Hypersequent Calculi for Propositional Modal Logic

Xuezhe Dang

Department of Philosophy
Peking University

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Outline

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The Sequent Calculus **Gcl**

Definition (Sequent)

Let $M, N, \cdots$ vary on finite or empty multisets of well-formed formulas; s *sequent* is an object of the form: $M \Rightarrow N$. M, N are called, respectively, the *antecedent* and the *succedent*.

Definition (Interpretation)

The *interpretation* τ of a sequent $M \Rightarrow N$ is : $(M \Rightarrow N)^\tau := \bigwedge M \rightarrow \bigvee N$

Definition (Inference rules)

The *inference rules* is arranged in either of these two forms:

$$\frac{s'}{s} \qquad \frac{s' \quad s''}{s}$$

s' and s'' are called the *upper* sequents or the *premises* of the rule; s is called the *lower* sequent or the *conclusion* of the rule.

The Sequent Calculus **Gcl**

Axiomas

$Ax : p \Rightarrow p$ $\qquad$ $L\bot : \bot \Rightarrow$

Structural Rules

$$\frac{M \Rightarrow N}{\alpha, M \Rightarrow N}\ LW \qquad \frac{M \Rightarrow N}{M \Rightarrow N, \alpha}\ RW$$

$$\frac{\alpha, \alpha, M \Rightarrow N}{\alpha, M \Rightarrow N}\ LC \qquad \frac{M \Rightarrow N, \alpha, \alpha}{M \Rightarrow N, \alpha}\ RC$$

$$\frac{M \Rightarrow N, \alpha \quad \alpha, P \Rightarrow Q}{M, P \Rightarrow N, Q}\ Cut_\alpha$$

Logical Rules

Propositional Rules

$$\frac{\alpha_i, M \Rightarrow N}{\alpha_0 \wedge \alpha_1, M \Rightarrow N}\ L\wedge \qquad \frac{M \Rightarrow N, \alpha \quad M \Rightarrow N, \beta}{M \Rightarrow N, \alpha \wedge \beta}\ R\wedge$$

$$\frac{\alpha, M \Rightarrow N \quad \beta, M \Rightarrow N}{\alpha \vee \beta, M \Rightarrow N}\ L\vee \qquad \frac{M \Rightarrow N, \alpha_i}{M \Rightarrow N, \alpha_0 \vee \alpha_1}\ R\vee$$

$$\frac{M \Rightarrow N, \alpha \quad \beta, M \Rightarrow N}{\alpha \to \beta, M \Rightarrow N}\ L\to \qquad \frac{\alpha, M \Rightarrow N, \beta}{M \Rightarrow N, \alpha \to \beta}\ R\to$$

Some Notions

Definition (Contexts, principal and auxiliary formula)

In rules, the formula occurences in $M, N, \cdots$ are called *contexts* or *side formulas*; the formula occurrence of the conclusion that is not a side formula is the *principal* or *main* formula; the formula occurrences in the premises that are not side formulas are called *auxiliary*.

Definition (Derivation)

A derivation is a tree of sequents satisfying:

1. The topmost sequents are Ax or $L\bot$.
2. Every sequent except the lowest one is an upper sequent of an instance of a inference rule.

Some Notions

Definition (Derivation height)

For a derivation d, the derivation height $h(d)$ of d is defined inductively as follows:

- $d \equiv M \Rightarrow N$, $h(d) = 0$
- $d \equiv \dfrac{M' \Rightarrow N'}{M \Rightarrow N}\,\mathcal{R}$, where the derivation height of $M' \Rightarrow N'$ is d_1, then $h(d) = h(d_1) + 1$
- $d \equiv \dfrac{M' \Rightarrow N' \quad M'' \Rightarrow N''}{M \Rightarrow N}\,\mathcal{R}$, where the derivation height of $M' \Rightarrow N'$ and $M'' \Rightarrow N''$ are d_1 and d_2, then $h(d) = max(h(d_1), h(d_2)) + 1$

Definition ((height-preserving) eliminable and admissible)

A rule $\mathcal{R}$ is said to be *(height-preserving) eliminable* if, whenever there exists a derivation of height n of the premise of $\mathcal{R}$, then there also exists a derivation of the conclusion of $\mathcal{R}$, that does not contain any application of $\mathcal{R}$(and with the height an most n).

If the rule $\mathcal{R}$ does not belong to the calculus, but the condition above still holds, then $\mathcal{R}$ is said to be *(height-preserving) admissible*.

Some Notions

Definition ((height-preserving) invertible)

For a logical rule $\mathcal{R}$ such that, given $M' \Rightarrow N'$, $\mathcal{R}$ allows to infer $M \Rightarrow N$. We called $\mathcal{R}$ a *(height-preserving) invertible* rule if when its *inverse*, i.e. the rule that allows us to infer $M' \Rightarrow N'$ from $M \Rightarrow N$, is (height-preserving) admissible.

Fact

For all formulas α, and for all sequents $M \Rightarrow N$,

*if $\vdash \alpha$ in **Hcl**, then $\vdash \Rightarrow \alpha$ in **Gcl**. If $\vdash M \Rightarrow N$ in **Gcl**, then $\vdash \bigwedge M \to \bigvee N$ in **Hcl**.*

Fact

- *The cut-rule is eliminable in **Gcl**.*
- *Each formula in the derivation of $\Gamma \Rightarrow \Delta$ in **Gcl** is a subformula of Γ, Δ.*
- *Derivability of a sequent $\Gamma \Rightarrow \Delta$ in the **Gcl** is decidable.*

We generate all possible finite derivation trees with endsequent $\Gamma \Rightarrow \Delta$. Starting with $\Gamma \Rightarrow \Delta$, we write all instance of rules that conclude it, then do the same for all the premisses of the last step. If there is one tree all leaves of which are axiom or conclusions of $L\bot$, the endsequent is derivable; if not, it is underivable.

Variants and Alternatives

We divide the possible reformulations of sequent calculus into *variants* and *alternatives*.
We call **Gcl**, **Gcl***, **Gcl**** the variant of the *multiset*/*set*/*sequence* alternative of Gentzen system for classical propositional logic.

$$\frac{M, \alpha, \beta, N \Rightarrow P}{M, \beta, \alpha, N \Rightarrow P}\ LE \qquad \frac{M \Rightarrow N, \beta, \alpha, P}{M \Rightarrow N, \alpha, \beta, P}\ RE$$

M, N in **Gcl*** are sets of formulas. **Gcl**** is obtained by adding the above two *exchange* rules to **Gcl** where M, N are sequences of formulas.
Give adequate translation functions between sequences, sets and multisets, we can show they porve the same theorems, cf. [Troelstra and Schwichtenberg, 1996, p. 77].

G3cp, a variant of the multiset alternative

Axiomas

$Ax : p, M \Rightarrow N, p$ $\qquad$ $L\bot : \bot, M \Rightarrow N$

Logical Rules

Propositional Rules

$$\frac{\alpha, \beta, M \Rightarrow N}{\alpha \wedge \beta, M \Rightarrow N}\ L\wedge \qquad \frac{M \Rightarrow N, \alpha \quad M \Rightarrow N, \beta}{M \Rightarrow N, \alpha \wedge \beta}\ R\wedge$$

$$\frac{\alpha, M \Rightarrow N \quad \beta, M \Rightarrow N}{\alpha \vee \beta, M \Rightarrow N}\ L\vee \qquad \frac{M \Rightarrow N, \alpha, \beta}{M \Rightarrow N, \alpha \vee \beta}\ R\vee$$

$$\frac{M \Rightarrow N, \alpha \quad \beta, M \Rightarrow N}{\alpha \to \beta, M \Rightarrow N}\ L\to \qquad \frac{\alpha, M \Rightarrow N, \beta}{M \Rightarrow N, \alpha \to \beta}\ R\to$$

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Theorem ([Negri, 2001])

*All rules of **G3cp** are height-preserving invertible.*

Theorem ([Negri, 2001])

The rules of weakening, contraction and cut are height-preserving admissible.

Sequent Calculus for Modal Logic

$$\frac{M \Rightarrow \alpha}{\Box M \Rightarrow \Box \alpha}\ k$$

Amongst others, [Leivant, 1981], [Mints, 1990] and [Sambin and Valentini, 1982] agree on adding the rule *k* to **Gcl** to obtain **Gk** for the system **K**, where $\Box M = \{\Box\alpha \mid \alpha \in M\}$

$$\frac{\alpha, M \Rightarrow}{\Box\alpha, \Box M \Rightarrow}\ d \quad \frac{M' \Rightarrow \alpha}{\Box M \Rightarrow \Box\alpha}\ d4$$

Goble in [Goble, 1974] introduced the calculus **Gkd**, resulting from **Gk** by the addition of the rule *d*. To obtain **Gkd4**, it suffices to substitute the rule *k* in **Gkd** with the rule *d*4, where M' results from *M* by prefixing zero or more formulas in *M* by the symbol $\Box$.

$$\frac{\alpha, M \Rightarrow N}{\Box\alpha, M \Rightarrow N}\ t$$

Gkt results from **Gk** by adjoining the rule *t*, [Ohnishi and Matsumoto, 1957].

$$\frac{M, \Box M \Rightarrow \alpha}{\Box M \Rightarrow \Box\alpha}\ 4$$

Adding 4 to **Gcl**, we have **Gk4**, [Sambin and Valentini, 1982].

Sequent Calculus for Modal Logic

$$\frac{M \Rightarrow \Box N, \alpha}{\Box M \Rightarrow N, \Box\alpha}\ b \quad \frac{M, \Box M \Rightarrow \Box N, \Box T, \alpha}{\Box M \Rightarrow \Box N, T, \Box\alpha}\ b4 \quad \frac{M \Rightarrow \Box N}{\Box M \Rightarrow N}\ db$$

Following [Takano, 1992], **Gkb** and **Gkb4** result from **Gcl** by including, respectively, the rules *b* and *b*4. **Gktb** and **Gkdb** are obtained from **Gkb** by adjoining the rule *t* and *db* respectively.

$$\frac{\Box M \Rightarrow \alpha}{\Box M \Rightarrow \Box\alpha}\ s4$$

Gs4 results from **Gcl** by including the rule *t* and the rule *s*4, [Ohnishi and Matsumoto, 1957].

Ohnishi and Matsmoto's Calculus for **S5**

Ohnishi and Matsmoto's Calculus for **S5** can be obtained from the calculus **Gs4** by modifying the rule *s*4 in the following way:

$$\frac{\Box M \Rightarrow \Box N, \alpha}{\Box M \Rightarrow \Box N, \Box\alpha}\ s5om$$

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Unfortunately this calculus is not cut-free as the following proof of the axiom *b* shows.

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Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha, \neg\alpha \Rightarrow}\ L\neg}{\alpha, \Box\neg\alpha \Rightarrow}\ t}{\alpha \Rightarrow \neg\Box\neg\alpha}\ R\neg \qquad \dfrac{\dfrac{\dfrac{\Box\neg\alpha \Rightarrow \Box\neg\alpha}{\Rightarrow \Box\neg\alpha, \neg\Box\neg\alpha}\ R\neg}{\Rightarrow \Box\neg\alpha, \Box\neg\Box\neg\alpha}\ s5om}{\neg\Box\neg\alpha \Rightarrow \Box\neg\Box\neg\alpha}\ L\neg}{\alpha \Rightarrow \Box\neg\Box\neg\alpha}\ cut_{\neg\Box\neg\alpha}}{\Rightarrow \alpha \rightarrow \Box\neg\Box\neg\alpha}\ R\rightarrow$$

It appears clearly that we come to a halt after the first inference: on the left side, α is not preceded by a connective or modal operator, on the right side, we cannot apply the rule *s*5*om*, since the antecedent is not boxed. Therefore, in order to reach the axioms, we need to use the cut-rule.

Ohnishi and Matsmoto's Calculus for **S5**

Theorem ([Ono, 1998])

*The sequent calculus **Gs5** has the subformula property.*

Definition (Acceptable or analytic cut)

$$\frac{M \Rightarrow N, \alpha \quad \alpha, M' \Rightarrow N'}{M, M' \Rightarrow N, N'}$$

An application of the cut rule is *acceptable*, if the cut formula α is a subformula of a formula in M, M', N, N'.

Theorem ([Ono, 1998])

*For any sequent s, if s is provable in **Gs5**, then there exists a proof of s in **Gs5** such that every application of the cut rule in it is acceptable.*

> *The most important proof-theoretic property is the subformula property, and the most convenient way of showing the subformula property is to show the cut elimination theorem.[Ono, 1998]*

What a "Good" Proof-Theoretical Framework Should Be Like

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1. It should be able to handle a great diversity of logics of different types.

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2. Because of the proof-theoretical nature and the expected generality, the framework should be independent of any particular semantics.

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3. The structures used in the framework should be built from the formulae of the logic and should not be too complicated.

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4. The rules of inference should have a small, fixed number of premises, and their application should have a local nature. In other words: the applicability of a rule should depend only on the structure of the premises and not on the way they have been obtained.

What a "Good" Proof-Theoretical Framework Should Be Like

In [Avron, 1996], Avron listed 6 properties that good proof-systems should have.

5. Since there should be something common to all the various connectives, we call "conjunction", "disjunction", "implication" and "negation", the corresponding rules should be as standard as possible. The difference between logics should be due to some other rules, which are independent of any particular connective. Such rules are usually called "structural rules". This is also known as **Došen's Principle**.

What a "Good" Proof-Theoretical Framework Should Be Like

In [Avron, 1996], Avron listed 6 properties that good proof-systems should have.

6. The proof systems constructed within the framework should give us better understanding of the corresponding logics and the difference between them.

Outline

Hypersequent

Hypersequent calculi is a "one step further" generalised form of ordinary sequent calculi invented independently by [Pottinger, 1983] and [Avron, 1987].
Here, we dene hypersequents as finite multisets of ordinary Gentzens sequents.

Definition (Hypersequent)

A *hypersequent* is a syntactic object of the form

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

where for all $i = 1, \cdots, n$, Γ_i, Δ_i are multisets of formulas, $\Gamma_i \Rightarrow \Delta_i$ is an ordinary sequent called a *component* of the hypersequent. We use

- G, H for hypersequents.
- $G \mid s$ or $s \mid G$ (resp. $G \mid \Gamma \Rightarrow \Delta$ or $\Gamma \Rightarrow \Delta \mid G$) for hypersequents with displayed sequent s (resp. $\Gamma \Rightarrow \Delta$).

The Hypersequent Calculus **HCcl**

Axiomas

$G \mid \alpha, \Gamma \Rightarrow \Delta, \alpha$ $\qquad$ $G \mid \Gamma, \bot \Rightarrow \Delta$

Logical Rules

Propositional Rules

$$\frac{G \mid \alpha_i, M \Rightarrow N}{G \mid \alpha_0 \wedge \alpha_1, M \Rightarrow N}\ L\wedge \qquad \frac{G \mid M \Rightarrow N, \alpha \quad G \mid M \Rightarrow N, \beta}{G \mid M \Rightarrow N, \alpha \wedge \beta}\ R\wedge$$

$$\frac{G \mid \alpha, M \Rightarrow N \quad G \mid \beta, M \Rightarrow N}{G \mid \alpha \vee \beta, M \Rightarrow N}\ L\vee \qquad \frac{G \mid M \Rightarrow N, \alpha_i}{G \mid M \Rightarrow N, \alpha_0 \vee \alpha_1}\ R\vee$$

$$\frac{G \mid M \Rightarrow N, \alpha \quad G \mid \beta, M \Rightarrow N}{G \mid \alpha \to \beta, M \Rightarrow N}\ L\to \qquad \frac{G \mid \alpha, M \Rightarrow N, \beta}{G \mid M \Rightarrow N, \alpha \to \beta}\ R\to$$

$$\frac{G \mid M \Rightarrow N, \alpha}{G \mid M, \neg\alpha \Rightarrow N}\ L\neg \qquad \frac{G \mid M, \alpha \Rightarrow N}{G \mid M \Rightarrow N, \neg\alpha}\ R\neg$$

The Hypersequent Calculus **HCcl**

Structural Rules

$$\frac{G \mid M \Rightarrow N, \alpha \quad H \mid \alpha, P \Rightarrow Q}{G \mid H \mid M, P \Rightarrow N, Q}\ Cut_\alpha$$

Internal Structural Rules

$$\frac{G \mid M \Rightarrow N}{G \mid \alpha, M \Rightarrow N}\ ILW \qquad \frac{G \mid M \Rightarrow N}{G \mid M \Rightarrow N, \alpha}\ IRW$$

$$\frac{G \mid \alpha, \alpha, M \Rightarrow N}{G \mid \alpha, M \Rightarrow N}\ ILC \qquad \frac{G \mid M \Rightarrow N, \alpha, \alpha}{G \mid M \Rightarrow N, \alpha}\ IRC$$

External Structural Rules

$$\frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\ EC \qquad \frac{G}{G \mid \Gamma \Rightarrow \Delta}\ EW$$

The Hypersequent Calculus for **S5**

$$\frac{G \mid \Box\Gamma_1, \Gamma_2 \Rightarrow \Box\Delta_1, \Delta_2}{G \mid \Box\Gamma_1 \Rightarrow \Box\Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ MS$$

The Hypersequent Calculus **HCs5** for **S5** results from the hypersequent version of the sequent calculus for **S4** by including the rule "modalized splitting rule"(*MS*). A survey of hypersequent calculus for S5 can be found in [Bednarska and Indrzejczak, 2015],[Poggiolesi, 2011].

Example

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\alpha \Rightarrow \alpha}{\alpha, \neg\alpha \Rightarrow}\ L\neg}{\alpha, \Box\neg\alpha \Rightarrow}\ t}{\alpha \Rightarrow \ \mid \Box\neg\alpha \Rightarrow}\ MS}{\alpha \Rightarrow \ \mid \ \Rightarrow \neg\Box\neg\alpha}\ R\neg}{\alpha \Rightarrow \ \mid \ \Rightarrow \Box\neg\Box\neg\alpha}\ k}{\alpha \Rightarrow \Box\neg\Box\neg\alpha \mid \alpha \Rightarrow \Box\neg\Box\neg\alpha}\ IW}{\alpha \Rightarrow \Box\neg\Box\neg\alpha}\ EC$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Box\alpha \Rightarrow \Box\alpha}{\Box\alpha, \neg\Box\alpha \Rightarrow}\ L\neg}{\Box\alpha \Rightarrow \ \mid \neg\Box\alpha \Rightarrow}\ MS}{\Rightarrow \neg\Box\alpha \mid \neg\Box\alpha \Rightarrow}\ R\neg}{\Rightarrow \Box\neg\Box\alpha \mid \neg\Box\alpha \Rightarrow}\ k}{\neg\Box\alpha \Rightarrow \Box\neg\Box\alpha \mid \neg\Box\alpha \Rightarrow \Box\neg\Box\alpha}\ IW}{\neg\Box\alpha \Rightarrow \Box\neg\Box\alpha}\ EC$$

Hypersequent Calculi for Modal Logics Extending **S4**,

The modal hyperstructural rule "Restricted Modal Splitting" and "Modal Communication"[Kurokawa, 2014].

$$\frac{G \mid \Box\Gamma, \Box\Delta \Rightarrow}{G \mid \Box\Gamma \Rightarrow \ \mid \Box\Delta \Rightarrow}\ RMS \qquad \frac{G \mid \Sigma, \Box\Gamma \Rightarrow \Pi \quad G \mid \Theta, \Box\Delta \Rightarrow \Lambda}{G \mid \Sigma, \Box\Delta \Rightarrow \Pi \mid \Theta, \Box\Gamma \Rightarrow \Lambda}\ MC$$

- Axioms for **S4.2**:**S4**+ **.2**: $\neg\Box\neg\Box A \to \Box\neg\Box\neg A$
- Axioms for **S4.3**:**S4**+ **.3**: $\Box(\Box A \to B) \vee \Box(\Box B \to A)$
- **HCs4.2**: **HCs4**+*RMS*
- **HCs4.3**: **HCs4**+*MC*

Outline

Simple Frame Properties

[Lahav, 2013] provides a general method for generating cut-free and/or analytic hypersequent Gentzen-type calculi for a variety of normal modal logics.
We use classical first-order language to formulate the frame properties.

Definition (Simple Frame Properties)

Simple frame properties are formulated by formulas of the form

$$\forall w_1 \cdots w_n \exists u \varphi$$

where φ consists of:

- Atomic formulas of the form $w_i R u$ or $w_i = u$.
- Conjunctions and disjunctions.

Reflexivity is simple: $\forall w_1 \exists u (w_1 R u \wedge w_1 = u)$

Some Examples of Simple Frame Properties

Seriality	$\forall w_1 \exists u (w_1 Ru)$
Directedness	$\forall w_1 \forall w_2 \exists u (w_1 Ru \wedge w_2 Ru)$
Degenerateness	$\forall w_1 \forall w_2 \exists u (w_1 = u \wedge w_2 = u)$
Universality	$\forall w_1 \forall w_2 \exists u (w_1 Ru \wedge w_2 = u)$
Linearity	$\forall w_1 \forall w_2 \exists u (w_1 Ru \wedge w_2 = u) \vee (w_2 Ru \wedge w_1 = u)$
Bounded Cardinality	$\forall w_1 \cdots \forall w_n \exists u \bigvee_{1 \leq i < j \leq n} (w_i = u \wedge w_j = u)$
Bounded Top Width	$\forall w_1 \cdots \forall w_n \exists u \bigvee_{1 \leq i < j \leq n} (w_i Ru \wedge w_j Ru)$
Bounded Width	$\forall w_1 \cdots \forall w_n \exists u \bigvee_{1 \leq i, j \leq n; i \neq j} (w_i Ru \wedge w_j = u)$

From Simple Frame Properties to Hypersequent Rules

Step 1. Extract the **normal form** of $\forall w_1 \cdots w_n \exists u \varphi$, a set $\{\langle R_1, E_1\rangle, \cdots, \langle R_m, E_m\rangle\}$ such that

$$\varphi \equiv \bigvee_{1 \leq i \leq m} (\bigwedge_{j \in R_i} w_j R u \wedge \bigwedge_{j \in E_i} w_j = u)$$

$\forall w_1 \forall w_2 \exists u (w_1 R u \wedge w_2 R u)$	$\{\langle\{1,2\}, \emptyset\rangle\}$
$\forall w_1 \forall w_2 \exists u (w_1 R u \wedge w_2 = u)$	$\{\langle\{1\}, \{2\}\rangle\}$
$\forall w_1 \forall w_2 \exists u (w_1 R u \wedge w_2 = u) \vee (w_2 R u \wedge w_1 = u)$	$\{\langle\{1\}, \{2\}\rangle, \langle\{2\}, \{1\}\rangle\}$
$\forall w_1 \cdots \forall w_n \exists u \bigvee_{1 \leq i < j \leq n} (w_i = u \wedge w_j = u)$	$\{\langle\emptyset, \{i,j\}\rangle \mid 1 \leq i < j \leq n\}$

From Simple Frame Properties to Hypersequent Rules

Step 2. For a normal form $\{\langle R_1, E_1\rangle, \cdots, \langle R_m, E_m\rangle\}$ construct the following rule and add it to **HCk**:

$$\frac{H \mid \Gamma_{E_1}, \Gamma'_{R_1} \Rightarrow \Delta_{E_1} \quad \cdots \quad H \mid \Gamma_{E_m}, \Gamma'_{R_m} \Rightarrow \Delta_{E_m}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n}$$

Notation: $\Gamma_{\{i_1,\cdots,i_k\}} := \Gamma_{i_1}, \cdots, \Gamma_{i_n}$

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Notation: $\Gamma_{\{i_1,\cdots,i_k\}} := \Gamma_{i_1}, \cdots, \Gamma_{i_n}$

Directedness	$\forall w_1 \forall w_2 \exists u (w_1 Ru \wedge w_2 Ru)$
Universality	$\forall w_1 \forall w_2 \exists u (w_1 Ru \wedge w_2 = u)$
Linearity	$\forall w_1 \forall w_2 \exists u (w_1 Ru \wedge w_2 = u) \vee (w_2 Ru \wedge w_1 = u)$
Bounded Cardinality	$\forall w_1 \cdots \forall w_n \exists u \bigvee_{1 \leq i < j \leq n} (w_i = u \wedge w_j = u)$

In the presence of the weakening rules, $\Gamma_i, \Gamma'_i, \Delta_i$'s that appear only in the conclusion can be discarded.

$$\frac{H \mid \Gamma'_1, \Gamma'_2 \Rightarrow}{H \mid \Box\Gamma'_1 \Rightarrow \ \mid \Box\Gamma'_2 \Rightarrow}$$
Directedness

$$\frac{H \mid \Gamma_2, \Gamma'_1 \Rightarrow \Delta_2}{H \mid \Box\Gamma'_1 \Rightarrow \ \mid \Gamma_2 \Rightarrow \Delta_2}$$
Universality

$$\frac{H \mid \Gamma_2, \Gamma'_1 \Rightarrow \Delta_2 \quad H \mid \Gamma_1, \Gamma'_2 \Rightarrow \Delta_1}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \Gamma_2, \Box\Gamma'_2 \Rightarrow \Delta_2}$$
Linearity

$$\frac{\{H \mid \Gamma_i, \Gamma_j \Rightarrow \Delta_i, \Delta_j \mid 1 \leq i < j <\leq n\}}{H \mid \Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n}$$
Bounded Cardinality

Main Result

Definition (Strong Soundness and Completeness)

$$\Gamma \vdash_{Local} A \ \textit{iff} \ \vdash \Gamma \Rightarrow A$$

$$\Gamma \vdash_{Global} A \ \textit{iff} \ \{\Rightarrow B \mid B \in \Gamma\} \vdash \Rightarrow A$$

where $\Gamma \vdash_{Local} A$ means A holds in every world in which Γ holds; $\Gamma \vdash_{Global} A$ means A holds in every world if Γ holds in every world.

Definition (Strong Cut-Admissibility)

Cut can be confined to apply only on formulas that appear in the assumptions.

Theorem

The constructed hypersequent calculus is sound and complete for the modal logic, and it enjoys cut-admissibility.

Decidability

Corollary

All modal logics characterized by finite sets of simple frame properties are decidable.

Proof.

Cut-admissibility $\longrightarrow$ Subformula property $\longrightarrow$ We can check one by one all possible proofs candidates. $\square$

Transitivity and Symmetry

Definition (Simple Frame Properties)

Simple frame properties are formulated by formulas of the form

$$\forall w_1 \cdots w_n \exists u \varphi$$

where φ consists of:

- Atomic formulas of the form $w_i R u$ or $w_i = u$.
- Conjunctions and disjunctions.

- Simple properties are *monotone increasing* (preserved under enrichment of R).
- Transitivity and symmetry are *not simple*.

We have to change the basic calculus:

$$\frac{H \mid \Gamma \Rightarrow A}{H \mid \Box\Gamma \Rightarrow \Box A}\ \mathbf{K} \qquad \frac{H \mid \Gamma, \Box\Gamma \Rightarrow A}{H \mid \Box\Gamma \Rightarrow \Box A}\ \mathbf{K4} \qquad \frac{H \mid \Gamma \Rightarrow A, \Box\Delta}{H \mid \Box\Gamma \Rightarrow \Box A, \Delta}\ \mathbf{KB}$$

Transitivity

For a normal form $\{\langle R_1, E_1\rangle, \cdots, \langle R_m, E_m\rangle\}$ construct the following rule :

$$\frac{H \mid \Gamma_{E_1}, \Gamma'_{R_1}, \Box\Gamma'_{R_1} \Rightarrow \Delta_{E_1} \quad \cdots \quad H \mid \Gamma_{E_m}, \Gamma'_{R_m}, \Box\Gamma'_{R_m} \Rightarrow \Delta_{E_m}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n}$$

For example:

$$\frac{H \mid \Gamma_2, \Gamma'_1, \Box\Gamma'_1 \Rightarrow \Delta_2 \quad H \mid \Gamma_1, \Gamma'_2, \Box\Gamma'_2 \Rightarrow \Delta_1}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1 \mid \Gamma_2, \Box\Gamma'_2 \Rightarrow \Delta_2}$$

Linearity

We have:

- Strong soundness and completeness.
- Strong cut-admissibility.
- Decidability.

Symmetry

For a normal form $\{\langle R_1, E_1\rangle, \cdots, \langle R_m, E_m\rangle\}$ construct the following rule :

$$\frac{H \mid \Gamma_{E_1}, \Gamma'_{R_1} \Rightarrow \Delta_{E_1}, \Box\Delta'_{R_1} \quad \cdots \quad H \mid \Gamma_{E_m}, \Gamma'_{R_m} \Rightarrow \Delta_{E_m}, \Box\Delta'_{R_m}}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1, \Delta'_1 \mid \cdots \mid \Gamma_n, \Box\Gamma'_n \Rightarrow \Delta_n, \Delta'_n}$$

For example:

$$\frac{H \mid \Gamma'_1 \Rightarrow \Box\Delta'_1}{H \mid \Box\Gamma'_1 \Rightarrow \Delta'_1} \qquad \frac{H \mid \Gamma_1, \Gamma'_1 \Rightarrow \Delta_1, \Box\Delta'_1}{H \mid \Gamma_1, \Box\Gamma'_1 \Rightarrow \Delta_1, \Delta'_1}$$

Seriality / Reflexivity

$$\frac{H \mid \alpha, M \Rightarrow}{H \mid \Box\alpha, \Box M \Rightarrow}\ HCd \qquad \frac{H \mid \alpha, M \Rightarrow N}{H \mid \Box\alpha, M \Rightarrow N}\ HCt$$

- Cut-admissibility does not hold (even for the basic calculus).
- All constructed calculi still enjoy the subformula property.
- Decidability still follows.

Outline

Summary and further interests

- The successful application of gentzen-style sequent calculus for classical propositional logic and modal logic till up to **S4**.
- The "one-step-further" proof theoretical framework, hypersequent calculus for propositional modal logic.
- A general method for generating cut-free and/or analytic hypersequent Gentzen-type calculi for a variety of normal modal logics.

Summary and further interests

- The successful application of gentzen-style sequent calculus for classical propositional logic and modal logic till up to **S4**.
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Further interests:

- Other proof theoretical framework that address modal logic.
- Methods of proving cut-elimination.
- Other consequences of the cut elimination theorem, interpolation theorem, conservativity, disjunction property. A brief discuss can be found in [Ono, 1998].
- Epistemic logic.

References I

Avron, A. (1987).
A constructive analysis of **RM**.
Journal of Symbolic Logic, 52(4):939–951.

Avron, A. (1996).
The method of hypersequents in the proof theory of propositional non-classical logics.
In *Logic: from foundations to applications (Staordshire, 1993)*, pages 1–32. Oxford Sci. Publ., Oxford Univ. Press, New York.

Bednarska, K. and Indrzejczak, A. (2015).
Hypersequent calculi for s5: The methods of cut elimination.
Logic and Logical Philosophy, 24.

Goble, L. F. (1974).
Gentzen systems for modal logic.
Notre Dame J. Formal Logic, 15(3):455–461.

Kurokawa, H. (2014).
Hypersequent calculi for modal logics extending s4.
In Nakano, Y., Satoh, K., and Bekki, D., editors, *New Frontiers in Artificial Intelligence*, pages 51–68, Cham. Springer International Publishing.

References II

Lahav, O. (2013).
From frame properties to hypersequent rules in modal logics.
IEEE Symposium on Logic in Computer Science, 29(3):408–417.

Leivant, D. (1981).
On the proof theory of the modal logic for arithmetic provability.
Journal of Symbolic Logic, 46(3):531538.

Mints, G. (1990).
Gentzen-type systems and resolution rules part i propositional logic.
In *International Conference on Computer Logic*.

Negri (2001).
Structural Proof Theory.
Cambridge University Press.

Ohnishi, M. and Matsumoto, K. (1957).
Gentzen method in modal calculi.
Osaka Math. J., 9(2):113–130.

References III

Ono, H. (1998).
Proof-Theoretic Methods in Nonclassical Logic –an Introduction, volume Volume 2 of *MSJ Memoirs*, pages 207–254.
The Mathematical Society of Japan, Tokyo, Japan.

Poggiolesi, F. (2011).
Gentzen Calculi for Modal Propositional Logic.
Springer.

Pottinger, G. (1983).
Uniform, cut-free formulations of t, s4 and s5 (abstract).
Journal of Symbolic Logic, 48(3).

Sambin, G. and Valentini, S. (1982).
The modal logic of provability. the sequential approach.
Journal of Philosophical Logic, 11(3):311–342.

Takano, M. (1992).
Subformula property as a substitute for cut-elimination in modal propositional logics.
Mathematica Japonica, 37.

References IV

Troelstra, A. S. and Schwichtenberg, H. (1996).
Basic proof theory.
Cambridge University Press, Cambridge.