Hypersequent Calculi for Propositional Modal Logic

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May 25, 2019
Outline

1. Preliminaries

2. Hypersequent Calculus for Modal Logics

3. From Frame Properties to Hypersequent Rules in Modal Logics

4. Summary and further interests
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2. Hypersequent Calculus for Modal Logics
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**Definition (Sequent)**

Let $M, N, \cdots$ vary on finite or empty multisets of well-formed formulas; a sequent $s$ is an object of the form: $M \Rightarrow N$. $M, N$ are called, respectively, the **antecedent** and the **succedent**.

**Definition (Interpretation)**

The interpretation $\tau$ of a sequent $M \Rightarrow N$ is: $(M \Rightarrow N)^\tau := \wedge M \rightarrow \vee N$

**Definition (Inference rules)**

The **inference rules** is arranged in either of these two forms:

$$
\frac{s'}{s} \qquad \frac{s' \quad s''}{s}
$$

$s'$ and $s''$ are called the **upper** sequents or the **premises** of the rule; $s$ is called the **lower** sequent or the **conclusion** of the rule.
The Sequent Calculus $\text{GcI}$

**Axioms**

$$Ax : p \Rightarrow p$$

**Structural Rules**

$$\frac{M \Rightarrow N}{\alpha, M \Rightarrow N} \quad \text{LW}$$

$$\frac{\alpha, M \Rightarrow N}{\alpha, \alpha, M \Rightarrow N} \quad \text{LC}$$

$$\frac{\alpha, M \Rightarrow N}{M \Rightarrow N, \alpha} \quad \text{Cut}_\alpha$$

$$\frac{M \Rightarrow N}{M \Rightarrow N, \alpha} \quad \text{RW}$$

$$\frac{M \Rightarrow N, \alpha, \alpha}{M \Rightarrow N, \alpha} \quad \text{RC}$$

**Logical Rules**

*Propositional Rules*

$$\frac{\alpha_i, M \Rightarrow N}{\alpha_0 \land \alpha_1, M \Rightarrow N} \quad \text{L}^\land$$

$$\frac{\alpha, M \Rightarrow N, \beta, M \Rightarrow N}{\alpha \lor \beta, M \Rightarrow N} \quad \text{L}^\lor$$

$$\frac{\alpha \rightarrow \beta, M \Rightarrow N}{L \rightarrow}$$

$$\frac{M \Rightarrow N, \alpha}{M \Rightarrow N, \alpha \land \beta} \quad \text{R}^\land$$

$$\frac{M \Rightarrow N, \alpha_i}{M \Rightarrow N, \alpha} \quad \text{R}^\lor$$

$$\frac{M \Rightarrow N, \alpha_0 \lor \alpha_1}{\alpha, M \Rightarrow N, \beta} \quad \text{R} \rightarrow$$

$$\frac{\alpha, M \Rightarrow N, \beta}{M \Rightarrow N, \alpha \rightarrow \beta} \quad \text{R} \leftarrow$$
Some Notions

**Definition (Contexts, principal and auxiliary formula)**

In rules, the formula occurrences in $M, N, \cdots$ are called *contexts* or *side formulas*; the formula occurrence of the conclusion that is not a side formula is the *principal* or *main* formula; the formula occurrences in the premises that are not side formulas are called *auxiliary*.

**Definition (Derivation)**

A derivation is a tree of sequents satisfying:

1. The topmost sequents are $Ax$ or $L\bot$.
2. Every sequent except the lowest one is an upper sequent of an instance of a inference rule.
Some Notions

**Definition (Derivation height)**

For a derivation $d$, the derivation height $h(d)$ of $d$ is defined inductively as follows:

- $d \equiv M \Rightarrow N$, $h(d) = 0$

- $\frac{M' \Rightarrow N'}{R}$, where the derivation height of $M' \Rightarrow N'$ is $d_1$, then $h(d) = h(d_1) + 1$

- $\frac{M' \Rightarrow N' \quad M'' \Rightarrow N''}{R}$, where the derivation height of $M' \Rightarrow N'$ and $M'' \Rightarrow N''$ are $d_1$ and $d_2$, then $h(d) = \max(h(d_1), h(d_2)) + 1$

**Definition ((height-preserving) eliminable and admissible)**

A rule $R$ is said to be *(height-preserving) eliminable* if, whenever there exists a derivation of height $n$ of the premise of $R$, then there also exists a derivation of the conclusion of $R$, that does not contain any application of $R$(and with the height an most $n$).

If the rule $R$ does not belong to the calculus, but the condition above still holds, then $R$ is said to be *(height-preserving) admissible*. 
Some Notions

### Definition ((height-preserving) invertible)

For a logical rule $\mathcal{R}$ such that, given $M' \Rightarrow N'$, $\mathcal{R}$ allows to infer $M \Rightarrow N$. We called $\mathcal{R}$ a (height-preserving) invertible rule if when its inverse, i.e. the rule that allows us to infer $M' \Rightarrow N'$ from $M \Rightarrow N$, is (height-preserving) admissible.

### Fact

*For all formulas $\alpha$, and for all sequents $M \Rightarrow N$,*

$\vdash \alpha$ in $\text{Hcl}$, then $\vdash \alpha$ in $\text{Gcl}$. If $\vdash M \Rightarrow N$ in $\text{Gcl}$, then $\vdash \wedge M \rightarrow \vee N$ in $\text{Hcl}$.

### Fact

- The cut-rule is eliminable in $\text{Gcl}$.
- Each formula in the derivation of $\Gamma \Rightarrow \Delta$ in $\text{Gcl}$ is a subformula of $\Gamma, \Delta$.
- Derivability of a sequent $\Gamma \Rightarrow \Delta$ in the $\text{Gcl}$ is decidable.

We generate all possible finite derivation trees with endsequent $\Gamma \Rightarrow \Delta$. Starting with $\Gamma \Rightarrow \Delta$, we write all instance of rules that conclude it, then do the same for all the premisses of the last step. If there is one tree all leaves of which are axiom or conclusions of $L\bot$, the endsequent is derivable; if not, it is underivable.
Variants and Alternatives

We divide the possible reformulations of sequent calculus into *variants* and *alternatives*. We call $\mathbf{Gcl}$, $\mathbf{Gcl}^*$, $\mathbf{Gcl}^{**}$ the variant of the *multiset/set/sequence* alternative of Gentzen system for classical propositional logic.

\[
\frac{M, \alpha, \beta, N \Rightarrow P}{M, \beta, \alpha, N \Rightarrow P} \quad LE \quad \frac{M \Rightarrow N, \beta, \alpha, P}{M \Rightarrow N, \alpha, \beta, P} \quad RE
\]

$M, N$ in $\mathbf{Gcl}^*$ are sets of formulas. $\mathbf{Gcl}^{**}$ is obtained by adding the above two *exchange* rules to $\mathbf{Gcl}$ where $M, N$ are sequences of formulas.

Give adequate translation functions between sequences, sets and multisets, we can show they prove the same theorems, cf. [Troelstra and Schwichtenberg, 1996, p. 77].
G3cp, a variant of the multiset alternative

Axioms
Ax : p, M ⇒ N, p
L⊥ : ⊥, M ⇒ N

Logical Rules
Propositional Rules

\[ \frac{\alpha, \beta, M \Rightarrow N}{\alpha \land \beta, M \Rightarrow N} \quad L^\land \]
\[ \frac{\alpha, M \Rightarrow N, \beta, M \Rightarrow N}{\alpha \lor \beta, M \Rightarrow N} \quad L^\lor \]
\[ \frac{M \Rightarrow N, \alpha, \beta, M \Rightarrow N}{M \Rightarrow N, \alpha \land \beta} \quad R^\land \]
\[ \frac{M \Rightarrow N, \alpha, \beta}{M \Rightarrow N, \alpha \lor \beta} \quad R^\lor \]
\[ \frac{M \Rightarrow N, \alpha \Rightarrow \beta, M \Rightarrow N}{M \Rightarrow N, \alpha \Rightarrow \beta} \quad R^\rightarrow \]
**G3cp, a variant of the multiset alternative**

**Axioms**

\[ Ax : p, M \Rightarrow N, p \quad L\bot : \bot, M \Rightarrow N \]

**Logical Rules**

**Propositional Rules**

\[
\begin{align*}
\frac{\alpha, \beta, M \Rightarrow N}{\alpha \land \beta, M \Rightarrow N} & \quad \text{L}^\land \\
\frac{\alpha \lor \beta, M \Rightarrow N}{\alpha \lor \beta, M \Rightarrow N} & \quad \text{L}^\lor \\
\frac{M \Rightarrow N, \alpha \quad M \Rightarrow N, \beta}{\alpha \rightarrow \beta, M \Rightarrow N} & \quad \text{L}^\rightarrow \\
\frac{M \Rightarrow N, \alpha, \beta}{M \Rightarrow N, \alpha \lor \beta} & \quad \text{R}^\lor \\
\frac{M \Rightarrow N, \alpha \lor \beta}{M \Rightarrow N, \alpha \rightarrow \beta} & \quad \text{R}^\rightarrow
\end{align*}
\]

**Theorem ([Negri, 2001])**

*All rules of G3cp are height-preserving invertible.*

**Theorem ([Negri, 2001])**

*The rules of weakening, contraction and cut are height-preserving admissible.*
Sequent Calculus for Modal Logic

\[
\frac{M \Rightarrow \alpha}{\Box M \Rightarrow \Box \alpha} \quad k
\]

Amongst others, [Leivant, 1981], [Mints, 1990] and [Sambin and Valentini, 1982] agree on adding the rule \(k\) to \(G\cl\) to obtain \(Gk\) for the system \(K\), where \(\Box M = \{\Box \alpha | \alpha \in M\}\)

\[
\frac{\alpha, M \Rightarrow \alpha}{\Box \alpha, \Box M \Rightarrow \Box \alpha} \quad d4
\]

Goble in [Goble, 1974] introduced the calculus \(Gkd\), resulting from \(Gk\) by the addition of the rule \(d\). To obtain \(Gkd4\), it suffices to substitute the rule \(k\) in \(Gkd\) with the rule \(d4\), where \(M'\) results from \(M\) by prefixing zero or more formulas in \(M\) by the symbol \(\Box\).

\[
\frac{\alpha, M \Rightarrow N}{\Box \alpha, M \Rightarrow N} \quad t
\]

\(Gkt\) results from \(Gk\) by adjoining the rule \(t\), [Ohnishi and Matsumoto, 1957].

\[
\frac{M, \Box M \Rightarrow \alpha}{\Box M \Rightarrow \Box \alpha} \quad 4
\]

Adding 4 to \(G\cl\), we have \(Gk4\), [Sambin and Valentini, 1982].
Sequent Calculus for Modal Logic

\[
\begin{align*}
M \Rightarrow \Box N, \alpha & \quad \Box M \Rightarrow N, \Box \alpha \\
M, \Box M \Rightarrow \Box N, \Box T, \alpha & \quad b4 \quad M \Rightarrow \Box N \\
\Box M \Rightarrow \Box N, T, \Box \alpha & \quad db \quad M \Rightarrow \Box N
\end{align*}
\]

Following [Takano, 1992], \textbf{Gkb} and \textbf{Gkb4} result from \textbf{Gcl} by including, respectively, the rules \textit{b} and \textit{b4}. \textbf{Gktb} and \textbf{Gkdb} are obtained from \textbf{Gkb} by adjoining the rule \textit{t} and \textit{db} respectively.

\[
\begin{align*}
\Box M \Rightarrow \alpha & \quad s4
\end{align*}
\]

\textbf{Gs4} results from \textbf{Gcl} by including the rule \textit{t} and the rule \textit{s4}, [Ohnishi and Matsumoto, 1957].
Ohnishi and Matsmoto’s Calculus for **S5**

Ohnishi and Matsmoto’s Calculus for **S5** can be obtained from the calculus **Gs4** by modifying the rule **s4** in the following way:

\[
\begin{array}{c}
\Box M \Rightarrow \Box N, \alpha \\
\hline
\Box M \Rightarrow \Box N, \Box \alpha
\end{array}
\]

\textit{s5om}
Ohnishi and Matsmoto’s Calculus for S5

Ohnishi and Matsmoto’s Calculus for S5 can be obtained from the calculus Gs4 by modifying the rule s4 in the following way:

\[
\begin{array}{c}
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\Box M \Rightarrow \Box N, \Box \alpha
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\]

s5om

Unfortunately this calculus is not cut-free as the following proof of the axiom b shows.
Ohnishi and Matsumoto’s Calculus for S5

Ohnishi and Matsumoto’s Calculus for S5 can be obtained from the calculus Gs4 by modifying the rule s4 in the following way:

\[
\frac{\Box M \Rightarrow \Box N, \alpha}{\Box M \Rightarrow \Box N, \Box \alpha} \quad s5om
\]

Unfortunately this calculus is not cut-free as the following proof of the axiom b shows.

Example

\[
\begin{array}{cccccc}
\alpha \Rightarrow \alpha & \text{L} \neg & \Box \neg \alpha \Rightarrow \Box \neg \Box \neg \alpha & \text{R} \neg & \Box \neg \alpha \Rightarrow \Box \neg \alpha, \neg \Box \neg \alpha & \text{L} \neg \\
\alpha, \neg \alpha \Rightarrow & \text{t} & \Box \neg \alpha \Rightarrow \Box \neg \alpha, \neg \Box \neg \alpha & \Box \neg \neg \neg \alpha \Rightarrow \Box \neg \Box \neg \alpha & \text{R} \neg \\
\alpha \Rightarrow \neg \Box \neg \alpha & \text{R} \neg & \neg \Box \neg \neg \neg \alpha \Rightarrow \neg \Box \neg \Box \neg \neg \alpha & \alpha \Rightarrow \Box \neg \neg \neg \neg \neg \neg \alpha & \text{cut} \neg \neg \neg \neg \alpha
\end{array}
\]

It appears clearly that we come to a halt after the first inference: on the left side, \( \alpha \) is not preceded by a connective or modal operator, on the right side, we cannot apply the rule s5om, since the antecedent is not boxed. Therefore, in order to reach the axioms, we need to use the cut-rule.
Ohnishi and Matsumoto’s Calculus for S5

**Theorem ([Ono, 1998])**

The sequent calculus $Gs5$ has the subformula property.

**Definition (Acceptable or analytic cut)**

$$M \Rightarrow N, \alpha \quad \alpha, M' \Rightarrow N'$$

$\frac{}{M, M' \Rightarrow N, N'}$

An application of the cut rule is acceptable, if the cut formula $\alpha$ is a subformula of a formula in $M, M', N, N'$.

**Theorem ([Ono, 1998])**

For any sequent $s$, if $s$ is provable in $Gs5$, then there exists a proof of $s$ in $Gs5$ such that every application of the cut rule in it is acceptable.

*The most important proof-theoretic property is the subformula property, and the most convenient way of showing the subformula property is to show the cut elimination theorem.* [Ono, 1998]
What a "Good" Proof-Theoretical Framework Should Be Like

1. It should be able to handle a great diversity of logics of different types.
2. Because of the proof-theoretical nature and the expected generality, the framework should be independent of any particular semantics.
3. The structures used in the framework should be built from the formulae of the logic and should not be too complicated.
4. The rules of inference should have a small, fixed number of premises, and their application should have a local nature. In other words: the applicability of a rule should depend only on the structure of the premises and not on the way they have been obtained.
5. Since there should be something common to all the various connectives, we call "conjunction", "disjunction", "implication" and "negation", the corresponding rules should be as standard as possible. The difference between logics should be due to some other rules, which are independent of any particular connective. Such rules are usually called "structural rules". This is also known as Došen's Principle.
6. The proof systems constructed within the framework should give us better understanding of the corresponding logics and the difference between them.
In [Avron, 1996], Avron listed 6 properties that good proof-systems should have.
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2 Hypersequent Calculus for Modal Logics

3 From Frame Properties to Hypersequent Rules in Modal Logics

4 Summary and further interests
Hypersequent calculi is a "one step further" generalised form of ordinary sequent calculi invented independently by [Pottinger, 1983] and [Avron, 1987]. Here, we define hypersequents as finite multisets of ordinary Gentzen's sequents.

**Definition (Hypersequent)**

A *hypersequent* is a syntactic object of the form

\[ \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n \]

where for all \( i = 1, \cdots, n \), \( \Gamma_i, \Delta_i \) are multisets of formulas, \( \Gamma_i \Rightarrow \Delta_i \) is an ordinary sequent called a *component* of the hypersequent. We use

- \( G, H \) for hypersequents.
- \( G \mid s \) or \( s \mid G \) (resp. \( G \mid \Gamma \Rightarrow \Delta \) or \( \Gamma \Rightarrow \Delta \mid G \)) for hypersequents with displayed sequent \( s \) (resp. \( \Gamma \Rightarrow \Delta \)).
The Hypersequent Calculus $\text{HCcl}$

**Axiomas**

\[
G | \alpha, \Gamma \Rightarrow \Delta, \alpha
\]

**Logical Rules**

*Propositional Rules*

\[
\begin{align*}
G | \alpha_i, M \Rightarrow N & \quad L^\wedge \\
G | \alpha_0 \land \alpha_1, M \Rightarrow N & \\
G | \alpha, M \Rightarrow N & \quad G | \beta, M \Rightarrow N \\
G | \alpha \lor \beta, M \Rightarrow N & \\
G | M \Rightarrow N, \alpha & \quad G | \beta, M \Rightarrow N \\
G | \alpha \rightarrow \beta, M \Rightarrow N & \\
G | M \Rightarrow N, \alpha & \quad G | M \Rightarrow N, \neg \alpha \\
G | M, \neg \alpha \Rightarrow N & \quad L^- \\
\end{align*}
\]

\[
\begin{align*}
G | M \Rightarrow N, \alpha & \quad G | M \Rightarrow N, \beta \\
G | M \Rightarrow N, \alpha \land \beta & \quad R^\wedge \\
G | M \Rightarrow N, \alpha_i & \quad R^\lor \\
G | M \Rightarrow N, \alpha_0 \lor \alpha_1 & \\
G | \alpha, M \Rightarrow N, \beta & \quad R^\rightarrow \\
G | \alpha \rightarrow \beta, M \Rightarrow N & \\
G | M, \alpha \Rightarrow N & \quad R^- \\
G | M \Rightarrow N, \neg \alpha & \\
\end{align*}
\]
The Hypersequent Calculus $\text{HC}$

**Structural Rules**

$$
\begin{align*}
\frac{G \mid M \Rightarrow N, \alpha \quad H \mid \alpha, P \Rightarrow Q}{G \mid H \mid M, P \Rightarrow N, Q} & \text{Cut}_\alpha \\
\end{align*}
$$

**Internal Structural Rules**

$$
\begin{align*}
\frac{G \mid M \Rightarrow N}{G \mid \alpha, M \Rightarrow N} & \text{ILW} \\
\frac{G \mid \alpha, \alpha, M \Rightarrow N}{G \mid \alpha, M \Rightarrow N} & \text{ILC} \\
\end{align*}
$$

**External Structural Rules**

$$
\begin{align*}
\frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta} & \text{EC} \\
\frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta} & \text{EW} \\
\end{align*}
$$
The Hypersequent Calculus for S5

The Hypersequent Calculus HC_{S5} for S5 results from the hypersequent version of the sequent calculus for S4 by including the rule "modalized splitting rule" (MS). A survey of hypersequent calculus for S5 can be found in [Bednarska and Indrzejczak, 2015], [Poggiolesi, 2011].

Example

\[
\begin{align*}
\frac{G \mid \Box \Gamma_1, \Gamma_2 \Rightarrow \Box \Delta_1, \Delta_2}{G \mid \Box \Gamma_1 \Rightarrow \Box \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2} \quad MS
\end{align*}
\]

The Hypersequent Calculus HC_{S5} for S5 results from the hypersequent version of the sequent calculus for S4 by including the rule "modalized splitting rule" (MS). A survey of hypersequent calculus for S5 can be found in [Bednarska and Indrzejczak, 2015], [Poggiolesi, 2011].

Example

\[
\begin{align*}
\frac{\alpha \Rightarrow \alpha}{\alpha, \neg \alpha \Rightarrow} \quad L_t \quad \frac{\Box \alpha, \neg \Box \alpha \Rightarrow}{\Box \alpha \Rightarrow} \quad MS \\
\frac{\alpha \Rightarrow | \Box \neg \alpha \Rightarrow}{\alpha \Rightarrow | \Rightarrow \neg \Box \neg \alpha} \quad R_k \\
\frac{\alpha \Rightarrow | \Rightarrow \neg \neg \neg \neg \alpha}{\Box \alpha \Rightarrow \Box \neg \Box \neg \neg \alpha \Rightarrow} \quad IW \\
\frac{\neg \Box \alpha \Rightarrow \Box \neg \Box \neg \neg \alpha \Rightarrow}{\neg \Box \alpha \Rightarrow \Box \neg \alpha \Rightarrow} \quad EC
\end{align*}
\]
Hypersequent Calculi for Modal Logics Extending $\mathbf{S4}$,

The modal hyperstructural rule "Restricted Modal Splitting" and "Modal Communication"[Kurokawa, 2014].

\[
\frac{G \mid \Box \Gamma, \Box \Delta \Rightarrow}{G \mid \Box \Gamma \Rightarrow \mid \Box \Delta \Rightarrow} \quad \text{RMS} \quad \frac{G \mid \Sigma, \Box \Gamma \Rightarrow \Pi}{G \mid \Sigma, \Box \Delta \Rightarrow \Pi} \quad \frac{G \mid \Theta, \Box \Delta \Rightarrow \Lambda}{G \mid \Theta, \Box \Gamma \Rightarrow \Lambda} \quad \text{MC}
\]

- Axioms for $\mathbf{S4.2}: \mathbf{S4+ .2}$: $\neg \square \neg \square A \rightarrow \square \neg \square \neg A$
- Axioms for $\mathbf{S4.3}: \mathbf{S4+ .3}$: $\square (\square A \rightarrow B) \vee \square (\square B \rightarrow A)$
- $\mathbf{HCs4.2}$: $\mathbf{HCs4+RMS}$
- $\mathbf{HCs4.3}$: $\mathbf{HCs4+MC}$
Outline

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[Lahav, 2013] provides a general method for generating cut-free and/or analytic hypersequent Gentzen-type calculi for a variety of normal modal logics. We use classical first-order language to formulate the frame properties.

**Definition (Simple Frame Properties)**

Simple frame properties are formulated by formulas of the form

$$\forall w_1 \ldots w_n \exists u \varphi$$

where $\varphi$ consists of:

- Atomic formulas of the form $w_i Ru$ or $w_i = u$.
- Conjunctions and disjunctions.

Reflexivity is simple: $\forall w_1 \exists u (w_1 Ru \land w_1 = u)$
Some Examples of Simple Frame Properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>Seriality</td>
<td>$\forall w_1 \exists u (w_1 Ru)$</td>
</tr>
<tr>
<td>Directedness</td>
<td>$\forall w_1 \forall w_2 \exists u (w_1 Ru \land w_2 Ru)$</td>
</tr>
<tr>
<td>Degenerateness</td>
<td>$\forall w_1 \forall w_2 \exists u (w_1 = u \land w_2 = u)$</td>
</tr>
<tr>
<td>Universality</td>
<td>$\forall w_1 \forall w_2 \exists u (w_1 Ru \land w_2 = u)$</td>
</tr>
<tr>
<td>Linearity</td>
<td>$\forall w_1 \forall w_2 \exists u (w_1 Ru \land w_2 = u) \lor (w_2 Ru \land w_1 = u)$</td>
</tr>
<tr>
<td>Bounded Cardinality</td>
<td>$\forall w_1 \cdots \forall w_n \exists u \bigvee_{1 \leq i &lt; j \leq n} (w_i = u \land w_j = u)$</td>
</tr>
<tr>
<td>Bounded Top Width</td>
<td>$\forall w_1 \cdots \forall w_n \exists u \bigvee_{1 \leq i &lt; j \leq n} (w_i Ru \land w_j Ru)$</td>
</tr>
<tr>
<td>Bounded Width</td>
<td>$\forall w_1 \cdots \forall w_n \exists u \bigvee_{1 \leq i, j \leq n; i \neq j} (w_i Ru \land w_j = u)$</td>
</tr>
</tbody>
</table>
Step 1. Extract the **normal form** of $\forall w_1 \cdots w_n \exists u \varphi$, a set $\{\langle R_1, E_1 \rangle, \cdots, \langle R_m, E_m \rangle\}$ such that

$$
\varphi \equiv \bigvee_{1 \leq i \leq m} \left( \bigwedge_{j \in R_i} w_j Ru \land \bigwedge_{j \in E_i} w_j = u \right)
$$

- $\forall w_1 \forall w_2 \exists u (w_1 Ru \land w_2 Ru)$
  - $\{\{1, 2\}, \emptyset\}$
- $\forall w_1 \forall w_2 \exists u (w_1 Ru \land w_2 = u)$
  - $\{\{1\}, \{2\}\}$
- $\forall w_1 \forall w_2 \exists u (w_1 Ru \land w_2 = u) \lor (w_2 Ru \land w_1 = u)$
  - $\{\{1\}, \{2\}, \{2\}, \{1\}\}$
- $\forall w_1 \cdots \forall w_n \exists u \bigvee_{1 \leq i < j \leq n} (w_i = u \land w_j = u)$
  - $\{\emptyset, \{i, j\} | 1 \leq i < j \leq n\}$
Step 2. For a normal form \( \{ \langle R_1, E_1 \rangle, \cdots, \langle R_m, E_m \rangle \} \) construct the following rule and add it to \( HC_k \):

\[
\begin{align*}
H \mid \Gamma_{E_1}, \Gamma'_{R_1} &\Rightarrow \Delta_{E_1} & \cdots & H \mid \Gamma_{E_m}, \Gamma'_{R_m} &\Rightarrow \Delta_{E_m} \\
H \mid \Gamma_1, \Box \Gamma'_{1} &\Rightarrow \Delta_1 & \cdots & \Gamma_n, \Box \Gamma'_{n} &\Rightarrow \Delta_n 
\end{align*}
\]

Notation: \( \Gamma \{ i_1, \cdots, i_k \} := \Gamma_{i_1}, \cdots, \Gamma_{i_n} \)
From Simple Frame Properties to Hypersequent Rules

**Step 2.** For a normal form \( \{ \langle R_1, E_1 \rangle, \cdots, \langle R_m, E_m \rangle \} \) construct the following rule and add it to \( \text{HCK} \):

\[
H \mid \Gamma_{E_1}, \Gamma'_{R_1} \Rightarrow \Delta_{E_1} \quad \cdots \quad H \mid \Gamma_{E_m}, \Gamma'_{R_m} \Rightarrow \Delta_{E_m}
\]

\[
H \mid \Gamma_1, \Box \Gamma'_1 \Rightarrow \Delta_1 \quad \cdots \quad H \mid \Gamma_n, \Box \Gamma'_n \Rightarrow \Delta_n
\]

Notation: \( \Gamma \{ i_1, \ldots, i_k \} := \Gamma_{i_1}, \ldots, \Gamma_{i_n} \)

- **Directedness**  \( \forall w_1 \forall w_2 \exists u \text{ } (w_1 Ru \land w_2 Ru) \)
- **Universality**  \( \forall w_1 \forall w_2 \exists u \text{ } (w_1 Ru \land w_2 = u) \)
- **Linearity**  \( \forall w_1 \forall w_2 \exists u \text{ } (w_1 Ru \land w_2 = u) \lor (w_2 Ru \land w_1 = u) \)
- **Bounded Cardinality**  \( \forall w_1 \cdots \forall w_n \exists u \text{ } \lor_{1 \leq i < j \leq n}(w_i = u \land w_j = u) \)

In the presence of the weakening rules, \( \Gamma_i, \Gamma'_i, \Delta_i \)'s that appear only in the conclusion can be discarded.
Main Result

Definition (Strong Soundness and Completeness)

\[
\Gamma \vdash_{Local} A \iff \vdash \Gamma \Rightarrow A \\
\Gamma \vdash_{Global} A \iff \{ \Rightarrow B \mid B \in \Gamma \} \vdash \Rightarrow A
\]

where $\Gamma \vdash_{Local} A$ means $A$ holds in every world in which $\Gamma$ holds; $\Gamma \vdash_{Global} A$ means $A$ holds in every world if $\Gamma$ holds in every world.

Definition (Strong Cut-Admissibility)

Cut can be confined to apply only on formulas that appear in the assumptions.

Theorem

*The constructed hypersequent calculus is sound and complete for the modal logic, and it enjoys cut-admissibility.*
Corollary

All modal logics characterized by finite sets of simple frame properties are decidable.

Proof.

Cut-admissibility $\rightarrow$ Subformula property $\rightarrow$ We can check one by one all possible proofs candidates.
Transitivity and Symmetry

### Definition (Simple Frame Properties)

Simple frame properties are formulated by formulas of the form

$$\forall w_1 \cdots w_n \exists u \varphi$$

where $\varphi$ consists of:

- Atomic formulas of the form $w_iRu$ or $w_i = u$.
- Conjunctions and disjunctions.

- Simple properties are *monotone increasing* (preserved under enrichment of $R$).
- Transitivity and symmetry are *not simple*.

We have to change the basic calculus:

\[
\begin{align*}
H \vdash \Gamma \Rightarrow A & \quad H \vdash \Gamma, \Box \Gamma \Rightarrow A & \quad H \vdash \Gamma \Rightarrow A, \Box \Delta \\
\frac{H \vdash \Box \Gamma \Rightarrow \Box A}{H \vdash \Gamma, \Box \Gamma \Rightarrow \Box A} & \quad \frac{H \vdash \Box \Gamma \Rightarrow \Box A}{H \vdash \Gamma \Rightarrow \Box A, \Box \Delta} & \quad \frac{H \vdash \Box \Gamma \Rightarrow \Box A, \Box \Delta}{H \vdash \Gamma \Rightarrow \Box A, \Box \Delta}
\end{align*}
\]

K

K4

KB
Transitivity

For a normal form \( \{ \langle R_1, E_1 \rangle, \ldots, \langle R_m, E_m \rangle \} \) construct the following rule:

\[
H \mid \Gamma_{E_1}, \Gamma'_{R_1}, \square \Gamma'_{R_1} \Rightarrow \Delta_{E_1} \quad \cdots \quad H \mid \Gamma_{E_m}, \Gamma'_{R_m}, \square \Gamma'_{R_m} \Rightarrow \Delta_{E_m}
\]

\[
H \mid \Gamma_1, \square \Gamma_1' \Rightarrow \Delta_1 \quad \cdots \quad H \mid \Gamma_n, \square \Gamma_n' \Rightarrow \Delta_n
\]

For example:

\[
H \mid \Gamma_2, \Gamma_1', \square \Gamma_1' \Rightarrow \Delta_2 \quad H \mid \Gamma_1, \Gamma_2', \square \Gamma_2' \Rightarrow \Delta_1
\]

\[
H \mid \Gamma_1, \square \Gamma_1' \Rightarrow \Delta_1 \quad H \mid \Gamma_2, \square \Gamma_2' \Rightarrow \Delta_2
\]

Linearity

We have:

- Strong soundness and completeness.
- Strong cut-admissibility.
- Decidability.
Symmetry

For a normal form \( \{ \langle R_1, E_1 \rangle, \cdots, \langle R_m, E_m \rangle \} \) construct the following rule:

\[
\frac{H \mid \Gamma_{E_1, R_1} \Rightarrow \Delta_{E_1, \Box \Delta'_{R_1}} \cdots H \mid \Gamma_{E_m, R_m} \Rightarrow \Delta_{E_m, \Box \Delta'_{R_m}}}{H \mid \Gamma_1, \Box \Gamma'_1 \Rightarrow \Delta_1, \Delta'_1 \mid \cdots \mid \Gamma_n, \Box \Gamma'_n \Rightarrow \Delta_n, \Delta'_n}
\]

For example:

\[
\begin{align*}
H \mid \Gamma'_1 \Rightarrow \Box \Delta'_1 & \\
\frac{H \mid \Box \Gamma'_1 \Rightarrow \Delta'_1}{H \mid \Box \Gamma'_1 \Rightarrow \Delta'_1} & \quad \text{Seriality} \\
H \mid \Gamma_1, \Box \Gamma'_1 \Rightarrow \Delta_1, \Box \Delta'_1 & \\
\frac{H \mid \Gamma_1, \Box \Gamma'_1 \Rightarrow \Delta_1, \Box \Delta'_1}{H \mid \Gamma_1, \Box \Gamma'_1 \Rightarrow \Delta_1, \Delta'_1} & \quad \text{Reflexivity} \\
H \mid \alpha, M \Rightarrow & \\
\frac{H \mid \alpha, M \Rightarrow \Box \Delta'_{R_1}}{H \mid \Box \alpha, \Box M \Rightarrow} & \quad HCd \\
H \mid \alpha, M \Rightarrow N & \\
\frac{H \mid \alpha, M \Rightarrow N}{H \mid \Box \alpha, M \Rightarrow N} & \quad HCt
\end{align*}
\]

- Cut-admissibility does not hold (even for the basic calculus).
- All constructed calculi still enjoy the subformula property.
- Decidability still follows.
Outline

1. Preliminaries
2. Hypersequent Calculus for Modal Logics
3. From Frame Properties to Hypersequent Rules in Modal Logics
4. Summary and further interests
Summary and further interests

- The successful application of gentzen-style sequent calculus for classical propositional logic and modal logic till up to $S4$.
- The "one-step-further" proof theoretical framework, hypersequent calculus for propositional modal logic.
- A general method for generating cut-free and/or analytic hypersequent Gentzen-type calculi for a variety of normal modal logics.
Summary and further interests

- The successful application of gentzen-style sequent calculus for classical propositional logic and modal logic till up to S4.
- The "one-step-further" proof theoretical framework, hypersequent calculus for propositional modal logic.
- A general method for generating cut-free and/or analytic hypersequent Gentzen-type calculi for a variety of normal modal logics.

Further interests:

- Other proof theoretical framework that address modal logic.
- Methods of proving cut-elimination.
- Other consequences of the cut elimination theorem, interpolation theorem, conservativity, disjunction property. A brief discuss can be found in [Ono, 1998].
- Epistemic logic.
References I

A constructive analysis of RM.

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Hypersequent calculi for s5: The methods of cut elimination.
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Gentzen systems for modal logic.

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Referenced II


References IV