A Super Introduction to Reverse Mathematics

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Outline

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- Background
- Second Order Arithmetic
- RCA_0 and Mathematics in RCA_0
- Other Important Subsystems
- Reverse Mathematics and Other Branches
- References

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What is Reverse Mathematics

- Reverse mathematics is a program in mathematical logic that seeks to determine which axioms are required to prove theorems of mathematics.
- The program was founded by Harvey Friedman (1975, 1976). A standard reference for the subject is Simpson's (2009).
- The object of reverse mathematics is non-set theoretic or ordinary. The distinction between set-theoretic and ordinary mathematics corresponds roughly to the distinction between "uncountable mathematics" and "countable mathematics".

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Reverse Methods

To show that a system S is required to prove a theorem T, two proofs are required. The first proof shows T is provable from S; this is an ordinary mathematical proof along with a justification that it can be carried out in the system S. The second proof, known as a reversal, shows that T itself implies S; this proof is carried out in the base system. The reversal establishes that no axiom system S' that extends the base system can be weaker than S while still proving T.

Second Order Arithmetic(Z_2)

- The language of second order arithmetic(L_2) is a two-sorted language. This means that there are two distinct sorts of variables which are intended to range over all natural numbers and all subsets of natural numbers. The first sort are called number variables, denoted i, j, k, m, n, the other are called set variables, denoted X, Y, Z. What's more, the language contains 2-nary functions $+$ and $\cdot$, constants 0 and 1 and a order $<$.
- The Numerical terms are number variables, the constant symbols 0 and 1, and $t_1 + t_2$ and $t_1 \cdot t_2$ whenever t_1 and t_2 are numerical terms.

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- The Numerical terms are number variables, the constant symbols 0 and 1, and $t_1 + t_2$ and $t_1 \cdot t_2$ whenever t_1 and t_2 are numerical terms.

- Atomic formulas are $t_1 = t_2$, $t_1 < t_2$ and $t_1 \in X$ where t_1, t_2 are numerical terms and X is any set variable. Formulas are built up from atomic formulas by means of propositional connectives and number quantifiers $\forall n$, $\exists n$, and set quantifiers $\forall X$, $\exists X$.

Second Order Arithmetic(Z_2)

- L_2-structures. A model for L_2 is an ordered 7-tuple

$$M = (|M|, \mathcal{S}(M), +_M, \cdot_M, 0_M, 1_M, <_M)$$

  Where $|M|$ is a set which serves as the range of the numbers, $\mathcal{S}(M)$ is a set of subsets of $|M|$ serving as the range of the set variables. $+_M$ and $\cdot_M$ are binary operations on $|M|$, 0_M and 1_M are distinguished elements of $|M|$, and $<_M$ is binary relation on $|M|$. We always assume that the sets $|M|$ and $\mathcal{S}(M)$ are disjoint and nonempty.

- Parameters. Let $\mathcal{B}$ be any subset of $|M| \cup \mathcal{S}(M)$. By a formula with parameters from $\mathcal{B}$ we mean a formula of the extended language $L_2(\mathcal{B})$. Here $L_2(\mathcal{B})$ consists of L_2 augmented by new constant symbols corresponding to the elements of $\mathcal{B}$.

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- Definable. A set $A \subseteq |M|$ is said to be definable over M allowing parameters from $\mathcal{B}$ if there exists a formula $\varphi(a)$ with parameters from $\mathcal{B}$ and no free variables other than n such that

$$A = \{a \in |M| : M \models \varphi(a)\}$$

Here $M \models \varphi(a)$ means that M satisfies $\varphi(a)$, i.e. $\varphi(a)$ is true in M.

Second Order Arithmetic(Z_2)

(i) Basic Axioms:

$$\begin{aligned}
&\forall m(m+1 \neq 0),\\
&\forall m(m \cdot 0 = 0),\\
&\forall m,n(m+1 = n+1 \to m = n),\\
&\forall m,n(m \cdot (n+1) = m \cdot n + m),\\
&\forall m(m+0 = m),\\
&\forall m(\neg m < 0),\\
&\forall m,n(m+(n+1) = (m+n)+1),\\
&\forall m,n(m < n+1 \leftrightarrow (m = n \vee m < n)).
\end{aligned}$$

(ii) Induction Axiom:

$$(0 \in X \wedge \forall n(n \in X \to n+1 \in X)) \to \forall n(n \in X).$$

(iii) Comprehension Scheme

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any formula of L_2 in which X does not occur freely.

Intuitively, the given instance of comprehension scheme says that there exists a set $X = \{n : \varphi(n)\}$ = the set of all n such that $\varphi(n)$ holds. This set is said to be definable by the given formula $\varphi(n)$.

In the comprehension scheme, $\varphi(n)$ may contain free variable in addition to n. These free variables may be referred to as parameters of this instance of the comprehension scheme.

Second Order Arithmetic(Z_2)

- Z_2 is strong enough to develop analysis.
- If T is any subsystem of Z_2, a model of T is any L_2-structure satisfying the axioms of T. By Gödel's completeness theorem applied to the two sorted language L_2. We have the following important principle: a given L_2-sentence σ is a theorem of T if and only if all model of T satisfies σ.
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RCA_0

- Recursive Comprehension Axiom(RCA). The RCA scheme consists of all formulas of the form

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n(n \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any Σ_1^0 formula, $\psi(n)$ is any Π_1^0, n is any number variable, and X is a set variable which does not occur freely in $\varphi(n)$.
In the RCA, note that $\varphi(n)$ and $\psi(n)$ may contain parameters, i.e., free set variables and free number variables in addition to n. Thus all L_2-structure satisfies RCA if and only if $\mathcal{S}(M)$ contains all subsets of $|M|$ which are Δ_1^0 definable over M allowing parameters from $|M| \cup \mathcal{S}(M)$.

RCA_0

- RCA_0 is the subsystems of Z_2 consisting of the basic axioms, the Σ^0_1 induction scheme, and the RCA scheme.
- The system RCA_0 plays two key roles in Reverse Mathematics. First, the development of ordinary mathematics within RCA_0 correspond roughly to the positive content of what is known as "computable mathematics" or "recursive analysis". Thus RCA_0 is a kind of formalized recursive mathematics. Second, RCA_0 frequently play the role of a weak base theory in Reverse Mathematics. Most of the results of Reverse Mathematics will be stated formally as theorems of RCA_0.

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Mathematics in RCA_0

- Within RCA_0, we define a paring map $(i, j) = (i + j)^2 + i$, where of course $i^2 = i \cdot i$.
- Within RCA_0, a finite sequence of natural numbers is a finite set X such that $\forall n(n \in X \to \exists i \exists j(n = (i, j)))$ and $\forall i \forall j \forall k((i, j) \in X \wedge (i, k) \in X \to j = k)$ and $\exists l \forall i(i < l \leftrightarrow \exists j((i, j) \in X))$.
- Function. The following definitions are made in RCA_0. Let X and Y be sets of natural numbers. We write $X \subseteq Y$ to mean $\forall n(n \in X \to n \in Y)$.
  We define $X \times Y$ to be the set of all k such that $\exists i \leq k \exists j \leq k(i \in X \wedge j \in Y \wedge (i, j) = k)$.
  We define a function $f : X \to Y$ to be a set $f \subseteq X \times Y$ such that $\forall i \forall j \forall k(((i, j) \in f \wedge (i, k) \in f) \to j = k)$ and $\forall i \exists j(i \in X \to (i, j) \in f)$.

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Lemma 1 (Composition)

The following is provable in RCA_0 If $f : X \to Y$ and $g : Y \to Z$ the there exists $h = g \circ f : X \to Z$ defined by $h(i) = g(f(i))$.

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Within RCA_0, the set of all $s \in \mathit{Seq}$ such that $\mathit{lh}(s)=k$ is denoted $\mathbb{N}^k$. This set exists by Σ^0_0 comprehension. If $f : \mathbb{N}^k \to \mathbb{N}$ and $s = \langle n_1, ..., n_k\rangle \in \mathbb{N}^k$, we sometimes write $f(n_1, ..., n_k)$ instead of $f(s)$.

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Lemma 2 (Primitive recursion)

The following is provable in RCA_0. Given $f : \mathbb{N}^k \to \mathbb{N}$ and $g : \mathbb{N}^{k+2} \to \mathbb{N}$, there exists a unique $h : \mathbb{N}^{k+1} \to \mathbb{N}$ defined by

$$h(0, n_1, ..., n_k) = f(n_1, ..., n_k)$$
$$h(m, n_1, ..., n_k) = g(h(m, n_1, ..., n_k), m, n_1, ..., n_k).$$

Lemma 3 (Minimization)

The following is provable in RCA$_0$. Let $f : \mathbb{N}^{k+1} \to \mathbb{N}$ be such that for all $\langle n_1, ..., n_k \rangle \in \mathbb{N}^k$ there exists $m \in \mathbb{N}$ such that $f(m, n_1, ..., n_k) = 1$. Then there exists $g : \mathbb{N}^k \to \mathbb{N}$ defined by $g(n_1, ..., n_k)$= least m such that $f(m, n_1, ..., n_k) = 1$.

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Bounded Σ_k^0 comprehension. For each $k \in \omega$ the scheme of bounded Σ_k^0 comprehension consists of all axioms of the form

$$\forall n \exists X \forall i (i \in X \leftrightarrow (i < n \wedge \varphi(i)))$$

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Theorem 4

RCA_0 proves bounded Σ^0_1 comprehension.

Mathematics in RCA_0

- Within RCA_0 we can define $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ obviously.
- A sequence of rational numbers is defined in RCA_0 to be a function $f : \mathbb{N} \to \mathbb{Q}$. We usually denote such a sequence as $\langle q_k : k \in \mathbb{N} \rangle$ where $q_k = f(k)$.
- A real number is defined in RCA_0 to be a sequence of rational numbers $\langle q_k : k \in \mathbb{N} \rangle$ such that $\forall k \forall i (|q_k - q_{k+i}| \leq 2^{-k})$. Two real numbers $\langle q_k : k \in \mathbb{N} \rangle$ and $q'_k : k \in \mathbb{N}$ are said to be equal if $\forall k(|q_k - q'_k| \leq 2^{-k+1})$.
- The sum of two real numbers $x = \langle q_k : k \in \mathbb{N} \rangle$ and $y = \langle q'_k : k \in \mathbb{N} \rangle$ is defined by

$$x + y = \langle q_{k+1} + q'_{k+1} : k \in \mathbb{N} \rangle$$

We note that $|(q_{k+1} + q'_{k+1}) - (q_{k+i+1} + q'_{k+i+1}| \leq |q_{k+1} - q_{k+i+1}| + |q'_{k+1} - q'_{k+i+1}| \leq 2^{-k-1} + 2^{-k-1} = 2^{-k}$, so $x + y$ is a real number.

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- Trivially $-x = \langle -q_k : k \in \mathbb{N} \rangle$.
- We define $x \leq y$ if and only if $\forall k(q_k \leq q'_k + 2^{-k+1})$.
- It is straightforward to verify in RCA_0 that system $(\mathbb{R}, +, -, 0, 1, <)$ obey all the axioms for an ordered Abelian group. Note that formulas such as $x \leq y$, $x = y$, $x + y = z$ are Π^0_1 while $x < y$, $x \neq 0$,... are Σ^0_1.
- Multiplication of real numbers $x = \langle q_k : k \in \mathbb{N} \rangle$ and $y = \langle q'_k : k \in \mathbb{N} \rangle$ is defined by

$$x \cdot y = \langle q_{n+k} \cdot q'_{n+k} : k \in \mathbb{N} \rangle$$

  where n is as small as possible such that $2^n \geq |q_0| + |q'_0| + 2$. It is easy to verify that $x \cdot y$ is a real number.
- More details in Simpson's 2009.

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$$x \cdot y = \langle q_{n+k} \cdot q'_{n+k} : k \in \mathbb{N} \rangle$$

  where n is as small as possible such that $2^n \geq |q_0| + |q'_0| + 2$. It is easy to verify that $x \cdot y$ is a real number.
- More details in Simpson's 2009.

ACA$_0$

- Arithmetical formulas. A formula of L_2, or more generally a formula of $L_2(|M| \cup \mathcal{S}(M))$ where M is any L_2-structure, is said to be arithmetical if it contains no set quantifiers, i.e., all of the quantifiers appearing in the formula are number quantifiers.
- Arithmetical comprehension. The arithmetical comprehension scheme is the restriction of the comprehension scheme to arithmetical formulas $\varphi(n)$. Thus we have the universal closure of

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n))$$

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- ACA_0 is the subsystem of Z_2 whose axioms are the arithmetical comprehension scheme, the induction axiom and the basic axioms.
- The first order arithmetic(Z_1) is sometimes known as Peano Arithmetic(PA), let L_1 be the language of Z_1. It's easy to see that for any L_1-sentence σ, σ is a theorem of ACA_0 if and only if σ is a theorem of Z_1. In other wards, for any L_1-sentence, ACA_0 is a conservative extension of first order arithmetic.
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- Trees. Within RCA_0 we let

$$Seq = \omega^{<\omega} = \bigcup_{k \in \omega} \omega^k$$

denote the set of (codes for) finite sequences of natural numbers. For $\sigma, \tau \in \omega^{<\omega}$, there is $\sigma^\frown\tau \in \omega^{<\omega}$ which is the concatenation, σ followed by τ.
A tree is a set $T \subseteq \omega^{<\omega}$ such that any initial segment of a sequence in T belongs to T.
A path or infinite path through T is a function $f : \omega \to \omega$ such that for all $k \in \omega$, the initial sequence

$$f[k] = \langle f(0), f(1), ..., f(k-1) \rangle$$

belong to T.

WKL_0

- Weak König's Lemma. The following definitions are made in RCA_0. We use $\{0,1\}^{<\omega}$ or $2^{<\omega}$ to denote the full binary tress. Weak König's lemma is the following statement: Every infinite subtree of $2^{<\omega}$ has an infinite path.
- WKL_0 is defined to be the subsystem of Z_2 consisting of RCA_0 plus weak König's lemma.
- In fact, WKL_0 is strong enough to prove many well known nonconstructive theorems that are extremely important for mathematical practice but not probable in RCA_0.

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WKL_0

Theorem 5

Within RCA_0 one can prove that WKL_0 is equivalent to each of the following ordinary mathematical statements:
1. The Heine/Borel covering lemma: Every covering of the closed interval [0,1] by a sequence of open intervals has a finite subcovering.
2. Every covering of a compact metric space by a sequence of open sets has a finite subcovering.
3. The maximum principle: Every continuous real-valued function on [0,1], or on any compact metric space has, or attains, a supremum.
4. Gödel's completeness theorem: every finite, or countable, set of sentences in the predicate calculus has a countable model.
5. Every countable commutative ring has a prime ideal.
6. The separable Hahn/Banach theorem.

WKL_0

- We have seen that WLK_0 is much stronger than RCA_0 with respect to mathematical practice. Nevertheless, it can be shown that WKL_0 is the same strength as RCA_0 in a proof theoretic sense. Namely, the first order part of WKL_0 is the same as that of RCA_0, *viz.* Σ^0_1-PA.
- Another key conservation result is that WKL_0 is conservative over the formal system known as PRA or primitive recursive arithmetic, with respect to Π^0_2 sentences. In particular, we can find a primitive recursive function $f : \omega \to \omega$ such that $\varphi(m, f(m))$ holds for all $m \in \omega$. It means that a large portion of infinitistic mathematical practice is in fact finitistically reducible. Thus we have a significant partial realization of Hilbert's program of finitistic reductionism.

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Seetapun Enigma

- Ramsey's Theorem. The following definitions are made in RCA_0. For any countable $X \subseteq \omega$ and $k \in \omega$, let $[X]^k$ be the set of all increasing sequences of length k of elements of X. In symbols, $s \in [X]^k$ if and only if $s \in \omega^k$ and $\forall j < k(s(j) \in X \wedge \forall i < j(s(i) < s(j)))$. By $\omega \to (\omega)^k_l$, we mean the assertion that for some $l \in \omega$ and all $f : [\omega]^k \to l$, there exists $i < l$ and an infinite set $X \subseteq \omega$ such that $f(m_1, ..., m_k) = i$ for all $\langle m_1, ..., m_2 \rangle \in [X]^k$.
- It's easy to show that for each $k, l \in \omega$, $\omega \to (\omega)^k_l$ is provable in ACA_0.
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- In 1980's English Logician Seetapun showed that there is an ω-model of $\mathrm{WKL}_0 + \omega \to (\omega)^2_l$ in which ACA_0 fails.
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- Π^1_1-CA$_0$ is the subsystems of Z$_2$ whose axioms are the basic axioms, the induction axiom, and the comprehension scheme restricted to L_2-formulas $\varphi(n)$ which are Π^1_1. Thus we have the universal closure of

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Example:

Within ACA$_0$ we define a countable linear ordering to be a structure $\langle A, <_A\rangle$, where $A \subseteq \omega$ and $<_A \subseteq A \times A$ is an irreflexive linear ordering of A. The countable linear ordering $\langle A, <_A\rangle$ is called countable well ordering if there is no sequence $\langle a_n : n \in \omega\rangle$ of elements of A such that $a_{n+1} <_A a_n$ for all $n \in \omega$. Two countable well ordering $\langle A, <_A\rangle$, $\langle B, <_B\rangle$ are said to be comparable if they are isomorphic if one of them is isomorphic to a proper initial segment of the order.
The fact that any countable well ordering are comparable turn out to be proved in Π^1_1-CA$_0$ but not in ACA$_0$. Thus Π^1_1-CA$_0$, but not ACA$_0$, is strong enough to develop a good theory of countable ordinal numbers.

ATR_0

- Arithmetical Transfinite Recursion(ATR). Consider an arithmetical formula $\theta(n, X)$ with a free number variable n and a free set variable X. Note that $\theta(n, X)$ may also contain parameters. Fixing these parameters, we may view θ as an "arithmetical operator" $\Theta : P(\omega) \to P(\omega)$, defined by

$$\Theta(X) = \{n \in \omega : \theta(n, X)\}.$$

  Now let $\langle A, <_A \rangle$ be any countable well ordering, and consider the set $Y \subseteq \omega$ obtained by transfinitely iterating the operator Θ along $\langle A, <_A \rangle$. This set Y is defined by the following conditions: $Y \subseteq \omega \times A$ and, for each $a \in A$, $Y_a = \Theta(Y^a)$, where $Y_a = \{m : (m, a) \in Y\}$ and
  $Y^a = \{(n, b) : n \in Y_a \wedge b <_A a\}$.
  ATR is the axiom scheme asserting that such a set Y exists.

ATR_0

- Informally, arithmetical transfinite recursion can be described as the assertion that the Turing jump operator can be iterated along any countable well ordering starting any set.
- We define ATR_0 to consist of ACA_0 plus the scheme of arithmetical transfinite recursion. It is easy to see that ATR_0 is a subsystem of Π^1_1-CA_0. Furthermore, it is a proper subsystem.
- ATR_0 is sufficiently strong to accommodate a large portion of mathematical practice beyond ACA_0, including many basic theorems of infinitary combinatorics and classical descriptive theory.

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Big Five and Programs of Foundation

As a perhaps not unexpected byproduct, we note that these same five systems turn out to correspond to various well known, philosophically motivated programs in foundations of mathematics, as indicated in following table.

RCA_0	Constructivism	Bishop
WKL_0	Finitistic reductionism	Hilbert
ACA_0	Predicativism	Weyl, Feferman
ATR_0	Predicative reductionism	Friedman, Simpson
Π^1_1-CA_0	Impredicativity	Feferman *et al.*

Reverse Mathematics and Other Branches

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