Kit Fine's Theory of Arbitrary Objects

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1 Introduction

As is well-known, there exists certain informal procedures for arguing to a universal conclusion and from an existential premiss. We may establish that all objects of a certain kind have a given property by showing that an arbitrary object of that kind has that property; and having shown that there exists an object with a given property, we feel entitled to give it a name and declare that it has the property. So we may establish that all triangles have interior angles summing to 180° by showing of an arbitrary triangle that its interior angles sum to 180°; and having established that there exists a bisector to an angle, we feel entitled to give it a name and declare that it *is* a bisector to the angle.

In these inferences, certain objects play a crucial role, and it is nature to ask how they are to be understood. What is to be made of our talk of arbitrary triangles or indefinite bisectors in ordinary reasoning? To solve this problems, Kit Fine developed the theory of arbitrary objects.

2 Arbitrary Objects Defended

Objection 1 There are no arbitrary objects.

Is it seriously to be supposed that, in addition to individual numbers, there are arbitrary numbers and that, in addition to individual men, there are arbitrary men? What strange sorts of objects are they? Can I count with arbitrary numbers or have tea with an arbitrary man?

Objection 1a Are there any arbitrary objects?

Fine's reply It is necessary to distinguish, in a way that is familiar from the philosophical literature, between two senses of the phrase "there are". If now I am asked whether there are arbitrary objects, I will answer according to the intended sense of "there are". If it is the ontologically significant sense, then I am happy to agree with my opponent and say "no". But if the intended sense is ontologically neutral, then my answer is a decided "yes". I have, it seems to me, as much reason to affirm that there are arbitrary objects in this sense as the nominalist has to affirm that there are numbers.

Objection 1b What are arbitrary objects like?

Fine's reply This question may be taken, in an ordinary, non-philosophical way, as a request for an explanation of what objects one is talking about. One can do no better than refer to the kind of role that arbitrary objects are intended to play, such as, each arbitrary object is associated with a range of values, that it possesses those properties common to its values, and so on. Then this question may also be taken, in a philosophical way, as a request for a category or kind to which the objects can be assigned. We may then supply the quite reasonable answer that arbitrary objects belong, like sets or propositions, to the category of abstractions.

Objection 2 The theory of arbitrary objects is logically incoherent.

For the key principle, without which a theory would be unthinkable, is that each arbitrary object should have the properties common to the individuals in its range. But formulate such a principle properly and it will be seen to lead, either on its own or in conjunction with innocuous premisses, to contradictory conclusions. There are two forms this line of argument can take.

Objection 2a Take an arbitrary number. Then it is odd or even, since each individual number is odd or even. But it is not odd, since some individual number is not odd; and it is not even, since some individual number is not even. Therefore it is odd or even, yet not odd and not even. A contradiction.

Fine's reply Crucial to this argument is a certain formulation of what I shall call the *principle of generic attribution*, the principle that any arbitrary object has those properties common to the individuals in its range. Let $\phi(x)$ be any condition with free variable x; let a be the name of an arbitrary object a; and let i be a variable that ranges over the individuals in the range of a. (We here follow a general convention whereby a names a. Then the required formulation of the principle is:

$$\text{(G1) } \phi(a) \equiv \forall i \phi(i) (\text{a } \phi\text{'s iff every individual } \phi\text{'s})$$

Upon letting a name the arbitrary number of the argument, the various claims made about that number can be obtained by making appropriate substitutions for the condition $\phi(x)$. For example, upon substituting Ex (x is even), we obtain:

$$\text{(G1)' } Ea \equiv \forall i Ei;$$

and the claim that a is not even ($-Ea$) then follows from the evident truth that not all numbers are even ($-\forall i Ei$).

Look at its consequence (G1)' and ask whether the arbitrary number a satisfies the condition ($Ex \equiv \forall i Ei$) of being even iff all individual numbers are even. What the intuitive principle of generic attribution seems to tell us is that an arbitrary number a should satisfy the condition iff all individual numbers do. But not all individual numbers satisfy it; no even number does. So far from being a correct formulation of the intuitive principle, (G1) is something that the principle requires us to judge false.

Of course, we may have here a further sign that the intuitive principle is contradictory. But another possibility is that it is only intended that the intuitive principle apply to the whole context in which the name of the arbitrary object appears. Under this proviso, the correct formulation of the principle is:

(G2) The sentence $\phi(a)$ is true iff the sentence $\forall i\phi(i)$ is true.

Or using satisfaction in place of truth, the formulation becomes

(G2)' a satisfies the condition $\phi(x)$ iff each individual i in the range of a satisfies the condition $\phi(x)$.

From this perspective, the original formulation (G1) rests on the fallacy of applying the principle *internally*, to only a part of the context in which the name of an arbitrary object appears. For although we may affirm $\forall i\phi(i) = \forall i\phi(i)$, we cannot correctly infer $\phi(a) = \forall i\phi(i)$.

In such a way, the argument from complex properties can be stymied. It has always been thought that we have in this, and related, arguments a knockdown case against the whole theory of arbitrary objects. But such arguments depend upon the failure to distinguish between two basically different formulations of the principle of generic attribution: one is merely a rule of equivalence and is stated in the material mode; the other is a rule of truth and is stated in the formal mode. Once the distinction is made, the arguments are seen to be without foundation.

Objection 2a' Consider again the statement $Oa \vee Ea$ ("a is odd or a is even"). From the modified formulation (G2), it follows that this statement is true and yet that neither disjunct is true. So the semantical rule for disjunction fails. Or consider the statements Oa and $-Oa$. From (G2), it follows that neither Oa nor $-Oa$ is true. So the semantical rule for negation fails. Suppose now Oa is false only when $-Oa$ is true. Then the Law of Bivalence also fails.

Fine's reply Perhaps the worry is not so much about indeterminacy in the truth-conditions as inconstancy in the use of the logical constants. When '$\vee$' occurs between statements concerning only individuals, we evaluate the result according to the classical truth-tables; when '$\vee$' occurs between statements concerning arbitrary objects, we evaluate the result according to the rule of generic attribution. Is not this to use '$\vee$' in two radically different ways?

It seems that we must either deny that $\Psi(a) \vee \chi(a)$ is a genuine disjunction or else give up the principle that disjunctions are evaluated directly by a disjunction rule.

Fine prefers to the second alternative. But for those who are not, he also offers an acceptable way of holding to the first alternative. For we may suppose that the statement $\Psi(a) \vee \chi(a)$ is syntactically ambiguous. It may either be formed by disjoining $\Psi(a)$ and $\chi(a)$ or by applying the property abstract $\lambda x(\Psi(x) \vee \chi(x))$ to a. In the first case we have a genuine disjunction, in the second case not. A more perspicuous notation would use $[\lambda(\Psi(x) \vee \chi(x))]a$ for the second case, thereby making it clear that the statement is not genuinely disjunctive in form. We may now evaluate genuine disjunctions directly in accordance with the classical rule. The extensions of property abstracts $\lambda x \Psi(x)$ may be evaluated in the usual way for individuals and in a way analogous to (G2) for arbitrary objects. Thus the effect of the principle (G2) is achieved without any damage to the principle of direct evaluation.

Objection 2b Another logical objection is the argument from special properties.Take the property of being an individual number. Then each individual number has this property. So from the principle of generic attribution, it follows that any arbitrary number has this property-which is absurd.

Fine's reply We must distinguish between two kinds of condition or predicate. There are first of all the generic conditions and predicates. These include all of the ordinary predicates, such as "being odd" or "being mortal", and all of the conditions obtainable from them by means of the classical operations of quantification and truth-functional composition. To these conditions, the principle of generic attribution, in its revised form (G2), is applicable. There are then the classical conditions or predicates. These include certain special predicates, such as "being an individual number" or "being in the range of", and the various conditions obtained with their help. To these, the principle (G2) is not applicable.

The principle of generic attribution should therefore now receive the following formulation:

(G3) for any *generic* condition $\phi(x)$, $\phi(a)$ is true iff $\forall i \phi(i)$ is true;

Objection 3 It's doubtful whether anything approaching a satisfactory theory can be obtained.

"This [Czuber's account] gives rise to a host of questions. The author obviously distinguishes two classes of numbers: the determinate and the indeterminate. We may then ask, say, to which of these classes the primes belong, or whether maybe some primes are determinate numbers and others indeterminate. We may ask further whether in the case of indeterminate numbers we must distinguish between the rational and the irrational, or whether this distinction can only be applied to determinate numbers. How many indeterminate numbers are there? How are they distinguished from one another? Can you add two indeterminate numbers, and if so, how? How do you find the number that is to be regarded as their sum? The same questions arise for adding a determinate number to an indeterminate one. To which class does the sum belong? Or maybe it belongs to a third?"(by Frege ([5], p.160))

Objection 3a To which of these classes the primes belong?

Fine's reply Suppose "prime" is taken, in its classical reading, to mean "individual prime". Then no arbitrary number is prime. Suppose now that "prime" is taken in its generic reading. Then the statement "a is prime", for a an arbitrary number, will not be true, since some individual numbers are not prime. Of course, neither will it be false, since some individual numbers are prime.

There is, however, a complication. We have so far taken an arbitrary number to have an unrestricted range, one that includes all individual numbers as values. But it also seems reasonable that there should be arbitrary numbers with a restricted range, one that includes only some of the individual numbers as values. Suppose we now ask, of these arbitrary numbers, whether any of them are prime. Then we should say: the statement "a is prime" is true iff all of the individual numbers in the range of a are prime. So an arbitrary prime is prime, but not an arbitrary (and unrestricted) even number or an arbitrary (and unrestricted) factor of 12.

Objection 3b How are they [the indeterminate numbers] to be distinguished from one another?

Fine's reply But why should there be a non-trivial difference between any two arbitrary objects? A theory does not require an identity criterion for its objects. But still, the present theory is able to provide one-not as it stands, but upon the introduction of two new elements. The first is the **simultaneous assignment** of values. We must allow an interdependence among the values assigned to the arbitrary objects, so that what individuals are assignable to one object may be constrained by the values assigned to others. The principle of generic attribution must accordingly be modified so as to apply to all of the arbitrary objects that might be mentioned.

(G4) If$\phi(x_l, x_2, \ldots, x_n)$ is a generic condition containing no names for arbitrary objects, then $\phi(a_l, a_2, \ldots, a_n$ is true iff $\phi(a_l, a_2, \ldots, a_n$ is true for all admissible assignments of individuals $i_1, i_2, \ldots, i_n$ to the objects $a_1, a_2, \ldots, a_n$.

We could think of the previous principle (G3) as applying to the objects $a_1, a_2, \ldots, a_n$ one at a time. But we must think of the present principle (G4) as applying to those objects simultaneously.

The second component of the new apparatus is the relation of dependence among arbitrary objects or what may be called object dependence. When b is an arbitrary object that depends only upon the arbitrary objects $a_1, a_2, \ldots$, then the values assignable to b must be determinable upon the basis of the values assigned to $a_l, a_2, \ldots$. Thus the relation of object dependence provides a principle for the local determination of admissible value assignments.

The problem of providing identity criteria for arbitrary objects can now be solved. Say that an arbitrary object is *independent* if it depends upon no other objects and that otherwise it is *dependent*. We distinguish two cases, according to whether the objects are independent or dependent. Suppose first that a and b are independent objects. Then we say that $a = b$ iff their ranges are the same. Suppose now that a and b are dependent objects. Then we shall say that $a = b$ iff two conditions are satisfied. The first is that they should depend upon the same arbitrary objects; their "dependency range" should be the same. The second is that they should depend upon these objects in the same way. Let us make the reasonable assumption that the relation of dependence is well-founded: any sequence of arbitrary objects $a_l, a_2, a_3, \ldots$, with a_1 depending upon a_2, a_2 upon a_3, and so on, must eventually come to an end. The above two criteria then enable us to distinguish any two arbitrary objects a and b. So suppose that a and b depend upon different arbitrary objects. There is then an arbitrary object c upon which a depends, let us say, but not b. We can then distinguish a and b if we can distinguish c from all of the objects $d_1, d_2, \dots$, upon which b depends. But the whole of the previous argument may be repeated for the pairs (c, d_l), (c, d_2), ... Of course, this may lead to yet further pairs of objects that stand in need of distinction. But the well-foundedness of the dependency relation will guarantee that the process eventually comes to an end.

In such a way, we may answer Frege's question about identity. But a doubt may remain. In a sentence such as "Let x and y be two arbitrary reals", we will want to say that the symbols "x" and "y" refer to two arbitrary reals. But to which? It is natural to suppose that "x" and "y" refer to two unrestricted and independent arbitrary reals. But by the first criterion there is only one such real. So do either of the symbols refer to it and, if so, to what does the other refer? There are various ingenious solutions to this problem. But perhaps the most natural is one that makes x and y not be independent at all. There is a unique arbitrary pair of reals, p; it is the independent arbitrary object whose values are all the pairs of reals. We may then take x and y to be the first and second components of this arbitrary pair. More exactly: x and y will each depend upon p and upon p alone: when p takes the value (i, j), then and only then will x take the value i and y the value j. By the identity criteria, x and y, as so defined, will be unique.

Objection 3c How many indeterminate numbers are there?

Fine's reply We may sink this into the more general question: what arbitrary objects are there? Consider first the arbitrary objects that take their values from a given set I of individuals. Such objects may be generated in stages, according to their "degree" of independence. At the first stage are the independent objects. Since there are no essential constraints on the existence of arbitrary objects, we should expect that to each set of individuals from I there will be an arbitrary object with that set as its range. At the second stage are the arbitrary objects that depend upon the objects generated at the first stage, but not on anything else. We should now expect that, for each set X of independent objects and each suitable relation R, there will be an arbitrary object that has X as the set of objects upon which it depends and that has R as the way it depends upon them. At the other stages, both finite and transfinite, the arbitrary objects will be determined in a similar manner.

For each set I of individuals, we thereby obtain a "system" A_I of arbitrary objects. Now the choice of I here is completely free. In particular, it may contain arbitrary objects, so that these may now figure as values to other arbitrary objects. Let us suppose that as the systems A_I are generated the base sets I are expanded with the objects so obtained. Then with this understanding, we may take the arbitrary objects to be those objects that belong to one of the systems A_I. The question of cardinality may now be considered. The general system A of arbitrary objects and, indeed, each of the individual systems A, is a proper class. Therefore the class of arbitrary objects has no cardinality or, if one likes, it has the same cardinality as the universe. However, for each I of given cardinality and each ordinal a, we may determine, by a simple combinatorial calculation, how many objects of A ,are generated by the stage a. Thus we are in as good a position to answer questions concerning cardinality in the case of arbitrary objects as in the case of sets.

Objection 3d Can you add two indeterminate numbers and, if so, how?

Fine's reply Now the theory does not require that the sum of two arbitrary numbers be defined. Consider the equation "$a + b = b + a$". Then the principle of generic attribution (G4) tells us that the equation is true iff "$i + j = j + i$" is true for each i and j that can be assigned to a and b. Thus at no point, in the evaluation of the sentence, need we consider a denotation for the complex terms.

However, this argument rests on the identity context "$x = y$" being generic. If, as in other parts of mathematics, a complex term is to be applied to classical conditions, then a denotation should be supplied. What then is a suitable denotation c for "$a + b$"? First, we would like i, j, k to be admissible values of a, b, c, iff i, j are admissible values for a, b and $k = i + j$. Second, we would like c to depend just upon a and b and upon whatever a or b depend upon. Now it follows from our discussion of cardinality, that such an object c exists; and it follows, from our discussion of identity, that c is unique. Therefore the problem of finding a suitable sum is solved. The sum of a and i, for a an arbitrary and i an individual number, is the unique arbitrary object b such that (1) k, j are admissible values for b, a iff $k = i + j$ and j is an admissible value for a, (2) b just depends upon a and upon whatever a depends upon. There is no need of a third category of objects here and so no danger of a proliferation of categories.

Objection 4 This theory is of no use.

Fine's reply I conceive of the theory as having application to four areas: (1) the logic of generality; (2) mathematical logic; (3) language; and (4) the history of ideas.

3 The General Framework

3.1 The Models

$\mathcal{L}$ is a first order language. Let $\mathcal{M}$ be a classical model for $\mathcal{L}$. So $\mathcal{M}$ will be of the form $(I, \dots)$, where $\mathcal{L}$ is a non-empty set and $\dots$ indicates the interpretation for the non-logical constants of the language. We use the prefix "I" (for "individual") in connection with the model $\mathcal{M}$ and its components. So $\mathcal{M}$ itself is an *I-model*, I is an *I-domain*, and the members of I are *individuals* or *I-objects*.

$\mathcal{M}$ may then be expanded to a generic model $\mathcal{M}^+$. Any such model $\mathcal{M}^+$ is of the form *$(I, \dots, A, \prec, V)$*, where

(i) *$(I, \dots)$* is the model $\mathcal{M}$;

(ii) A is a set of objects disjoint from I

(iii) $\prec$ is a relation on A

(iv) V is a non-empty set of partial functions from A into I, i.e. function v for which $\mathrm{Dm}(v) \subseteq A$ and $\mathrm{Rg}(v) \subseteq I$.

A is the set of arbitrary or variable objects. It is assumed that these are disjoint from the set of individuals. $\prec$ is the relation of dependence between arbitrary objects. "$a \prec b$" indicates that the value of a depends upon the value of b. When the relationship $a \prec b$ holds, we call a the *dependent* object and b the *dependee*. V is the family of value assignments. The presence of v in V indicates that the arbitrary objects $a_1, a_2, \ldots$ in the domain of v can simultaneously assume the respective values $v(a_1)$, $v(a_2)$, ... This may be pictured as follows:

$$v{:}\frac{a_1 a_2 \ldots a_n}{i_1 i_2 \ldots i_n}$$

The members of V are called the *admissible* value-assignments. An arbitrary partial function from A into I might be called a *possible* value-assignment.

Then we use the prefix "A" for generic or arbitrary-object-related terms. So $\mathcal{M}^+$ itself is a *possible A-model*, A is an *A-domain*, and the members of A are *A-objects*. The variables "i", "j", "k", are used for I-objects, and the variables "a", "b", "c" for A-objects.

An A-model $(I, \ldots, A, \prec, V)$ shorn of its structural component ... is called an *A-structure*; and an A-structure $(I, A, \prec, V)$ shorn of its value-theoretic component V is called an *A-frame*. For $a \in A$, the *value-range* $VR(a)$ of a is $\{v(a) : v \in V\}$. Thus the value-range of an A-object consist of all the values it can assume. If $VR(a) \neq I$, then a is said to be *value-restricted* and otherwise to be *universal* or *value-unrestricted*. In the special case i which $VR(a) = \emptyset$, we say a is *null*. An A-object $a \in A$ is *dependent* if $a \prec b$ for some $b \in A$, and otherwise it is *independent*. An A-object is *restricted* if it is either value-restricted or dependent and otherwise it is *unrestricted*.

A subset B of A is *closed* if whenever $a \in B$ and $a \prec b$ then $b \in B$. A closed set therefore contains the dependees of any member. We use $[B]$ for the closure of B, *i.e.* the smallest closed set to contain B;and we use $|B|$ for $[B] - B$. In case $B = \{a\}$, we use $[a]$ and $|a|$ for $[B]$ and $|B|$ respectively.

We say that a value-assignment $v \in V$ is *defined on* $B \subseteq A$ if $Dm(v) = B$ and *defined over* B if $B \subseteq Dm(v)$. For $B \subseteq A$, we let V_B be the set of value-assignments of V defined on B; and in case $B = [a]$, we write V_B as V_a.

For $a \in A$, the *value dependence* $VD(a, B)$ *of* a *upon* B is the function f defined, for $v \in V_B$, by:

$$f(v) = \{i \in I : v \cup \{(a, i)\} \in V\}$$

The function f tells us which values a can receive for given values of the B's. The *value dependence* $VD(a)$ or VD_a of a simpliciter is $VD(a, |a|)$. The function VD_a tells us which values a can receive for given values to its dependees.

In order for the possible A-model $\mathcal{M}^+ = (I, \ldots, A, \prec, V)$ to be an *actual* A-model, it must be subject to further conditions (and similarly for A-structures and A-frames). These goes as follows:

(v)(a)(Transitivity) $a \prec b \wedge b \prec c$ implies $a \prec c$;

(b)(Foundation) The converse of the relation $\prec$ is well-founded, *i.e.* there is no infinite sequence of A-subsequence of A-objects $a_1, a_2, \ldots,$ for which $a_1 \prec a_2 \prec a_3 \prec \ldots.$

(vi) (Restriction) V is closed under restriction, *i.e.* $v \in V$ and $B \subseteq A$ implies that $v \upharpoonright B \in V$ (where $v \upharpoonright B$ is the restriction of v to B).

(vii) (Partial Extendibility) If $b \in V$, then there is a $v^+ \in V$ for which $v^+ \supseteq v$ and $Dm(v^+) \supseteq [Dm(v)]$

(viii) (Piecing) Let $\{v_\xi : \xi \in \Omega\}$, for $\Omega \neq \emptyset$, be an indexed subset of V subject to the requirements that (a) each $Dm(v_\xi)$ is closed and (b) the union $\bigcup v_\xi$ is a function. Then $v = \bigcup v_\xi$ is also a member of V.

3.2 Conditions

We shall be interested in imposing additional conditions on an A-model. Four kinds of conditions will be considered.

Extendibility

We say that an A-model $\mathcal{M}^+$ is *extensibility over* a subset B of A if, for every subset B' of B and value assignment $v' \in V$ with domain B', there is a $v \in V$ that extends v' and defined over B. We say that an A-model $\mathcal{M}^+$ is *extendible* if it is extendible over its A-domain A.

Existence

Existence (Ind)

For each $J \subseteq I$, there is an independent A-object $a \in A$ for which $VR(a) = J$.

Existence (Dep)

For each non-empty closed subset B of A and for each function f from V_B into $\beta(I)$, there is an $a \in A$ for which: (a) $|a| = B$, and (b) $VD_a = f$.

***d**-Existence (Dep)*

For each non-empty closed subset B of A of cardinality less than $\mathbf{d}$ and for each function f from V_B into $\beta(I)$, there is an $a \in A$ for which: (a) $|a| = B$, and (b) $VD_a = f$.

Identity

Identity (Ind)

For any two independent A-objects a and b of A, $a = b$ if $VR(a) = VR(b)$.

Identity (Dep)

For any two dependent A-objects a and b of A, $a = b$ if (a) $|a| = |b|$ and (b)$VD_a = VD_b$.

Multiplicity

***c**-Multiplicity (Ind)*

For any independent A-object a there are at least $\mathbf{c}$ independent A-objects b for which $VR(b) = VR(a)$;

$\boldsymbol{c}$-Multiplicity (Dep)

For any dependent A-object a there are at least $\mathbf{c}$ dependent A-objects b for which $|a| = |b|$ and (b)$VD_a = VD_b$.

3.3 Existence

We are here concerned to prove the existence of various sorts of A-model.

We may call an assignment $v \in V$ *basic* in an A-model or A-structure if $Dm(v) = [a]$ for some $a \in A$. In terms of our previous notation, the basic assignment are the members of $\bigcup\limits_{a \in A} V_a$. Call a subset U of V a *basis for* V if V is the closure U^+ of U under Restriction and Piecing. We can prove that:

(a) The basic assignments form a basis U for any A-structure $\mathcal{F} = (I, A, \prec, V)$.

(b) Each I-model $\mathcal{M} = (I, \dots)$ underlies a **c,d** - standard model $\mathcal{M}^+ = (I, \dots, A, \prec, V)$.

3.4 Truth

Now let's show how a language may secure reference to A-objects and how the sentences of the language may be evaluated for truth and falsity. The basic principle is that a sentence concerning A-objects is true (false) just in case it is true (false) for all of their values.

Reference to A-objects is secured in our given first-order language $\mathcal{L}$ through the addition of a new category of symbols. These might be called *A-letters*. The resulting language will be called $\mathcal{L}^*$. In the syntax of $\mathcal{L}^*$, the A-letters will be treated in exactly the same way as individual names.

Formulas from $\mathcal{L}^*$ will be called *A-formulas*, formulas from $\mathcal{L}$ alone *I-formulas*. We shall adopt the usual terminology of symbolic logic. However, it will be supposed that formulas contain no free occurrences of variables. Expressions that would be formulas, were it not for the presence of free variables, will instead be called *psedo*-formulas.

The A-models for $\mathcal{L}$ may now be extended to $\mathcal{L}^*$. An A-model $\mathcal{M}^*$ is of the form $(I, \dots, A, \prec, V, d)$, where $(I, \dots, A, \prec, V)$ is an A-model $\mathcal{M}^+$ for the language $\mathcal{L}$, as before, and d is a function from the set of A-names for $\mathcal{L}^*$ into A. Intuitively, d provides each A-name with a designation.

relative truth: Let $\phi = \phi(a_1, a_2, \dots, a_n)$ be a formula whose A-letters are as displayed. Then ϕ is true in the A-model $\mathbb{M}^*$ *relative* to $v \in V$ – in symbols, $\mathcal{M}^* \models_v \phi$ –if $a_1, a_2, \dots, a_n \in Dm(v)$ and $\mathcal{M} \models \phi[v(a_1), v(a_2), \dots, v(a_n)]$.

absolute truth: $\mathcal{M}^* \models \phi$ if $\mathcal{M}^* \models_v \phi$ for any $v \in V$ for which $a_1, a_2, \dots, a_n \in Dm(v)$.

Let A_ϕ be $\{a : aisanA - letterof\phi\}$. Say that v is defined on (over) ϕ if v is defined on (over) A_ϕ. We use $v(\phi)$ for the result of substituting $v(a)$ for each occurrence of an A-letter a in ϕ. The principle of generic attribution now takes the following simple form:

$$\mathcal{M}^* \models \phi \text{ iff } \mathcal{M} \models v(\phi) \text{ for each } v \in V \text{ defined over } \phi.$$

The principle may be extended to a set Δ of formulas in the obvious way. Say that $v \in V$ is defined over Δ if v is defined over A_Δ; and let $v(\Delta) = \{v(\phi) : \phi \in \Delta\}$. Then we stipulate that:

$$\mathcal{M}^* \models \Delta \text{ iff } \mathcal{M} \models v(\Delta) \text{ for each } v \in V \text{ defined over } \Delta.$$

If the language $\mathcal{L}^*$ contained classical as well as generic predicates or if it embodied in some other way the distinction between classical and generic occurrences of A-letters, then the truth-definition would need to be modified. Suppose that $\phi(a_1, \ldots, a_m, b_1, \ldots, b_m)$ is a sentence of $\mathcal{L}^*$ in which $a_1, \ldots, a_m$ are the A-letters that occur generically and $b_1, \ldots, b_m$ are the A-letters that occur classically. Then, we may say:

$$\mathcal{M}^* \models \phi(a_1, \ldots, a_m, b_1, \ldots, b_n) \text{ iff } \mathcal{M}^* \models \phi[v(a_1), \ldots, v(a_n), b_1, \ldots, b_n] \text{ for all } v \in V \text{ defined over } \{a_1, \ldots, a_m\}.$$

3.5 Validity

Given a sentence ϕ of $\mathcal{L}^*$ and a collection X of A-models for $\mathcal{L}^*$, we say that ϕ is *valid relative to* X – in symbols, $\models_X \phi$ – if , for each $\mathcal{M}^* \in X$, $\mathcal{M}^* \models \phi$. We say that ϕ is *generically valid* – in symbols, $\models_G \phi$ – if ϕ is valid relative to the class of all A-models for $\mathcal{L}^*$. We say that a sentence ϕ of $\mathcal{L}$ is *classical valid*, $\models_C \phi$ if ϕ is true in all classical models. Then, we have:

$$\text{For any sentence } \phi \text{ of } \mathcal{L}^*, \models_G \phi \text{ iff } \models_C \phi.$$

An inference concerning A-objects is truth-to-truth valid if the truth of the premisses logically guarantees the truth of the conclusion. Take an inference to be an ordered pair (Δ, ϕ), where ϕ is a sentence and Δ a set of sentences from $\mathcal{L}^*$. Let X be a collection of A-models for $\mathcal{L}^*$. Then the inference (Δ, ϕ) is said to be truth-to-truth valid in X if, for any model $\mathcal{M}^*$ of X, $\mathcal{M}^* \models \phi$ whenever $\mathcal{M}^* \models \Delta$.

And an inference is case-to-case valid if every case in which the premisses are true is a case in which the conclusion is true. More formally, the inference (Δ, ϕ) is *case-to-case valid relative to* the class of A-models X for $\mathcal{L}^*$ if, for any model $\mathcal{M}^*$ of X and assignment v for $\mathcal{M}^*$, $\mathcal{M} \models v(\phi)$, whenever $\mathcal{M} \models v(\Delta)$ and v is defined on both Δ and ϕ.

4 Systems

4.1 The HIlbert System H

This has the following axioms and rules:

(1) All tautologous formulas

(2) $\forall x\phi(x) \supset \phi(t)$

(3) $\forall x(\phi \supset \psi) \supset (\forall x\phi \supset \forall x\psi)$

(4) $\phi \supset \forall x\phi$

MP $\phi, \phi \supset \psi/\psi$

Gen. $\phi(a)/\forall x\phi(x)$, for a not in $\phi(x)$

In (2), it is assumed that t is a term (containing no variables) and that $\phi(t)$ comes from $\phi(x)$ upon replacing all free occurrences of x which t. In Gen., it is assumed that a is an A-letter and that $\phi(a)$ comes from $\phi(x)$ upon replacing all free occurrences of x with a.

The notions of *proof* and *theorem* for such a system are defined in the usual way.

Under the generic semantics, the A-letters are conceived as designating suitable A-objects. What is required of these A-objects is given by the following definition. An A-model $\mathcal{M}^*$ for $\mathcal{L}^*$ is *suitable* if :

(i) $a \neq b$ for a and b distinct A-letters of $\mathcal{L}^*$;

(ii) each a, for a an A-letter of $\mathcal{L}^*$, is unrestricted.

It will be recalled that an A-object is unrestricted if it is both independent and value-unrestricted.

Then we can prove that this system is sound and complete.

5 References

- Kit Fine, “A defense of arbitrary objects”, Proceedings of the Aristotelian Society Supp. Vol. 58: 55–77, 1983.
- Kit Fine, Reasoning with Arbitrary Objects. New York: Blackwell, 1985.