets can be taken as states in canonical model as well .
- We need extend Γ to maximal consistent set Γ^+ which satisfies property required both by PML and FOL.
- As FOL, domain of canonical model can be taken as equivalence classes over $\approx$ and let canonical assignments be similar as assignments in FOL.
- Notice the details about the extended language.

Completeness of System **QS+BF**

First, we introduce some notions here.

Definition

- A set Γ of formulas is maximal consistent w.r.t a system S iff it is S-consistent and for every formula φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.
- $\mathcal{L}^+$ is a **FOML** language which extends $\mathcal{L}$ by adding infinitely many new variables.
- A set Γ of formulas has the $\forall$-property iff for every variable x and every formula φ, there is some variable y such that $\varphi[y/x] \to \forall x\varphi \in \Gamma$.
- Λ is a set of formulas. $\Box^-(\Lambda) := \{\varphi \mid \Box\varphi \in \Lambda\}$.
- Γ is a set of formulas. $\sim$ is a binary relation on variables defined as: $x \sim_\Gamma y$ iff $x \approx y \in \Gamma$. Obviously, $\sim_\Gamma$ is an equivalent relation. Let $[x]$ be the equivalent class where x is in.

Lemma

If Λ is a consistent set of $\mathcal{L}$-formulas then there is a consistent set Δ of $\mathcal{L}^+$ with the $\forall$-property such that $\Lambda \subseteq \Delta$.

Proof.

Since $\mathcal{L}^+$ is a countable language, we can enumerate all variables and all formulas of the form $\forall x \varphi$ for any formula φ of $\mathcal{L}^+$ as $\forall x_1 \varphi_1, \forall x_2 \varphi_2, \cdots$. We define a sequent of sets as follows:

$$\Delta_0 = \Lambda$$
$$\Delta_1 = \Delta_0 \cup \{\varphi_1[y_1/x_1] \to \forall x_1 \varphi_1\}$$
$$\cdots$$
$$\Delta_{n+1} = \Delta_n \cup \{\varphi_{n+1}[y_{n+1}/x_{n+1}] \to \forall x_{n+1} \varphi_{n+1}\}$$

where for any $n (\geq 1)$, y_{n+1} is the first variable not in Δ_n or φ_{n+1}. Obviously, we can always find such a new variable y_{n+1}.

Completeness of System **QS+BF**

cont'd.

We show that every set of this sequent is consistent. $\Delta_0 = \Lambda$ is consistent. Suppose Δ_n is consistent while Δ_{n+1} is not. Then there will be $\psi_1, \cdots, \psi_k \in \Delta_n$ such that

$$(1)\ \vdash (\psi_1 \wedge \cdots \wedge \psi_k) \to \varphi_{n+1}[y_{n+1}/x_{n+1}]$$
$$(2)\ \vdash (\psi_1 \wedge \cdots \wedge \psi_k) \to \neg\forall x_{n+1}\varphi_{n+1}$$

Since y_{n+1} does not occur in Δ_n and φ_{n+1}, from (1) we have

$$(3)\ \vdash (\psi_1 \wedge \cdots \wedge \psi_k) \to \forall x_{n+1}\varphi_{n+1}$$

which contradicts with (2). It means that Δ_n is inconsistent. Therefore Δ_{n+1} is consistent.
Let $\Delta = \bigcup_{n\in\omega} \Delta_n$. It's not hard to see that Δ is a consistent set of $\mathcal{L}^+$ with the $\forall$-property such that $\Lambda \subseteq \Delta$. □

Completeness of System **QS+BF**

This theorem holds for any **FOML**-system. By Lindenbaum Lemma, Δ can extended to a maximal consistent set Δ^+ with the $\forall$-property in $\mathcal{L}^+$.

Lemma

Γ *is a maximal* **QS** + **BF***-consistent set of* $\mathcal{L}^+$*-formulas with the* $\forall$*-property,* φ *is a formula such that* $\Box\varphi \notin \Gamma$*, then there is a consistent set* Δ *of* $\mathcal{L}^+$*-formulas with the* $\forall$*-property such that* $\Box^-(\Gamma) \cup \{\neg\varphi\} \subseteq \Delta$.

Proof.

We enumerate all formulas of the form $\forall x\varphi$ as $\forall x_1\alpha_1, \forall x_2\alpha_2, \cdots$, and define a sequent of formulas $\beta_1, \beta_2, \cdots$ as:

$$
\begin{aligned}
&\beta_0 = \neg\varphi \\
&\beta_1 = \beta_0 \wedge (\alpha_1[y_1/x_1] \to \forall x_1\alpha_1) \\
&\cdots \\
&\beta_{n+1} = \beta_n \wedge (\alpha_{n+1}[y_{n+1}/x_{n+1}] \to \forall x_{n+1}\alpha_{n+1})
\end{aligned}
$$

where for every $n(n \geq 1)$, y_{n+1} is the first variable such that

$$(*) \quad \Box^-(\Gamma) \cup \{\beta_{n+1}\} \text{ is consistent}$$

cont'd.

It's not hard to prove that $\Box^{-}(\Gamma) \cup \{\beta_0\}$ is consistent. We need to show there always will be a y_{n+1} satisfying $(*)$. Suppose there were not. Then there would be some $\{\Box\gamma_1, \cdots, \Box\gamma_k\} \subseteq \Gamma$ such that for every variable y_{n+1} in $\mathcal{L}^{+}$, (we use $\vdash$ to denote $\vdash_{\mathbf{QS+BF}}$)

$$\vdash (\gamma_1 \wedge \cdots \wedge \gamma_k) \to (\beta_n \to \neg(\alpha_{n+1}[y_{n+1}/x_{n+1}] \to \forall x_{n+1}\alpha_{n+1})$$

then we have

$$\vdash (\Box\gamma_1 \wedge \cdots \wedge \Box\gamma_k) \to \Box(\beta_n \to \neg(\alpha_{n+1}[y_{n+1}/x_{n+1}] \to \forall x_{n+1}\alpha_{n+1})$$

Since Γ is a maximal consistent set,

$$\Box(\beta_n \to \neg(\alpha_{n+1}[y_{n+1}/x_{n+1}] \to \forall x_{n+1}\alpha_{n+1})) \in \Gamma$$

Let z be a variable does not occur in α_{n+1} or β_n. By $\forall$-property, there will be a y_{n+1} such that

cont'd.

$$\Box(\beta_n \to \neg(\alpha_{n+1}[y_{n+1}/x_{n+1}] \to \forall x_{n+1}\alpha_{n+1})) \to \\ \forall z\Box(\beta_n \to \neg(\alpha_{n+1}[z/x_{n+1}] \to \forall x_{n+1}\alpha_{n+1})) \in \Gamma$$

Then

$$\forall z\Box(\beta_n \to \neg(\alpha_{n+1}[z/x_{n+1}] \to \forall x_{n+1}\alpha_{n+1})) \in \Gamma$$

By BF,

$$\Box\forall z(\beta_n \to \neg(\alpha_{n+1}[z/x_{n+1}] \to \forall x_{n+1}\alpha_{n+1})) \in \Gamma$$

Since z does not occur in β_n or a_{n+1},

$$\Box(\beta_n \to \forall z\neg(\alpha_{n+1}[z/x_{n+1}] \to \forall x_{n+1}\alpha_{n+1})) \in \Gamma$$

But

$$\vdash \neg\forall z\neg((\alpha_{n+1}[z/x_{n+1}] \to \forall x_{n+1}\alpha_{n+1}))$$

Completeness of System **QS+BF**

cont'd.

Then $\Box\neg\beta_n \in \Gamma$, and $\neg\beta_n \in \Box^-(\Gamma)$. $\Box^-(\Gamma) \cup \{\beta_0\}$ is inconsistent. Contradiction. Then we have that every $\Delta_n = \Box^- \cup \{\beta_n\}$ is consistent.

Let $\Delta = \bigcup_{n\in\omega} \Delta_n$, then Δ has all the required properties. $\Box$

Completeness of System **QS+BF**

Definition (Canonical Model)

A canonical model (constant domain) of a system **QS+BF** in language $\mathcal{L}$ with an extension $\mathcal{L}^+$ is a tuple $\mathcal{M}^c = \langle W^C, R^C, D^C, \{V^C_\Gamma\}\Gamma \in W\rangle$ where

- W^C is the set of all maximal consistent sets with the $\forall$-property in $\mathcal{L}^+$.
- $R^C \in W^C \times W^C$ defined by $R^C\Gamma\Gamma'$ iff $\Box^-(\Gamma) \subseteq \Gamma$.
- $D^C = \{[x] \mid x \text{ is a variable in } \mathcal{L}^+\}$.
- For each $\Gamma \in W^C$, V^C_Γ is a function that for each n-place predicate P, $\langle x_1, \cdots, x_n\rangle \in V^C_\Gamma(P)$ iff $Px_1 \cdots x_n \in \Gamma$.

A canonical assignment σ is defined as for every $x \in D^C$, $\sigma(x) = [x]$.

Lemma (Truth Lemma)

$\mathcal{M}^C$ is a canonical model of **QS+BF**, *σ is a canonical assignment. For every $\mathcal{L}^+$ formular φ, any $\Gamma \in W^C$, $\mathcal{M}^C, \Gamma, \sigma \vDash \varphi$ iff $\varphi \in \Gamma$.*

Proof.

By deduction on φ. We only show some featured cases here.

- $\varphi = Px_1 \cdots x_n$: $\mathcal{M}^c, \Gamma, \sigma \vDash Px_1 \cdots P_n \Leftrightarrow \langle \sigma(x_1), \cdots, \sigma(x_n) \rangle \in V^C_\Gamma(P) \Leftrightarrow \langle x_1, \cdots, x_n \rangle \in V^C_\Gamma(P) \Leftrightarrow Px_1 \cdots x_n \in \Gamma$.
- $\varphi = x \approx y$: $\mathcal{M}^c, \Gamma, \sigma \vDash x \approx y$ iff $\sigma(x) = \sigma(y)$ iff $[x] = [y]$ iff $x \approx y \in \Gamma$
- Suppose $\varphi = \forall x \psi \notin \Gamma$. Then $\neg \forall x \psi \in \Gamma$. By Γ's $\forall$-property, there is a y such that $\neg \psi[y/x] \in \Gamma$. By IH, $\mathcal{M}^C, \Gamma, \sigma \nvDash \psi[y/x]$, then by $\forall 1$, $\mathcal{M}^C, \Gamma, \sigma \nvDash \forall x \psi$.
- Suppose $\varphi = \Box \psi \notin \Gamma$. Then $\neg \Box \psi \in \Gamma$. By Theorem, there is a $\Gamma' \in W^C$ such that $\neg \psi \in \Gamma'$. Then by IH, $\mathcal{M}, \Gamma', \sigma \nvDash \psi$, thus $\mathcal{M}^C, \Gamma, \sigma \nvDash \Box \psi$.

□

Completeness of System **QS+BF**

Theorem (Model Completeness)

Every **QS+BF** *logic is (strongly) complete with respect to* $\mathcal{M}^{\mathcal{C}}$*.*

Proof.

For any set Γ of consistent formulas, it can be extended to a maximal consistent set Γ^+ with $\forall$-property which is a state of $\mathcal{M}^C$. Then by Truth Lemma, $\mathcal{M}^C, \Gamma^+, \sigma \vDash \Gamma$. □

Theorem (Frame Completeness)

If the frame of the canonical model for QS+BF is a frame for **S***, then* **QS+BF** *is characterized by any class of frames for* **S** *which contains the frame of the canonical model for* **QS+BF***.*

Proof.

It's an analogue to the proof in propositional modal logic. □

BF and BFc

Now we should look back to BF formulas which we take as an axiom. Here is an example:

$$\text{BF} \quad \forall x \Box Px \rightarrow \Box \forall x Px$$

It can be read as: If everything is necessarily P, then it is necessary that everything is P.
We may not accept it intuitively. On the contrary, consider Converse BF formulas such as:

$$\text{BFc} \quad \Box \forall x Px \rightarrow \forall x \Box Px$$

which means: If it is necessary that everything is P, then everything is necessarily P. It seems more reasonable.
So could we find a logic which rules BF formulas out while keeps Converse BF formulas?

Increasing Domain

Now we introduce another kind of models in which the domain of worlds is not constant. We will not repeat conventions and definitions that are the same as in the constant domain case.

Definition (Increasing Domain Model)

An increasing domain model is a tuple $\langle W, R, D, \delta, \{V_w\}_{w \in W}\rangle$, where

- $\langle W, R, D, \{V_w\}w \in W\rangle$ is a constant domain model.
- $\delta : W \to 2^D$ assigns to each $w \in W$ an non-empty subset of D such that $\delta(w) \subseteq \delta(v)$ whenever wRv. We also write $\delta(w)$ as D_w.

The tuple $\langle W, R, D, \delta\rangle$ is called an increasing domain *skeleton*. Note that a constant domain model can be considered as an increasing domain model by letting $D_w = W$ for every $w \in W$.

Increasing Domain

Definition (Semantics)

Given a increasing domain model $\mathcal{M} = \langle W, R, D, \delta, \{V_w\}w \in W \rangle$ and an assignment σ, we define $\mathcal{M}, w, \sigma \vDash \varphi$ in the same way as in the constant domain case, except

- $\mathcal{M}, w, \sigma \vDash \forall x \varphi$ iff for any $a \in D_w$, $\mathcal{M}, w, \sigma(a/x) \vDash \varphi$

A formula φ is valid in a model $\mathcal{M}$ iff for every world w and every assignment σ such that for every variable x, $\sigma(x) \in D_w$, $\mathcal{M}, w, \sigma \vDash \varphi$. (This convention is crucial. Consider $\forall x \varphi \to \varphi[y/x]$.) That is to say, the only assignments considered are assignments where every variable is assigned an element of the local domain.

Increasing Domain

BF's Validity

Consider the following increasing domain model $\mathcal{M}$:

$$w_1\{a\} : Pa \longrightarrow w_2\{a, b\} : Pa$$

Clearly, for any assignment σ, $\mathcal{M}, w_1, \sigma \vDash \forall x \Box Px$ but $\mathcal{M}, w_1, \sigma \nvDash \Box \forall x Px$. Hence, $\mathcal{M} \nvDash \forall x \Box Px \to \Box \forall x Px$.

It shows that BF is invalid in this model.
However, BF is valid in symmetrical models which are actually constant domain models.
The system of increasing domain FOML is **QS** (without **BF**).

Theorem (Soundness)

If $\mathcal{F}$ is frame for a normal propositional modal logic system **S**, *then an increasing skeleton $\mathcal{S} = \langle \mathcal{F}, D, \delta \rangle$ is a skeleton for* **QS**.

Proof.

We only show the parts that are different with the constant domain case.

- $\forall 1$: Suppose that $\mathcal{M}, w, \sigma \vDash \forall x \varphi$ where $\sigma(x) \in Dw$ for any variable x. Then for any $a \in D_w$, $\mathcal{M}, w, \sigma(a/x) \vDash \varphi$. Let $\sigma(y) = a$, then $\mathcal{M}, w, \sigma \vDash \varphi[y/x]$.
- N: Suppose φ is valid on $\mathcal{M}$ while there is a $w \in W$ such that $\mathcal{M}, w, \sigma \nvDash \Box \varphi$ where $\sigma(x) \in D_W$ for any variable x. Then there exists a $w' \in W$ such that wRw', $\mathcal{M}, w', \sigma \nvDash \varphi$. Since wRw', $D_w \subseteq D_{w'}$. Thus $\sigma(x) \in D_{w'}$ for any variable x which means that φ is not valid in $\mathcal{M}$. Contradiction.

□

Completeness of System **QS**

As before, we assume two languages $\mathcal{L}$ and $\mathcal{L}^+$. To form the canonical model $\mathcal{M}^c$, We use $\mathcal{L}^+$ as D^c as well. But we need to make this canonical model as an increasing domain model. For any given state Γ, let $\mathcal{L}_\Gamma$ be a language which contains all variables in $\mathcal{L}$ and possibly some of the new variables of $\mathcal{L}^+$. Then let $\mathcal{L}_{\Gamma'}$ contain infinitely many of the variable of $\mathcal{L}^+$ not in $\mathcal{L}_\Gamma$. Using all variables of $\mathcal{L}_\Gamma$ and $\mathcal{L}_{\Gamma'}$ as D_Γ and $D_{\Gamma'}$ correspondingly, we guarantee the canonical model is an increasing domain model. Since $\mathcal{L}_{\Gamma'}$ contains infinitely many variables not in $\mathcal{L}_\Gamma$, we can prove the following theorem straightforwardly.

Lemma

Γ *is maximal* **QS**-*consistent set with the* $\forall$-*property,* φ *is a formula. If* $\Box\varphi \notin \Gamma$, *then there is a maximal consistent set* Γ' *with the* $\forall$-*property in a language* $\mathcal{L}_{\Gamma'}$ *containing* $\mathcal{L}_\Gamma$ *such that* $\Box - (\Gamma) \cup \{\neg\alpha\} \subseteq \Gamma'$.

Definition (Canonical Model)

A canonical model (increasing domain) of a system **QS** in language in language $\mathcal{L}$ with an extension $\mathcal{L}^+$ is a tuple $\mathcal{M}^c = \langle W^C, R^C, D^C, \delta^c, \{V^C_\Gamma\}_{\Gamma \in W} \rangle$ where

- W^C is the set of all maximal consistent sets with the $\forall$-property in $\mathcal{L}^+$.
- $R^C \in W^C \times W^C$ defined by $R^C \Gamma\Gamma'$ iff $\Box^-(\Gamma) \subseteq \Gamma'$.
- $D^C = \{[x] \mid x \text{ is a variable in } \mathcal{L}^+\}$
- $\delta^c(\Gamma) = \{[x] \mid x \text{ is a variable in } \mathcal{L}_w\}$.
- For each $\Gamma \in W^C$, V^C_Γ is a function that for each n-place predicate P, $\langle x_1, \cdots, x_n \rangle \in V^C_\Gamma(P)$ iff $Px_1 \cdots x_n \in \Gamma$.

A canonical assignment σ is an assignment such that $\sigma(x) = [x]$ for every $x \in D$.

It's not hard to check that this canonical model is indeed an increasing domain model.

Lemma (Truth Lemma)

For any $\Gamma \in W^c$, any $\mathcal{L}_\Gamma$-formula φ, $\mathcal{M}^c, \Gamma, \sigma \models \varphi$ iff $\varphi \in \Gamma$.

Proof.

By induction on φ. We show some cases in detail:

- Suppose $\neg\varphi$ is a $\mathcal{L}_\Gamma$-formula. Then φ is also a $\mathcal{L}_\Gamma$-formula. Then $\mathcal{M}^c, \Gamma, \sigma \models \neg\varphi$ iff $\mathcal{M}^c, \Gamma, \sigma \not\models \varphi$ iff $\varphi \notin \Gamma$ iff $\neg\varphi \in \Gamma$.
- Suppose $\Box\varphi \in \Gamma$. Then $\varphi \in \Gamma'$ for any $R\Gamma\Gamma'$ and so φ is a $\mathcal{L}_{\Gamma'}$-formula. Then by IH, $\mathcal{M}^c, \Gamma', \sigma \models \varphi$. Since this is so for any Γ' with $R\Gamma\Gamma'$, $\mathcal{M}^C, \Gamma, \sigma \models \Box\varphi$.
- Suppose $\Box\varphi$ is a $\mathcal{L}_\Gamma$-formula such that $\Box\varphi \notin \Gamma$. Then by the above theorem, there is a $\Gamma' \in W^c$ with $\forall$-property such that $\neg\varphi \in \Gamma'$ and $R\Gamma\Gamma'$. So $\varphi \notin \Gamma'$. Since φ is a $\mathcal{L}_{\Gamma'}$-formula, by IH, we have that $\mathcal{M}^c, \Gamma', \sigma \not\models \varphi$. Then $\mathcal{M}^c, \Gamma, \sigma \not\models \Box\varphi$.

□

Completeness of System **QS**

Theorem (Model Completeness)

For any formula φ, φ is valid in the canonical model of **QS** *iff* $\vdash_{\mathbf{QS}} \varphi$.

For frame completeness, note that R^c cannot be symmetrical, therefore it doesn't work on systems with respect to symmetrical frames such as B and S5. But since BF is provable on those frames, we can use the method in constant domain case to establish completeness. There also are systems for which neither method will work, cf. Hughes and Cresswell[1996], pp.282.

BF and BFc

Although Converse Barcan Formula are not taken as an axiom in any **QS** system, it's not hard to prove the following theorem:

Theorem

A model is an increasing domain model iff Converse Barcan formula is valid in it.

In fact, we have a similar theorem with respect to Barcan Formula:

Theorem

A model is an decreasing domain model iff Barcan formula is valid in it.

BF and BFc

Someone may argue that he thinks Converse Barcan Formula is not plausible neither. For consider

$$\Box\forall x Px \rightarrow \forall x \Box Px$$

Suppose that in every world everything which exists in that world is P, but that something in our world fails to be P in some other world.
What if we also abandon Converse Barcan Formula ?

Varying Domain

The semantics of varying domain is almost the same as increasing domain except we do not have inclusion requirement(if wRw' then $D_w \subseteq D_{w'}$) here.

Definition (Varying Domain Model)

An varying domain model is a tuple $\langle W, R, D, \delta, \{V_w\}_{w \in W} \rangle$, where

- $\delta : W \to 2^D$ assigns to each $w \in W$ an non-empty subset of D. We also write $\delta(w)$ as D_w.

Definition (Semantics)

Given a varying domain model $\mathcal{M} = \langle W, R, D, \delta, \{V_w\}_{w \in W} \rangle$ and an assignment σ, we define $\mathcal{M}, w, \sigma \vDash \varphi$ in the same way as in the increasing domain case.

Varying Domain

A formula φ is valid in a model $\mathcal{M}$ iff for every world w and every assignment σ, $\mathcal{M}, w, \sigma \vDash \varphi$. (NOT the increasing domain case).

- $\forall 1$ is not valid. Consider the following varying domain model $\mathcal{M}$:

$$w_1\{a, b\} : PaPb \longrightarrow w_2\{a\} : Pa$$

Let $\sigma(y) = b$. Then $\mathcal{M}, w_2, \sigma \vDash \forall xPx$, but $\mathcal{M}, w_2, \sigma \nvDash Py$. Thus $\mathcal{M}, w_2, \sigma \nvDash \forall xPx \to Py$.
(If we require $\sigma(y)$ in local domain, then $\mathtt{N}$ will no longer preserve validity.)

- $\forall 1$ can be modified in two ways, which gives us different varying domain axiom systems.

System $\mathbf{Q_K S}$

- The first system $\mathbf{Q_K S}$ is due to Kripke.
- $\mathbf{Q_k S}$ is a system for the logic without identity predicate,
- $\mathbf{Q_K S}$ is sound and complete with respect to varying domain skeletons. But we do not have the completeness result for $\mathbf{Q_K S}$ containing B.

System $\mathbf{Q_K S}$

Axiom Schemas

S'	All **FOML** substitution instances of a theorem of **S**.
∀1K	$\forall y \forall z(\forall x \varphi \to \varphi[y/x])$ (where $\varphi[y/x]$ is φ with free y replacing every free x)
∀-DIST	$\forall x(\varphi \to \beta) \to (\forall x \varphi \to \forall x \beta)$
VQ	$\forall x \varphi \leftrightarrow \varphi$ (x is not free in φ)

Rules

MP	$\dfrac{\varphi \to \psi \quad \varphi}{\psi}$
N	$\dfrac{\varphi}{\Box \varphi}$
UG	$\dfrac{\varphi}{\forall x \varphi}$ (x is not free in $\varphi_1, \cdots, \varphi_n$)

System $\mathbf{Q_K S}$

Theorem ($\forall$-Permutation)

$\vdash_{Q_K S} \forall x \forall y \varphi \to \forall y \forall x \varphi$

Proof.

1	$\forall y \forall x(\forall y \varphi \to \varphi)$	($\forall$1K)
2	$\forall x(\forall y \varphi \to \varphi) \to (\forall x \forall y \varphi \to \forall x \varphi)$	($\forall$-DIST)
3	$\forall y \forall x(\forall y \varphi \to \varphi) \to \forall y(\forall x \forall y \varphi \to \forall x \varphi)$	(UG,$\forall$-DIST,2)
4	$\forall y(\forall x \forall y \varphi \to \forall x \varphi)$	(1,3)
5	$\forall y \forall x \forall y \varphi \to \forall y \forall x \varphi$	($\forall$-DIST4)
6	$\forall x \forall y \varphi \to \forall y \forall x \forall y \varphi$	(VQ)
7	$\forall x \forall y \varphi \to \forall y \forall x \varphi$	(5,6)

$\square$

System $\mathbf{Q_K S}$

Theorem (Soundness)

If $\mathcal{F}$ is frame for a normal propositional modal logic system **S**, *then a varying skeleton $\mathcal{S} = \langle \mathcal{F}, D, \delta \rangle$ is a skeleton for $Q_K S$.*

Proof.

- $\forall 1\mathrm{K}$: Suppose $\mathcal{M}, w, \sigma \not\models \forall y \forall z(\forall x \varphi \to \varphi[y/x])$. By semantics, for any $a, b \in D_w$, we have $\mathcal{M}, w, \sigma(a/y)(b/z) \models \forall x \varphi$ and $\mathcal{M}, w, \sigma(a/y)(b/z) \not\models \varphi[y/x]$. Then $\mathcal{M}, w, \sigma(\sigma(y)/x)(a/y)(b/z) \not\models \varphi$. Contradiction.

□

System $\mathbf{Q_K S}$

Canonical models is produced by the same method of the increasing domain case. But we need to modify the $\forall$-property.

Definition (Extended $\forall$-property)

Λ is a set of $\mathcal{L}$-formulas, and Y is a set of variables of $\mathcal{L}$. Λ has the extended $\forall$-property in $\mathcal{L}$ with respect to Y iff

- Λ has the $\forall$-property with respect to Y.
- For every formula φ and variable y of Y, $\forall x \varphi \to \varphi[y/x] \in \Lambda$.

We will not show details of the proof of completeness of $\mathbf{Q_K S}$. One can find it on Hughes and Cresswell [1996], pp.306-309.

System $\mathbf{Q_E S}$

- Another system is $\mathbf{Q_E S}$.
- We will use an *existence predicate* E:

$$E(x) = \exists y(x = y)$$

- Obviously, $V_w(E) = D_w$.
- Logic with existence predicate in its axioms is called *free logic* (logic "free" of existential assumptions in FOL).

System $\mathbf{Q_E S}$

Axiom Schemas

S'	All FOML substitution instances of a theorem of **S**.
∀1E	$(\forall x\varphi \wedge E(y)) \rightarrow \varphi[y/x]$
I1	$\forall x(\varphi \rightarrow \beta) \rightarrow (\forall x\varphi \rightarrow \forall x\beta)$
I2	$x \approx y \rightarrow (\varphi \rightarrow \psi)$(where φ and ψ differ only in that φ has free x where ψ has free y)
□NI	$x \not\approx y \rightarrow \Box(x \not\approx y)$

Rules

MP N UG $\dfrac{\varphi}{\forall x\varphi}$

∀2E $\dfrac{(\varphi \wedge E(y)) \rightarrow \psi[y/x]}{\varphi \rightarrow \forall x\psi}$
(where y is not free in ψ and $\forall x\varphi$).

System $\mathbf{Q_E S}$

It's not hard to show that $\mathbf{Q_E S}$ is sound with respect to varying domain semantics.

Theorem (Soundness)

If $\mathcal{F}$ is frame for a normal propositional modal logic system **S**, *then an varying skeleton $\mathcal{S} = \langle \mathcal{F}, D, \delta \rangle$ is a skeleton for* $\mathbf{Q_E S}$.

To completeness, we could continue to use the same strategy in the increasing domain model case, which extends every $\mathcal{L}_w$ to $\mathcal{L}_{w'}$ by adding infinite new variables. We need to modify $\forall$-property to $E\forall$-property as: there is some y such that $Ey \wedge (\varphi[y/x] \rightarrow \forall x \varphi) \in \Delta$. The proof is routine.
Note that this method cannot give us result of completeness based on symmetrical frames.

System $\mathbf{Q_G S}$

- $\mathbf{Q_G S}$ is another system with existence predicate.
- We introduce some notions:

$$\varphi \rightarrowtail \psi := \Box(\varphi \to \psi)$$

System $\mathbf{Q_GS}$

Axiom Schemas

S'	All FOML substitution instances of a theorem of **S**.
∀1E	$(\forall x\varphi \wedge E(y)) \to \varphi[y/x]$
I1 I2 □NI	
∀-DIST	$\forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall y\varphi)$
VQ	$\varphi \leftrightarrow \forall x\varphi$ (x is not free in φ)
UE	$\forall x Ex$

Rules

MP N UG	$\dfrac{\varphi}{\forall x\varphi}$
G^n	$\dfrac{\varphi_1 \to (\varphi_2 \rightarrowtail (\cdots \rightarrowtail (\varphi_n \rightarrowtail \Box\psi)\cdots)}{\varphi_1 \to (\varphi_2 \rightarrowtail (\cdots \rightarrowtail (\varphi_n \rightarrowtail \Box\forall x\psi)\cdots)}$ (where x is not free in $\varphi_1, \cdots, \varphi_n$).

System $\mathbf{Q_G S}$

Theorem (QR)

$\vdash_{\mathbf{Q_G S}} \exists y(\varphi[y/x] \to \forall x \varphi)$

Proof.

1	$\forall y(\varphi[y/x] \wedge \neg\forall x\varphi) \to \forall y \varphi[y/x]$	(UG, PL)
2	$\forall y(\varphi[y/x] \wedge \neg\forall x\varphi) \to \forall x \varphi$	(FOL,2)
3	$\forall y(\varphi[y/x] \wedge \neg\forall x\varphi) \to \forall y \neg\forall x \varphi$	(UG,PL)
4	$\forall y(\varphi[y/x] \wedge \neg\forall x\varphi) \to \neg\forall x \varphi$	(VQ,3)
5	$\neg\forall y \neg(\varphi[y/x] \to \forall x \varphi)$	(PL,2,4)

□

System $\mathbf{Q_G S}$

Theorem (Soundness)

If $\mathcal{F}$ is frame for a normal propositional modal logic system **S**, *then a varying skeleton $\mathcal{S} = \langle \mathcal{F}, D, \delta \rangle$ is a skeleton for $Q_G S$.*

Proof.

- $\mathtt{G^n}$: Suppose that $\mathcal{M}, w, \sigma \nvDash \varphi_1 \to (\varphi_2 \rightarrowtail (\cdots \rightarrowtail (\varphi_n \rightarrowtail \Box \forall x \psi) \cdots)$. Then there is a sequent $w1, \cdots, w_{n+1}$ where $w_1 = w$ and $\mathcal{M}, w_i, \sigma \vDash \varphi (1 \leq i \leq n)$ and $\mathcal{M}, w_{n+1}, \sigma \nvDash \forall x \psi$. Then there is $a \in D_{w_{n+1}}$ such that $\mathcal{M}, w_{n+1}, \sigma(a/x) \vDash \psi$. Since x is not free in $\varphi_1, \cdots, \varphi_n$, for any $i \leq n$, $\mathcal{M}, w_i, \sigma(a/x) \vDash \varphi_i$. Then $\mathcal{M}, w, \sigma(a/x) \nvDash \varphi_1 \to (\varphi_2 \rightarrowtail (\cdots \rightarrowtail (\varphi_n \rightarrowtail \Box \psi) \cdots)$. Contradiction.

□

System $\mathbf{Q_G S}$

Let $\mathcal{L}$ and $\mathcal{L}^+$ be set as before. We introduce a modification of $\forall$-property:

Definition ($\Box\forall$-property)

Δ is a set of $\mathcal{L}^+$ formulas, Δ has the $\Box\forall$-property iff

- For every $\mathcal{L}^+$ formula φ and variable x, there is some variable y such that $Ey \wedge (\varphi[y/x] \to \forall x\varphi) \in \Delta$.
- For all $\mathcal{L}^+$ formulas $\psi_1, \cdots, \psi_n (n \geq 0)$ and φ, and every variable x not free in $\psi_1, \cdots, \psi_n$, there is some variable z such that $\psi_1 \rightarrowtail (\cdots \rightarrowtail \psi_n \rightarrowtail (Ez \rightarrowtail \varphi[z/x]) \cdots) \to (\psi_1 \rightarrowtail (\psi_2 \rightarrowtail (\cdots \rightarrowtail (\psi_n \rightarrowtail \Box\forall x\varphi) \cdots) \in \Delta$.

Lemma

If Λ is a consistent set of $\mathcal{L}$, then there is a consistent set Δ of $\mathcal{L}^+$ with $\Box\forall$-property such that $\Lambda \subseteq \Delta$.

Proof.

We enumerate all the formulas of the form $\forall x\varphi$ and the form $\varphi_1 \rightarrowtail (\varphi_2 \rightarrowtail (\cdots \rightarrowtail (\varphi_n \rightarrowtail \Box\forall x\psi)\cdots)$. Suppose that $\forall s\alpha$ and $\gamma_1 \rightarrowtail (\gamma_2 \rightarrowtail (\cdots \rightarrowtail (\gamma_k \rightarrowtail \Box\forall x\delta)\cdots)$ (x is not free in $\gamma_1, \cdots, \gamma_k$) are the n+1th formula respectively. Let Y and Z are disjoint sets of variables of $\mathcal{L}^+$. We define Δ_n as:

- $\Delta_0 = \Lambda$
- $\Delta_{n+1} = \Delta_n \cup \{Ey, \alpha[y/x] \to \forall x\alpha\} \cup \{\gamma_1 \rightarrowtail (\gamma_2 \rightarrowtail (\cdots \rightarrowtail (\gamma_k \rightarrowtail (Ez \rightarrowtail \delta[z/x])\cdots) \to (\gamma_1 \rightarrowtail (\gamma_2 \rightarrowtail (\cdots \rightarrowtail (\gamma_k \rightarrowtail \Box\forall x\delta)\cdots)\}$

where y, z are the first variables in Y and Z not occurring in Δ_n or in $\gamma_1, \cdots, \gamma_k$ or in δ.

Cont'd.

We show that Δ_n is consistent by induction on n. We give a sketch to the proof of Δ_{n+1}'s consistency here.

- Suppose it inconsistent. Use G^{k+1} to $\cdots \gamma_1 \rightarrowtail (\gamma_2 \rightarrowtail (\cdots \rightarrowtail (\gamma_k \rightarrowtail \Box(Ez \rightarrow \delta[z/x]))$ to have $\cdots\cdots (\gamma_k \rightarrow \Box\forall z(Ez \rightarrow \delta[z/x])$
- Move $\forall zEz$ to the front of formula and eliminate $\Box^k\forall zEz$ as an axiom.
- As a result, $\cdots \rightarrow (\gamma_1 \cdots \rightarrowtail (\gamma_k \rightarrowtail \Box\forall x\delta)$. Then we could deny the other parts in the antecedent.
- Similarly, we can eliminate Ey. And by QR, $\exists y(\alpha[y/x] \rightarrow \forall s\alpha$ is a theorem. Then Δ_n must be inconsistent. Contradiction.

□

As *Delta* has the $\Box\forall$-property, we can prove the following lemma:

Lemma

If Γ is a maximal-consistent set of $\mathcal{L}^+$ formulas, and Γ has the $\Box\forall$-property, and φ is a formula such that $\Box\varphi \notin \Gamma$, then there is a consistent set Δ of $\mathcal{L}^+$ formulas with $\Box\forall$-property such that $\Box^-(\Gamma) \cup \{\neg\varphi\} \subseteq \Delta$.

Proof.

- Define $\gamma_0 = \neg\varphi$. For any γ_n, let $\gamma_n^+ = \gamma_n \wedge Ey \wedge (\delta[y/x] \to \forall x\delta)$ where y is the first variable such that $\Box^-(\gamma) \cup \{\gamma_n^+\}$ is consistent. As before, we need to there always will be an appropriate y. Then show that $\Box^-(\Gamma) \cup \{\gamma_n^+\}$ is consistent.
- Let $\gamma_{n+1} = \gamma_n^+ \wedge (\psi_1 \rightarrowtail (\cdots \rightarrowtail \psi_n \rightarrowtail (Ez \rightarrowtail \chi[z/x]) \cdots) \to (\psi_1 \rightarrowtail (\psi_2 \rightarrowtail (\cdots \rightarrowtail (\psi_n \rightarrowtail \Box\forall x\varphi) \cdots)$. Show that there always will be an appropriate z such that $\Box^-(\Gamma) \cup \{\gamma_{n+1}$ is consistent. Then show that $\Box^-(\Gamma) \cup \{\gamma_n\}$ is consistent.
- Let Δ be the union of $\Box^-(\Gamma)$ and all the γ_ns.

Definition (Canoncial Model of $\mathbf{Q_G S}$)

A canonical model (increasing domain) of a system **QS** in language in language $\mathcal{L}$ with an extension $\mathcal{L}^+$ is a tuple $\mathcal{M}^c = \langle W^C, R^C, D^C, \delta^c, \{V_\Gamma^C\}_{\Gamma \in W} \rangle$ where

- W^C is the set of all maximal consistent sets with the $\forall$-property in $\mathcal{L}^+$.
- $R^C \in W^C \times W^C$ defined by $R^C \Gamma\Gamma'$ iff $\Box^-(\Gamma) \subseteq \Gamma'$.
- $D^C = \{[x] \mid x \text{ is a variable in } \mathcal{L}^+\}$
- $\delta^c(\Gamma) = \{[x] \mid x \text{ is a variable in } \mathcal{L}_w\}$.
- For each $\Gamma \in W^C$, V_Γ^C is a function that for each n-place predicate P, $\langle x_1, \cdots, x_n \rangle \in V_\Gamma^C(P)$ iff $Px_1 \cdots x_n \in \Gamma$. And for existence predicate E, $[x] \in D_\Gamma$ iff $Ex \in \Gamma$.

A canonical assignment σ is an assignment such that $\sigma(x) = [x]$ for every $x \in D$.

Completeness of $\mathbf{Q_GS}$

It's routine to prove the Truth Lemma.

Lemma (Truth Lemma)

For any $\Gamma \in W$ *and any* $\varphi \in \mathcal{L}^+$, $\mathcal{M}^c, \Gamma, \sigma \vDash \varphi$ *iff* $\varphi \in \Gamma$.

Theorem (Completeness)

$\mathbf{Q_GS}$ *is complete with respect to a varying domain canonical model of* $\mathbf{Q_GS}$.

Like the constant domain case, this can give us frame completeness results.

- One may find that there are so many similarities between constant domain models and varying domain models. Actually, A varying domain model can be viewed as a constant domain model.

From Varying Domain To Constant Domain

Definition

Given a varying domain model $\mathcal{M} = \langle W, R, D, \delta, \{V_w\}_{w \in W} \rangle$, a constant domain model $\mathcal{M}^*$ for the language extended with a unary predicate symbol E is defined by letting V_w^* of $\mathcal{M}^*$ be the extension of V_w such that $V_w(E) = \delta(w)$.
We can define a translation T from the original language to the extended language as:

$$
\begin{aligned}
T(Px_1 \cdots x_n) &= Px_1 \cdots x_n \\
T(x \approx y) &= x \approx y \\
T(\varphi \vee \psi) &= T(\varphi) \vee T(\psi) \\
T(\neg\varphi) &= \neg T(\varphi) \\
T(\Box\varphi) &= \Box T(\varphi) \\
T(\forall x \varphi) &= \forall x (E(x) \to T(\varphi))
\end{aligned}
$$

From Varying Domain To Constant Domain

Theorem

$\mathcal{M}$ is a varying domain model. For any FOML-formula φ. any world w, and any assignment σ, $\mathcal{M}, w, \sigma \vDash \varphi$ iff $\mathcal{M}^*, w, \sigma \vDash T(\varphi)$.

Proof.

By induction on φ. We only show the last case: $\varphi = \forall x \varphi$

- Suppose that $\mathcal{M}, w, \sigma \vDash \forall x \varphi$. For any $b \in D$, if $b \in D_w$), then by IH, $\mathcal{M}^*, w, \sigma(b/x) \vDash \varphi$. If $b \in D \setminus D_w$, then $\mathcal{M}^*, w, \sigma(b/x) \nvDash E(x)$. In either case, we both have $\mathcal{M}^*, w, \sigma(b/x) \vDash E(x) \to T(\varphi)$. Thus $\mathcal{M}^*, w, \sigma \vDash \forall x(E(x) \to T(\varphi))$.
- Suppose that $\mathcal{M}^*, w, \sigma \vDash \forall x(E(x) \to T(\varphi))$. Then for any $a \in D_w$, $a \in D$ as well. Thus $\mathcal{M}^*, w, \sigma(a/x) \vDash E(x) \to T(\varphi))$. Since $V_w(E) = D_w$, $\mathcal{M}^*, w, \sigma(a/x) \vDash E(x)$. Therefore, $\mathcal{M}^*, w, \sigma(a/x) \vDash T(\varphi)$. By IH, $\mathcal{M}, w, \sigma(a/x) \vDash \varphi$. Then $\mathcal{M}, w, \sigma \vDash \forall x \varphi$.

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