Knowing That, Knowing What, and Public Communication: Public Announcement Logic with Kv Operators*

Yanjing Wang, Jie Fan
Department of Philosophy, Peking University
{y.wang,fanjie}@pku.edu.cn

Abstract

In his seminal work [Pla89], Plaza proposed the public announcement logic (**PAL**), which is considered as the pilot logic in the field of dynamic epistemic logic. In the same paper, Plaza also introduced an interesting "know-value" operator *Kv* and listed a few valid formulas of **PAL**+*Kv*. However, it is unknown that whether these formulas, on top of the axioms for **PAL**, completely axiomatize **PAL**+*Kv*. In this paper, we first gave a negative answer to this open question. Moreover, we generalize the *Kv* operator and show that in the setting of **PAL**, replacing the *Kv* operator with its generalized version does not increase the expressive power of the resulting logic. This suggests that we can simply use the more flexible generalization instead of the original **PAL**+*Kv*. As the main result, we give a complete proof system for **PAL** plus the generalized operator based on a complete axiomatization of epistemic loic with the same operator in the single-agent setting.

Key words: public announcement logic, know-value operator, modal logic, multi-agent system

1 Introduction

As originally proposed in [VW51, Hin62], classic epistemic logic (**EL**) focuses on *propositional knowledge* of agents, expressed as *knowing that* p. Such formalism of knowledge turns out to be very successful in the research of AI and theoretical computer science, demonstrated by the widely use of various **EL**-based multi-agent systems, such as the epistemic temporal approach proposed in [FHMV95, PR85] and the dynamic epistemic approach proposed in [Pla89, GG97, BMS98].

On the other hand, there are other types of knowledge which are relevant to AI, such as *procedural knowledge* (*knowing how* to do a), and *descriptive knowledge* (*knowing*

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what is d). Compared to the heated discussions between propositional knowledge and procedural knowledge in epistemology (cf. e.g., [Ryl49, Car79, SW01]), the distinctions between "knowing that" and "knowing what" received relatively little attention, although some early work in AI, such as [McC79], claimed that "knowing what" is the most useful notion among the three.

At the first glance, knowing what may seem to be reducible to knowing that, for example, instead of saying (1): "He knows [*what*] Peter's phone number [is]", we can also say (2): "He knows *that* Peter's phone number is 1234" if we know Peter's phone number. However, if we replace the "phone number" in (1) by "password" which is usually not known to us, then to express the exact meaning of the revised (1) in terms of knowing that, we may need to say the following:

(3): "He knows *that* Peter's password is 1111 *or* he knows *that* Peter's password is 1112 *or* ... *or* he knows *that* Peter's password is 9999"

Clearly, in every day communication, we prefer the succinctness of sentence (1) compared to (3). Actually the scenario behind (3) is crucial in the setting of information security, where it is very common to assume that each agent has its private key while others have no idea what it is. We would like to express "I know that he knows his password, but I do not know it." However, the literal translation in terms of propositional knowledge ($K_1K_2\phi \wedge \neg K_1\phi$) is clearly inconsistent in classic epistemic logic. Such concerns inspired different formalisms of knowledge in epistemic logic related to security verification (cf. e.g., [RS05, HP03, DW10]).

Independently from all the above mentioned works, Plaza proposed a very natural knowing what operator *Kv* in the seminal work [Pla89], which is well-known for its contribution to *public announcement logic* (**PAL**) and the reduction-style axiomatization.[1] Intuitively, $Kv_i d$ expresses exactly that "Agent i *knows what* d is (or i knows the *value* of d, in Plaza's original term)." The *Kv* operator was used in [Pla89] to handle the following puzzle:[2]

Example 1. *(Sum and Product) Mr. A chooses two natural numbers x and y and tell their sum to S only, and their product to P only. Now the following conversation takes place:*

P: I don't know the numbers.
S: I knew you didn't know.
P: But now I know the numbers!
S: So do I!
Question: What are x and y if they are not greater than 100?

Note that in this puzzle it is crucial to distinguish the two kinds of knowledge: S *knows that* P does not *know what* are the numbers (formally, $K_S\neg Kv_P d$). With the

[1] **PAL** was also independently proposed in [GG97, Ger99].

[2] This example is also discussed using **PAL** in [vDRV08], where the knowing-what formulas are encoded by disjunctions of knowing-that formulas with special basic propositions representing values of numbers within a fixed finite upper bound. For discussions on the origin of the sum-and-product riddle, please also refer to [vDRV08].

corresponding semantics, formulas like $K_S Kv_P d \wedge \neg Kv_S d$ are perfectly consistent, in contrast to the inconsistency of $K_S K_P q \wedge \neg K_S q$ in the standard epistemic logic.

Plaza gave an axiomatization of **ELKv** (the epistemic logic with the new *Kv* operator) and proposed a few axioms for **PALKv** (**PAL** with *Kv*). However, it is unknown whether these axioms are enough to axiomatize **PALKv**(cf. [Pla89] and [vD07]).

In this paper, we will answer this open question and continue Plaza's work on *Kv* in the context of public announcement logic. Our main technical contributions of this paper are summarized as below:

- The axioms proposed in [Pla89] do *not* axiomatize **PALKv** completely;
- We introduce the *relativized Kv* operator which is a generalization of the original *Kv* operator. We give a highly non-trivial complete axiomatization of the *single-agent* epistemic logic with this new operator (**ELKv**^{r}), based on which a complete axiomatization of single-agent **PAL** with this operator (**PALKv**^{r}) is provided;
- **PALKv**, **PALKv**^{r} and **ELKv**^{r} are equally expressive but **ELKv** is strictly less expressive than them. Therefore it is impossible to reduce **PALKv** to **ELKv**, and we can simply use the more felxible **PALKv**^{r} instead of **PALKv** without increasing the expressive power.

The rest of the paper is organized as follows: In Section 2 we review the basic results of **PAL**, and introduce the *Kv* operator and the proof system $\mathbb{PALKV}_p$ that Plaza proposed. Section 3 proves the incompleteness of $\mathbb{PALKV}_p$. In Section 4 we introduce the relativized *Kv* operator and show that **PALKv**^{r} is equally expressive as **ELKv**^{r}. We then axiomatize **PALKv**^{r} based on a complete axiomatization of **ELKv**^{r}, which is given in Section 5 for the single-agent case. Section 6 discusses the expressivity of the various logics we introduced and we conclude with future work in Section 7.

2 Preliminaries

2.1 Public Announcement Logic

Definition 2 (Logical Languages **EL**(**I**,**P**) and **PAL**(**I**,**P**))**.** *Given a set **P** of proposition letters and a set **I** of agent names, the language of public announcement logic **PAL(I,P)** is defined as follows:*

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid K_i\phi \mid \langle\phi\rangle\phi$$

*where $p \in$ **P**, $i \in$ **I**. The language of epistemic logic (**EL(I,P)**) is the announcement-free fragment of **PAL(I,P)**.*

$K_i\phi$ is read "agent i knows that ϕ". $\langle\psi\rangle\phi$ says that " ψ can be truthfully announced publicly, and after its announcement ϕ holds." As usual, we define $\bot$, $\phi \vee \psi$, $\phi \to \psi$,

$\hat{K}_i\phi$ and $[\psi]\phi$ as the abbreviations of $\neg\top$, $\neg(\neg\phi \wedge \neg\psi)$, $\neg\phi \vee \psi$, $\neg K_i\neg\phi$ and $\neg\langle\psi\rangle\neg\phi$ respectively.

In the sequel, we fix **I** and **P** thus simply writing **PAL** and **EL** for **PAL**(**I**,**P**) and **EL**(**I**,**P**) respectively.

The semantics of PAL is defined on (S5) Kripke structures. An *epistemic model* for **PAL** is a triple $\mathcal{M} = \langle S, \{\sim_i \mid i \in \mathbf{I}\}, V\rangle$ where S is a non-empty set of possible worlds, $\sim_i$ is an equivalence relation over S, and V is a valuation function assigning a set of worlds $V(p) \subseteq S$ to each $p \in \mathbf{P}$. Given an epistemic model $\mathcal{M}$, the semantics is defined as follows:

$\mathcal{M}, s \vDash \top$	$\Leftrightarrow$	always
$\mathcal{M}, s \vDash p$	$\Leftrightarrow$	$s \in V(p)$
$\mathcal{M}, s \vDash \neg\phi$	$\Leftrightarrow$	$\mathcal{M}, s \nvDash \phi$
$\mathcal{M}, s \vDash \phi \wedge \psi$	$\Leftrightarrow$	$\mathcal{M}, s \vDash \phi$ and $\mathcal{M}, s \vDash \psi$
$\mathcal{M}, s \vDash K_i\psi$	$\Leftrightarrow$	for all t such that $s \sim_i t : \mathcal{M}, t \vDash \psi$
$\mathcal{M}, s \vDash \langle\psi\rangle\phi$	$\Leftrightarrow$	$\mathcal{M}, s \vDash \psi$ and $\mathcal{M}\|_\psi, s \vDash \phi$

where $\mathcal{M}|_\psi = (S', \{\sim'_i \mid i \in \mathbf{I}\}, V')$ such that:

$$S' = \{s \mid \mathcal{M}, s \vDash \psi\}, \sim'_i = \sim_i |_{S' \times S'}, \text{ and } V'(p) = V(p) \cap S'.$$

An announcement $\langle\psi\rangle$ is interpreted as a *model transformer* which deletes the worlds that do not satisfy ψ.

It is shown in [Pla89] via a reduction to **EL** that the following axiomatization is sound and complete (cf. also [vDvdHK07] for details):

System $\mathbb{PAL}$

Axiom Schemas		Rules	
`TAUT`	all the instances of tautologies	`MP`	$\dfrac{\phi, \phi \to \psi}{\psi}$
`DISTK`	$K_i(\phi \to \psi) \to (K_i\phi \to K_i\psi)$	`NECK`	$\dfrac{\phi}{K_i\phi}$
`!ATOM`	$\langle\psi\rangle p \leftrightarrow (\psi \wedge p)$	`RE`	$\dfrac{\psi \leftrightarrow \chi}{\phi(\psi) \leftrightarrow \phi(\chi)}$
`!NEG`	$\langle\psi\rangle\neg\phi \leftrightarrow (\psi \wedge \neg\langle\psi\rangle\phi)$		
`!CON`	$\langle\psi\rangle(\phi \wedge \chi) \leftrightarrow (\langle\psi\rangle\phi \wedge \langle\psi\rangle\chi)$		
`!K`	$\langle\psi\rangle K_i\phi \leftrightarrow (\psi \wedge K_i(\psi \to \langle\psi\rangle\phi))$		
`T`	$K_i\phi \to \phi$		
`4`	$K_i\phi \to K_iK_i\phi$		
`5`	$\neg K_i\phi \to K_i\neg K_i\phi$		

where $p \in \mathbf{P} \cup \{\top\}$.

Note that in [Pla89] Plaza did not make it clear whether the `RE` rule (*replacement of equivalences*) allows the replacements for the formulas inside the announcements. Here we suppose so.[3]

[3] This will make our incompleteness result stronger.

It can be shown that the following schemata are derivable/admissible in $\mathbb{PAL}$:[4]

Axiom Schema		Rule	
DIST!	$[\chi](\phi \to \psi) \to ([\chi]\phi \to [\chi]\psi)$	NEC!	$\dfrac{\phi}{[\chi]\phi}$

We will be using DIST! and NEC! in the later discussions. In the sequel, we use $\mathbb{EL}$ to denote $\mathbb{PAL}$ without the so-called "reduction axioms" (i.e., !ATOM, !NEG, !CON, and !K). It is well-known that $\mathbb{EL}$ completely axiomatizes **EL**.

2.2 Adding the *Kv* operator

Given **I**, **P** and a set **D** of names, the language $\mathbf{PALKv}(\mathbf{I}, \mathbf{P}, \mathbf{D})$ (**PALKv** for short) is defined as:

$$\phi ::= \top \mid p \mid Kv_i d \mid \neg\phi \mid \phi \wedge \phi \mid K_i\phi \mid \langle\phi\rangle\phi$$

where $p \in \mathbf{P}$, $i \in \mathbf{I}$, and $d \in \mathbf{D}$. As before, we obtain an epistemic language **ELKv**(**I**,**P**,**D**)(**ELKv** for short) when we ignore the announcements in **PALKv**.

To interpret the new $Kv_i d$ formulas, we need the assignment function $V_\mathbf{D} : \mathbf{D} \times S \to O$ in the epistemic models, where O is a non-empty set of *objects*. The semantics of $Kv_i d$ is as follows:

$$\mathcal{M}, s \vDash Kv_i d \iff \text{for any } t_1, t_2 : \text{ if } s \sim_i t_1, s \sim_i t_2, \text{ then } V_\mathbf{D}(d, t_1) = V_\mathbf{D}(d, t_2).$$

Kv acts as a mixture of a modality and a predicate. It is not hard to see that Kv_i also obeys the positive and negative introspections, namely the following two axioms are valid:

Kv4	$Kv_i d \to K_i Kv_i d$
Kv5	$\neg Kv_i d \to K_i \neg Kv_i d$

It is claimed (without a proof) in [Pla89] that adding Kv4 and Kv5 to $\mathbb{EL}$ completely axiomatizes **ELKv**.

Now a natural question is: how can we axiomatize **PALKv**? [Pla89] proposed the following extra axioms on top of $\mathbb{PAL}$ with Kv4, Kv5:

KKv	$\langle K_i\phi\rangle Kv_i d$	$\leftrightarrow$	$K_i\phi \wedge Kv_i d$
KvKv	$\langle Kv_i c\rangle Kv_i d$	$\leftrightarrow$	$Kv_i c \wedge Kv_i d$
!Kv	$\langle\phi\rangle Kv_i d$	$\to$	$K_i(\phi \to \langle\phi\rangle Kv_i d)$
!nKv	$\langle\phi\rangle\neg Kv_i d$	$\to$	$K_i(\phi \to \langle\phi\rangle\neg Kv_i d)$

It is unknown whether the above axiom schemata are enough to completely axiomatize **PALKv**.[5]

In fact, the last three axioms above are superfluous. In the sequel, let us denote $\mathbb{PAL} + \mathtt{Kv4} + \mathtt{Kv5} + \mathtt{KKv}$ as $\mathbb{PALKV}_p$. It is not hard to see that KvKv, !Kv, !nKv can all be derived in $\mathbb{PALKV}_p$:

[4]For a thorough discussion on axiomatizations of **PAL**, see [WC13].

[5]Note that there is no obvious reduction axiom for $Kv_i d$ and we will show it is impossible to reduce **PALKv** to **ELKv** in Section 6.

Proposition 3. (1) $\vdash_{\mathbb{PALKV}_p} \langle Kv_i c\rangle Kv_i d \leftrightarrow (Kv_i c \wedge Kv_i d)$;
(2) $\vdash_{\mathbb{PALKV}_p} \langle\phi\rangle Kv_i d \to K_i(\phi \to \langle\phi\rangle Kv_i d)$;
(3) $\vdash_{\mathbb{PALKV}_p} \langle\phi\rangle\neg Kv_i d \to K_i(\phi \to \langle\phi\rangle\neg Kv_i d)$.

Proof. For (1):

(i)	$Kv_i c \leftrightarrow K_i Kv_i c$	`Kv4,T`
(ii)	$\langle Kv_i c\rangle Kv_i d \leftrightarrow \langle K_i Kv_i c\rangle Kv_i d$	`RE`
(iii)	$\langle K_i Kv_i c\rangle Kv_i d \leftrightarrow K_i Kv_i c \wedge Kv_i d$	`KKv`
(iv)	$\langle Kv_i c\rangle Kv_i d \leftrightarrow (Kv_i c \wedge Kv_i d)$	`MP`(ii) (iii)

For (2):

(i)	$Kv_i d \to K_i Kv_i d$	`Kv4`
(ii)	$\langle\phi\rangle Kv_i d \to \langle\phi\rangle K_i Kv_i d$	`DIST!,NEC!`
(iii)	$\langle\phi\rangle K_i Kv_i d \leftrightarrow (\phi \wedge K_i(\phi \to \langle\phi\rangle Kv_i d))$	`!K`
(iv)	$\langle\phi\rangle Kv_i d \to K_i(\phi \to \langle\phi\rangle Kv_i d)$	`MP`(ii)(iii)

The proof of (3) is similar as (2). □

In the later discussion we will focus on $\mathbb{PALKV}_p$.

3 Incompleteness of $\mathbb{PALKV}_p$

To prove the incompleteness of $\mathbb{PALKV}_p$, we will show that the following valid formula is not derivable in $\mathbb{PALKV}_p$:

$$\theta : \langle p\rangle Kv_i d \wedge \langle q\rangle Kv_i d \to \langle p \vee q\rangle Kv_i d$$

In order to show this, we design a semantics $\Vdash$ on a class $\mathbb{C}$ of special models such that all the axiom schemata and rules of $\mathbb{PALKV}_p$ are valid w.r.t. $\Vdash$ on $\mathbb{C}$ but not θ. Namely we have for all **PALKv** formulas $\phi : \vdash_{\mathbb{PALKV}_p} \phi \implies \mathbb{C} \Vdash \phi$ and $\mathbb{C} \nVdash \theta$, thus we have $\nvdash_{\mathbb{PALKV}_p} \theta$.

Inspired by the semantics developed in [WC13], we treat the announcement operator $\langle\phi\rangle$ as a standard modality interpreted on models with ϕ transitions.

Definition 4. *(Extended model) An extended (epistemic) model for **PALKv** is a tuple* $\langle S, \{\sim_i \mid i \in \boldsymbol{I}\}, \{\xrightarrow{\phi} \mid \phi \in \textbf{\textit{PALKv}}\}, V, V_{\boldsymbol{D}}\rangle$ *where*

- $\langle S, \{\sim_i \mid i \in \boldsymbol{I}\}, V, V_{\boldsymbol{D}}\rangle$ *is a standard epistemic model for **PALKv**.*
- *For each **PALKv** formula* ϕ, $\xrightarrow{\phi}$ *is a binary relation over* S.

We now define the truth conditions of **PALKv** formulas on extended models (w.r.t. $\Vdash$) as follows:

$$
\begin{array}{lcl}
\mathcal{M}, s \Vdash \top & \iff & \text{always} \\
\mathcal{M}, s \Vdash p & \iff & s \in V(p) \\
\mathcal{M}, s \Vdash \neg\phi & \iff & \mathcal{M}, s \nVdash \phi \\
\mathcal{M}, s \Vdash \phi \wedge \psi & \iff & \mathcal{M}, s \Vdash \phi \text{ and } \mathcal{M}, s \Vdash \psi \\
\mathcal{M}, s \Vdash K_i\phi & \iff & \text{for all } t \in W \text{ such that } s \sim_i t, \mathcal{M}, t \Vdash \phi \\
\mathcal{M}, s \vDash Kv_i d & \iff & \text{for any } t_1, t_2: \text{ if } s \sim_i t_1, s \sim_i t_2, \\
 & & \text{then } V_{\mathbf{D}}(d, t_1) = V_{\mathbf{D}}(d, t_2). \\
\mathcal{M}, s \Vdash \langle\phi\rangle\psi & \iff & \text{for some } t \text{ such that } s \xrightarrow{\phi} t \text{ and } t \Vdash \psi.
\end{array}
$$

Compared with the standard semantics w.r.t. $\vDash$, the only change is the interpretation of $\langle\phi\rangle\psi$. It is clear that

Proposition 5. *$\Vdash$ coincides with $\vDash$ on **ELKv** formulas.*

Now we try to define a class of extended models where these two semantics get even closer on **PALKv** formulas.

Definition 6. *(Normal extended epistemic model) An extended model $\mathcal{M} = \langle S, \{\sim_i \mid i \in \boldsymbol{I}\}, \{\xrightarrow{\phi} \mid \phi \in \boldsymbol{PALKv}\}, V, V_{\boldsymbol{D}}\rangle$ is called normal if the following properties hold for any $s, t \in S$, any $d \in \boldsymbol{D}$ and $\phi \in \boldsymbol{PALKv}$:*

P-Invariance (P-INV) *if $s \xrightarrow{\phi} t$, then for all $p \in P$: $s \in V(p) \iff t \in V(p)$.*

Partial Functionality (PFUNC) *If $\mathcal{M}, s \Vdash \phi$ then s has a unique ϕ-successor. If $\mathcal{M}, s \nVdash \phi$ then s has no ϕ-successor.*

No Miracle (NM) *if $s \sim_i s'$, $s \xrightarrow{\phi} t$ and $s' \xrightarrow{\phi} t'$, then $t \sim_i t'$.*

Perfect Recall (PR) *if $t \sim_i t'$ and $s \xrightarrow{\phi} t$, then there exists an s' such that $s \sim_i s'$ and $s' \xrightarrow{\phi} t'$.*

D-Invariance (D-INV) *if $s \xrightarrow{K_i\phi} t$, then $V_{\boldsymbol{D}}(d, s) = V_{\boldsymbol{D}}(d, t)$.*

U-Replacement (U-RE) *If $\mathcal{M} \Vdash \phi \leftrightarrow \psi$ then $s \xrightarrow{\phi} t \iff s \xrightarrow{\psi} t$.*

Note that PFUNC and U-RE refer to $\Vdash$. Let $\mathbb{C}$ be the class of all the normal extended models. Below we will show that $\mathbb{PALKV}_p$ is sound on $\mathbb{C}$ w.r.t. $\Vdash$. Based on Proposition 5, we have:

Lemma 7. *`TAUT`, `MP`, `DISTK`, `NECK`, `T`, `4`, `5`, `Kv4`, and `Kv5` are all valid in $\mathbb{C}$ w.r.t. $\Vdash$.*

We also need the validity of `RE`:

Lemma 8. *`RE` is valid in $\mathbb{C}$ w.r.t. $\Vdash$.*

Proof. To prove RE is valid, it suffices to show that the $\mathbb{C}$-validity of $\phi \leftrightarrow \psi$ will be preserved by using connectives and operators in **PALKv**.

First note that due to the similarity between $\Vdash$ and $\vDash$, it is not hard to show that if $\mathbb{C} \Vdash \phi \leftrightarrow \psi$ then the following are also valid on $\mathbb{C}$ w.r.t. $\Vdash$:

$$\neg\phi \leftrightarrow \neg\psi, (\phi \wedge \chi) \leftrightarrow (\psi \wedge \chi), K_i\phi \leftrightarrow K_i\psi$$

Now we check the cases w.r.t. the announcements. Suppose $\mathbb{C} \Vdash \phi \leftrightarrow \psi$ and $\mathcal{M}, s \Vdash \langle\chi\rangle\phi$ for some $\mathcal{M} \in \mathbb{C}$, then there exists a t in $\mathcal{M}$ such that $s \xrightarrow{\chi} t$ and $\mathcal{M}, t \Vdash \phi$. Thus there exists a t in $\mathcal{M}$ such that $s \xrightarrow{\chi} t$ and $\mathcal{M}, t \Vdash \psi$. Therefore $\mathcal{M}, s \Vdash \langle\chi\rangle\psi$.

Finally, suppose $\mathbb{C} \Vdash \phi \leftrightarrow \psi$ and $\mathcal{M}, s \Vdash \langle\phi\rangle\chi$ for some $\mathcal{M} \in \mathbb{C}$ then there is some t in $\mathcal{M}$ such that $s \xrightarrow{\phi} t$ and $\mathcal{M}, t \Vdash \chi$. Since $\mathbb{C} \Vdash \phi \leftrightarrow \psi$, we have $\mathcal{M} \Vdash \phi \leftrightarrow \psi$, thus by U-RE $s \xrightarrow{\psi} t$. Therefore $\mathcal{M}, s \Vdash \langle\psi\rangle\chi$.

□

Lemma 9. *!ATOM, !NEG, !CON, and !K are valid in $\mathbb{C}$ w.r.t. $\Vdash$.*

Proof. In the following we fix an $\mathcal{M}, s$ in $\mathbb{C}$, thus write $s \Vdash \phi$ for $\mathcal{M}, s \Vdash \phi$.

For !ATOM: $s \Vdash \langle\phi\rangle p \iff$ (there exists a t such that $s \xrightarrow{\phi} t$ and $t \Vdash p$) $\iff$ ($s \Vdash \phi$ and $s \Vdash p$) (by P-INV and PFUNC) $\iff s \Vdash \phi \wedge p$.

For !NEG: Due to PFUNC it is easy to verify $\mathbb{C} \Vdash \langle\phi\rangle\chi \rightarrow \phi \wedge [\phi]\chi$ and $\mathbb{C} \Vdash [\phi]\chi \wedge \phi \rightarrow \langle\phi\rangle\chi$. Now let $\chi = \neg\psi$ it is easy to see !NEG is valid.

For !CON: PFUNC ensures the uniqueness of the ϕ-successor of any state thus the validity of !CON is straightforward.

For !K: It is not hard to see that PR and NM guarantee the validity of the following two formulas respectively:

$$\langle\phi\rangle\hat{K}_i\psi \rightarrow \hat{K}_i\langle\phi\rangle\psi \quad \hat{K}_i\langle\phi\rangle\psi \rightarrow [\phi]\hat{K}_i\psi.$$

Taking the contrapositives and replacing $\neg\psi$ by ψ we have the validity of:

$$K_i[\phi]\psi \rightarrow [\phi]K_i\psi \quad \langle\phi\rangle K_i\psi \rightarrow K_i[\phi]\psi.$$

Then due to PFUNC, the following are valid:

$$\phi \wedge K_i[\phi]\psi \rightarrow \langle\phi\rangle K_i\psi \quad \langle\phi\rangle K_i\psi \rightarrow \phi \wedge K_i[\phi]\psi.$$

Namely, $\mathbb{C} \Vdash \langle\phi\rangle K_i\psi \leftrightarrow \phi \wedge K_i[\phi]\psi$. Now from the case of !NEG, we know that $\mathbb{C} \Vdash [\phi]\psi \leftrightarrow (\phi \rightarrow \langle\phi\rangle\psi)$, therefore by the validity of RE we have $\mathbb{C} \Vdash$!K. □

Lemma 10. *KKv ($i.e.\langle K_i\phi\rangle Kv_i d \leftrightarrow (K_i\phi \wedge Kv_i d)$) is valid on $\mathbb{C}$ w.r.t. $\Vdash$.*

Proof. Suppose $\mathcal{M}, s \Vdash \langle K_i\phi\rangle Kv_i d$ for some $\mathcal{M} \in \mathbb{C}$ and some **PALKv** formula ϕ. Then there exists a t such that $s \overset{K_i\phi}{\to} t$ and $\mathcal{M}, t \Vdash Kv_i d$. By PFUNC, we have $\mathcal{M}, s \Vdash K_i\phi$. Now take two arbitrary points s_1 and s_2 such that $s \sim_i s_1$ and $s \sim_i s_2$, we need to show $V_{\mathbf{D}}(d, s_1) = V_{\mathbf{D}}(d, s_2)$. Since $\mathcal{M}, s \Vdash K_i\phi$, it is easy to see that $\mathcal{M}, s_1 \Vdash K_i\phi$ and $\mathcal{M}, s_2 \Vdash K_i\phi$. Again by PFUNC, there are t_1 and t_2 such that $s_1 \overset{K_i\phi}{\to} t_1$ and $s_2 \overset{K_i\phi}{\to} t_2$. Now by NM, we have $t_1 \sim_i t \sim_i t_2$. Since $\mathcal{M}, t \Vdash Kv_i d$, we know that $V_{\mathbf{D}}(d, t_1) = V_{\mathbf{D}}(d, t_2)$. Now by D-INV, $V_{\mathbf{D}}(d, t_1) = V_{\mathbf{D}}(d, s_1)$ and $V_{\mathbf{D}}(d, t_2) = V_{\mathbf{D}}(d, s_2)$. Therefore $\mathcal{M}, s \Vdash Kv_i d$.

For the converse, suppose that $\mathcal{M}, s \Vdash K_i\phi \wedge Kv_i d$. By PFUNC, s has a unique $K_i\phi$-successor, say t. We need to show $\mathcal{M}, t \Vdash Kv_i d$. Now take two arbitrary points t_1, t_2 such that $t_1 \sim_i t \sim_i t_2$. Due to PR, there exist s_1, s_2 such that $s_1 \sim_i s \sim_i s_2$, $s_1 \overset{K_i\phi}{\to} t_1$, and $s_2 \overset{K_i\phi}{\to} t_2$. Due to D-INV and the assumption that $\mathcal{M}, s \Vdash Kv_i d$, it is clear that $V_{\mathbf{D}}(d, t_1) = V_{\mathbf{D}}(d, t_2)$ thus $\mathcal{M}, t \Vdash Kv_i d$. □

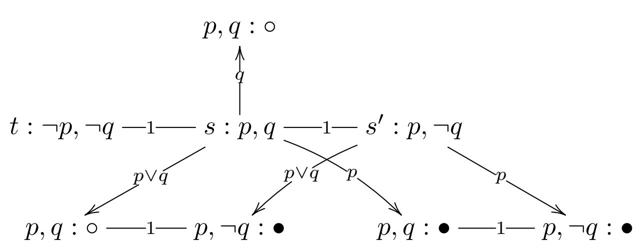
Lemmata 7,8,9, and 10, implies the soundness:

Lemma 11. *$\mathbb{PALKV}_p$ is sound w.r.t $\Vdash$ on $\mathbb{C}$.*

However, the $\vDash$-valid formula θ is not $\Vdash$-valid on $\mathbb{C}$.

Lemma 12. *$\mathbb{C} \nVdash \langle p\rangle Kv_i d \wedge \langle q\rangle Kv_i d \to \langle p \vee q\rangle Kv_i d$.*

Proof. For simplisity we assume $\mathbf{I} = \{1\}, \mathbf{P} = \{p, q\}$ and $\mathbf{D} = \{d\}$. Consider the following model $\mathcal{M}$ (we omit the reflexive epistemic links and only show the necessary parts for evaluating the desired formula, with $\bullet$ and $\circ$ denoting the objects assigned to d):



Note that for any ϕ: $\mathcal{M} \nVdash (p \vee q) \leftrightarrow K_1\phi$, since $K_1\phi$ will have a uniform truth value on t, s, s' while $p \vee q$ is false only on the leftmost world t. Similarly, for any ϕ: $\mathcal{M} \nVdash p \leftrightarrow K_1\phi$ and $\mathcal{M} \nVdash q \leftrightarrow K_1\phi$. Thus although U-RE and D-INV hold in $\mathcal{M}$, the assignments of d at s and s' can be different from the assignments at the bottom four worlds and the top world. It is not hard to see that (a proper completion of) $\mathcal{M}$ is a normal extended model and

$$\mathcal{M}, s \Vdash \langle p\rangle Kv_i d \wedge \langle q\rangle Kv_i d \wedge \neg\langle p \vee q\rangle Kv_i d.$$

Therefore θ is not valid in $\mathbb{C}$. □

From Lemmata 11 and 12 we have:

Theorem 13. *$\langle p\rangle Kv_i d \wedge \langle q\rangle Kv_i d \to \langle p \vee q\rangle Kv_i d$ is not derivable in $\mathbb{PALKV}_p$, thus $\mathbb{PALKV}_p$ is not complete w.r.t. $\vDash$ on epistemic models.*

4 PAL with the relativized *Kv* operator

To prevent the counter example in the previous section, we need an axiom to guarantee that the assignments of the names in **D** will essentially stay the same after any announcements. A similar issue applies to the basic propositions but we can use `!ATOM` to guarantee the propositional invariance. In contrast, we cannot directly talk about the assignments in **PALKv**.

In this section, instead of trying to give an axiomatization of **PALKv** we will discuss the language **PALKv**^{r} where *Kv* is replaced by a more general operator, which can be viewed as a relativized version of *Kv*. The idea of relativizing a modal operator also appeared in [vBvEK06] where the authors gave an axiomatization of **PALC**^{r} which is **PAL** extended with a relativized version of the common knowledge operator.

Formally the language of **PALKv**^{r} is defined as follows:

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid K_i\phi \mid Kv_i(\phi, d) \mid \langle\phi\rangle\phi$$

where $p \in \mathbf{P}, i \in \mathbf{I}, d \in \mathbf{D}$.

$Kv_i(\phi, d)$ says that the agent i would know what d is if he were informed that ϕ is true. It is different from $\phi \to Kv_i d$, $K_i\phi \to Kv_i(\phi, d)$, and $K_i(\phi \to Kv_i d)$. The distinction will become clear after understanding the following semantics w.r.t. $\mathcal{M} = \langle S, \{\sim_i \mid i \in \mathbf{I}\}, V, V_{\mathbf{D}}\rangle$:

$\mathcal{M}, s \vDash Kv_i(\phi, d)$	$\iff$	for every $t_1, t_2 \in S$ such that $s \sim_i t_1$ and $s \sim_i t_2$: if $\mathcal{M}, t_1 \vDash \phi, \mathcal{M}, t_2 \vDash \phi$, then $V_{\mathbf{D}}(d, t_1) = V_{\mathbf{D}}(d, t_2)$

Now the original Kv_i operator can be viewed as $Kv_i(\top, \cdot)$. Therefore **PALKv**^{r} is indeed an extension of **PALKv**. As before, we denote the announcement-free part of **PALKv**^{r} as **ELKv**^{r}.

We can see there is a similarity between an announcement and a condition in the relativized version of *Kv* operator, demonstrated by the following validity:

$$\texttt{!Kv}^r: \quad \langle\phi\rangle Kv_i(\psi, d) \leftrightarrow (\phi \wedge Kv_i(\langle\phi\rangle\psi, d))$$

Note that $\texttt{!Kv}^r$ is also in the shape of the reduction axioms which push the announcement to the "inner" part of the formulas. Based on this observation, we can show, as what Plaza showed for **PAL** and **EL**, that **PALKv**^{r} and **ELKv**^{r} are equally expressive.

Theorem 14. ***PALKv***^{r} *is equally expressive as* ***ELKv***^{r}.

Proof. Since **PALKv**^{r} is an extension of **ELKv**^{r} thus **PALKv**^{r} is no less expressive than **ELKv**^{r}. We show that **PALKv**^{r} is no more expressive than **ELKv**^{r} by giving the following truth preserving translation t : **PALKv**^{r} → **ELKv**^{r}:

$$
\begin{array}{lcl}
t(\top) & = & \top \\
t(p) & = & p \\
t(\neg\phi) & = & \neg t(\phi) \\
t(\phi \wedge \psi) & = & t(\phi) \wedge t(\psi) \\
t(K_i\phi) & = & K_i t(\phi) \\
t(Kv_i(\phi, d)) & = & Kv_i(t(\phi), d) \\
t(\langle\phi\rangle\top) & = & t(\phi) \\
t(\langle\phi\rangle p) & = & t(\phi \wedge p) \\
t(\langle\phi\rangle\neg\psi) & = & t(\phi \wedge \neg\langle\phi\rangle\psi) \\
t(\langle\phi\rangle(\psi \wedge \chi)) & = & t(\langle\phi\rangle\psi \wedge \langle\phi\rangle\chi) \\
t(\langle\phi\rangle K_i\psi) & = & t(\phi \wedge K_i(\phi \to \langle\phi\rangle\psi)) \\
t(\langle\phi\rangle Kv_i(\psi, d)) & = & t(\phi \wedge Kv_i(\langle\phi\rangle\psi, d))
\end{array}
$$

By defining a suitable complexity measure of **PALKv**^{r} formulas as in the case for **PAL**(cf. [vDvdHK07]), we can show that the translation can eventually eliminate the announcement operator. We can also prove $\phi \equiv t(\phi)$ for every $\phi \in$ **PALKv**^{r} by induction on the structure of ϕ. □

Now, as in the case for **PAL**, if we have a complete axiomatization of **ELKv**^{r}, then we can also axiomatize **PALKv**^{r} by using the reduction axioms:

Theorem 15. *If* ***ELKv***^{r} *is completely axiomatized by* $\mathbb{S}$ *then* $\mathbb{S}$ *plus* `RE`*,* `!ATOM`*,* `!CON`*,* `!NEG`*,* `!K`*, and* `!Kv`^{r} *completely axiomatizes* ***PALKv***^{r}.

Note that we need to include the rule `RE` to facilitate the completeness proof via reductions (cf.[WC13] for detailed discussions in the context of **PAL**). In the next section we will axiomatize **ELKv**^{r} in the single agent case.

5 Axiomatization of ELKv$^r_1$

As the reader may expect, axiomatizing **ELKv**^{r} is much harder than the case of **ELKv** due to the condition in the relativized operator. In the rest of this section, we will provide a complete axiomatization of **ELKv**^{r} for the *single agent* case (call it **ELKv**$^r_1$). Although all the new axioms that we will propose also hold in the multi-agent case, there are still some difficulties in the completeness proof related to multiple agents, to which we will come back at the end of the paper.

5.1 System $\mathbb{ELKV}^r_1$

Based on single-agent $\mathbb{EL}$, we propose the following extra axiom schemata and name the resulting system $\mathbb{ELKV}^r_1$:

`DISTKv`^{r}	$K(\phi \to \psi) \to (Kv(\psi, d) \to Kv(\phi, d))$
`Kv`^{r}`4`	$Kv(\phi, d) \to KKv(\phi, d)$
`Kv`$^r\bot$	$Kv(\bot, d)$
`Kv`$^r\vee$	$\hat{K}(\phi \wedge \psi) \wedge Kv(\phi, d) \wedge Kv(\psi, d) \to Kv(\phi \vee \psi, d)$

$\texttt{DISTKv}^r$ is the distribution axiom for the relativised *Kv* operator (pay attention to the positions of ψ and ϕ in the consequent). $\texttt{Kv}^r4$ is a variation of the positive introspection axiom. $\texttt{Kv}^r\bot$ stipulates the precondition of the *Kv* operator. Maybe the most interesting axiom is $\texttt{Kv}^r\vee$ which will play a very important role in the later completeness proof. Intuitively, it handles the composition of the conditions in the relativized operator: suppose all the possible ϕ worlds agree on what d is and all the possible ψ worlds also agree on d, then the overlap between ϕ possibilities and ψ possibilities implies all the $\phi \vee \psi$ possibilities also agree on what d is.

It is not hard to see that the extra axiom schemata are all valid:

Theorem 16. *$\mathbb{ELKV}^r_1$ is sound with respect to the class of epistemic models with assignments.*

In the sequel, for readability, we will sometimes write $\vdash$ for $\vdash_{\mathbb{ELKV}^r_1}$.

Seeing $\texttt{Kv}^r4$, the reader may miss a version of negative introspection $\neg Kv(\phi, d) \to K\neg Kv(\phi, d)$. Actually it can be derived in $\mathbb{ELKV}^r_1$:

Proposition 17. $\vdash_{\mathbb{ELKV}^r_1} \neg Kv(\phi, d) \to K\neg Kv(\phi, d)$

Proof.

$KKv(\phi, d) \leftrightarrow Kv(\phi, d)$	$\texttt{T}, \texttt{Kv}^r4$
$\neg KKv(\phi, d) \to K\neg KKv(\phi, d)$	5
$\neg Kv(\phi, d) \to K\neg Kv(\phi, d)$	RE

□

$\texttt{Kv}^r\vee$ can be generalized to arbitrary finite disjunctions due to an easy induction proof based on $\texttt{Kv}^r\vee$.

Proposition 18. *For any finite set of **ELKv**$^r_1$ formulas U:*

$$\vdash_{\mathbb{ELKV}^r_1} \hat{K}(\bigwedge U) \wedge \bigwedge_{\phi \in U} Kv(\phi, d) \to Kv(\bigvee U, d).$$

To ease the later proof we have the following results:

Proposition 19. *In the system $\mathbb{ELKV}^r_1$, the following theorem and inference rule can be derived:*

1. $\vdash_{\mathbb{ELKV}^r_1} K\neg\phi \to Kv(\phi, d)$
2. *If* $\vdash_{\mathbb{ELKV}^r_1} \phi \leftrightarrow \psi$, *then* $\vdash_{\mathbb{ELKV}^r_1} Kv(\phi, d) \leftrightarrow Kv(\psi, d)$.

Proof. For (1):

(i)	$\neg\phi \leftrightarrow (\phi \to \bot)$	TAUT
(ii)	$K\neg\phi \leftrightarrow K(\phi \to \bot)$	NECK, DISTK
(iii)	$K(\phi \to \bot) \to Kv(\bot, d) \to Kv(\phi, d)$	$\texttt{DISTKv}^r$
(iv)	$Kv(\bot, d)$	$\texttt{Kv}^r\bot$
(v)	$K(\phi \to \bot) \to Kv(\phi, d)$	TAUT(iii)(iv)
(vi)	$K\neg\phi \to Kv(\phi, d)$	MP(ii)(v)

For (2), If $\vdash \phi \leftrightarrow \psi$, then $\vdash \phi \to \psi$ and $\vdash \psi \to \phi$. From the former and rule NECK, it follows that $\vdash K(\phi \to \psi)$. Using the axiom DISTKv^r, we have $\vdash Kv(\psi, d) \to Kv(\phi, d)$. Similarly, we can get $\vdash Kv(\phi, d) \to Kv(\psi, d)$ from $\vdash \psi \to \phi$, and thus obtain $\vdash Kv(\phi, d) \leftrightarrow Kv(\psi, d)$. □

5.2 Completeness of $\mathbb{ELKV}^r_1$

In this subsection we prove that $\mathbb{ELKV}^r_1$ is complete w.r.t. the class of epistemic models. As in normal modal logics, we will show that every consistent set of $\mathbf{ELKv}^r_1$ formulas is satisfiable at some state in a canonical model. In defining the canonical model we need to borrow some ideas from the Henkin construction in first-order logic. The difficulties lie in the definition of the canonical model and the proof of the truth lemma for the new *Kv* operators, as we will explain later.

5.2.1 Canonical model

The canonical model $\mathcal{M}^c$ of $\mathbb{ELKV}^r_1$ is a tuple $\langle S^c, \sim^c, V^c, V^c_{\mathbf{D}}\rangle$ where:

- $S^c = MCS \times \{0, 1\}$, where MCS is the set of $\mathbb{ELKV}^r$-maximal consistent sets. That is, every maximal consistent set has two copies in S^c. We write $\phi \in s$ if ϕ is in the maximal consistent set of s. We write $\phi \in s \cap t$ if $\phi \in s$ and $\phi \in t$.
- $s \sim^c t$ iff $\{\phi \mid K\phi \in s\} \subseteq t$.
- $V^c(p) = \{s \in S^c \mid p \in s\}$.
- $V^c_{\mathbf{D}}(d, s) = |(d, s)|_R$. That is, $V^c_{\mathbf{D}}(d, s)$ is the equivalence class under the equivalence relation R defined below over $\{(d, s) \mid d \in \mathbf{D}, s \in S^c\}$:

$$R = \{((d, s), (e, t)) \mid d = e, s \sim^c t \text{ and } Kv(\phi, d) \in s \text{ for some } \phi \in s \cap t\} \cup \{((d, s), (d, s)) \mid d \in \mathbf{D}, s \in S^c\}$$

To show the above definition is well-defined, we need the following propositions:

Proposition 20. *$s \sim^c t$ iff $\{\phi \mid K\phi \in s\} = \{\phi \mid K\phi \in t\}$, thus $\sim^c$ is an equivalence relation.*

Proof. A standard exercise as in the case for the canonical model of $\mathbb{EL}$. □

Based on the above proposition and Kv^{r}4, the following is immediate:

Proposition 21. *For any $s, t \in S^c$, if $s \sim^c t$, then $Kv(\phi, d) \in s$ iff $Kv(\phi, d) \in t$.*

Now we can show the following:

Proposition 22. *R is an equivalence relation on $\{(d, s) \mid s \in S^c, d \in \boldsymbol{D}\}$.*

Proof. Reflexivity is obvious by the definition of R. Symmetry is also straightforward based on Propositions 20 and 21.

For transitivity, assume that $(d,s)R(e,t)$ and $(e,t)R(f,u)$, we need to show $(d,s)R(f,u)$. First note that if $s=t, d=e$ or $t=u, e=f$, then obviously $(d,s)R(f,u)$. Now let us consider the other case, i.e., $d=e, s\sim^c t$ and $Kv(\phi,d)\in s$ for some $\phi\in s\cap t$, $e=f, t\sim^c u$ and $Kv(\psi,e)\in t$ for some $\psi\in t\cap u$. By $d=e$ and $e=f$, we have $d=f$. By $s\sim^c t$, $t\sim^c u$ and Proposition 20, $s\sim^c u$. Since $\phi\in s\cap t$ and $\psi\in t\cap u$, $\phi\vee\psi\in s\cap u$ and $\phi\wedge\psi\in t$, and thus $\hat{K}(\phi\wedge\psi)\in t$. Now from Proposition 21 and the fact that $d=e$ and $Kv(\phi,d)\in s$ we have $Kv(\phi,e)\in t$, thus $\hat{K}(\phi\wedge\psi)\wedge Kv(\phi,e)\wedge Kv(\psi,e)\in t$. From $\texttt{Kv}^r\vee$, it follows that $Kv(\phi\vee\psi,d)\in t$. By Proposition 21 $Kv(\phi\vee\psi,d)\in s$, thus $(d,s)R(f,u)$. □

Remark 23. *The readers may wonder why we used two copies of each maximal consistent set. Consider the following model:*

$$p, d\mapsto\circ \text{———} p, d\mapsto\bullet$$

It is not hard to see that both worlds satisfy exactly the same ***PALKv***^{r} *formulas. However, we do need these two worlds to differentiate the assignments of the name d. These two copies of each maximal consistent set will play an important role in our proof of the completeness.*

5.2.2 Completeness

In order to establish the truth lemma, we need to show that if $Kv(\phi,d)\notin s$ then $\mathcal{M}^c, s\vDash\neg Kv(\phi,d)$. This is the most difficult part in the completeness proof which requires a few results below.

Given a state $s\in S^c$ such that $Kv(\phi,d)\notin s$, let $Z=\{\psi\mid K\psi\in s\}\cup\{\phi\}$ and $X=\{\neg\chi\mid Kv(\chi,d)\in s\}$. We have the following observations.

Observation 24. *1. For any $\neg\chi\in X$, $\{\neg\chi\}\cup Z$ is consistent.*

2. Z is consistent, and every element in X is also consistent.

Proof. In the sequel, we will write $\vdash$ as the shorthand for $\vdash_{\mathbb{ELKV}^r}$.

For (1): Suppose not, then there exist $\psi_1,\dots,\psi_m$ such that $\vdash\psi_1\wedge\cdots\wedge\psi_m\wedge\phi\to\chi$, equivalently, $\vdash\psi_1\wedge\cdots\wedge\psi_m\to(\phi\to\chi)$, and thus $\vdash K\psi_1\wedge\cdots\wedge K\psi_m\to K(\phi\to\chi)$ by $\texttt{DISTK}$ and $\texttt{NECK}$. Since $K\psi_1,\dots,K\psi_m\in s$, $K(\phi\to\chi)\in s$. Now by $\texttt{DISTKv}^r$ and the fact that $Kv(\chi,d)\in s$, it follows that $Kv(\phi,d)\in s$, contradiction.

For (2): followed immediately from (1) based on the non-emptiness of X due to $\texttt{Kv}^r\bot$. □

Recall that we want to show that $Kv(\phi,d)\notin s$ implies $\mathcal{M}^c, s\vDash\neg Kv(\phi,d)$. To show that $Kv(\phi,d)$ does not hold on s, we need to construct two ϕ worlds which are linked with s but with different assignments for d. In order to do this, according to the definition of R, we have to make sure that these two worlds do not share any ψ such that $Kv(\psi,d)\in s$.

Now since $\mathbf{ELKv}_1^r$ is denumerable, we can enumerate all formulas in X as $\neg\chi_0, \neg\chi_1, \dots$. Then we start to add these $\neg\chi_i$ one by one to two copies of Z. The procedure is described as follows:

1. Let $B_0 = Z \cup \{\neg\chi_0\}$ and let $C_0 = Z$.
2. If B_k and C_k are defined then we take $\neg\chi_{k+1} \in X$, and try to add it into B or C as follows: if it is consistent with B_k then let $B_{k+1} = B_k \cup \{\neg\chi_{k+1}\}$ and let $C_{k+1} = C_k$; otherwise let $B_{k+1} = B_k$ and let $C_{k+1} = C_k \cup \{\neg\chi_{k+1}\}$.
3. Let $B = \bigcup_{k<\omega} B_k$ and $C = \bigcup_{k<\omega} C_k$.

To construct two worlds out of B and C, we need to show B and C are consistent. From Observation 24, B_0 and C_0 are consistent. We only need to show that the second step preserves consistency, which amounts to the following lemma:

Lemma 25. *Suppose $Kv(\phi, d) \notin s$. For any k, if B_k and C_k are consistent, then $B_k \cup \{\neg\chi_{k+1}\}$ is inconsistent implies $C_k \cup \{\neg\chi_{k+1}\}$ is consistent.*

Proof. Suppose not, then there is a k such that both $B_k \cup \{\neg\chi_{k+1}\}$ and $C_k \cup \{\neg\chi_{k+1}\}$ are inconsistent. Now let $\overline{U} = B_k \backslash Z$, $\overline{V} = C_k \backslash Z$, $U = \{\chi \mid \neg\chi \in \overline{U}\}$, and $V = \{\chi \mid \neg\chi \in \overline{V}\}$. Note that $U, V, \overline{U}, \overline{V}$ are all finite.

We claim: there exist $\psi_1, \dots, \psi_l, \psi'_1, \dots, \psi'_m, \psi''_1, \dots, \psi''_n \in \{\psi \mid K\psi \in s\}$ such that

$$\begin{array}{ll} (i) & \vdash \psi_1 \wedge \cdots \wedge \psi_l \wedge \phi \wedge \bigwedge \overline{U} \to \chi_{k+1}, \\ (ii) & \vdash \psi'_1 \wedge \cdots \wedge \psi'_m \wedge \phi \wedge \bigwedge \overline{V} \to \chi_{k+1}, \\ (iii) & \vdash \psi''_1 \wedge \cdots \wedge \psi''_n \wedge \phi \wedge \bigwedge \overline{U} \to \bigwedge V. \end{array}$$

(i) and (ii) are immediate from the inconsistency of $B_k \cup \{\neg\chi_{k+1}\}$ and $C_k \cup \{\neg\chi_{k+1}\}$. For (iii), first note that for any $\chi \in V$, $\{\neg\chi\} \cup B_k$ is inconsistent due to the construction of B_k. Therefore for each $\chi \in V$ there exist $\theta_1, \dots, \theta_h \in \{\psi \mid K\psi \in s\}$ such that:

$$\vdash (\theta_1 \wedge \cdots \wedge \theta_h \wedge \phi \wedge \bigwedge \overline{U}) \to \chi$$

Now since V is a finite set, we collect all such θ for each $\chi \in V$ to obtain (iii).

From $(i) - (iii)$, `NECK`, `DISTK` and the fact that

$$K\psi_1, \dots, K\psi_l, K\psi'_1, \dots, K\psi'_m, K\psi''_1, \dots, K\psi''_n \in s,$$

we can show the following:

$$\begin{array}{ll} (iv) & K((\phi \wedge \bigwedge \overline{U}) \to \chi_{k+1}) \in s, \\ (v) & K((\phi \wedge \bigwedge \overline{V}) \to \chi_{k+1}) \in s, \\ (vi) & K((\phi \wedge \bigwedge \overline{U}) \to \bigwedge V) \in s. \end{array}$$

In the following, we will show that $\hat{K}(\chi_{k+1} \wedge \bigwedge V) \in s$. First we claim $\hat{K}(\phi \wedge \bigwedge \overline{U}) \in s$. Suppose not, then $K\neg(\phi \wedge \bigwedge \overline{U}) \in s$, thus $\neg(\phi \wedge \bigwedge \overline{U}) \in B_k$. Due to the construction of B_k we know ϕ and $\overline{U}$ are in B_k, thus B_k is inconsistent, contradicting the assumption. Therefore $\hat{K}(\phi \wedge \bigwedge \overline{U}) \in s$ thus by (iv), (vi) we have $\hat{K}(\chi_{k+1} \wedge \bigwedge V) \in s$.

By our assumption, for any $\chi \in V \cup \{\chi_{k+1}\}$ we have $Kv(\chi, d) \in s$. Now based on this fact and $\hat{K}(\chi_{k+1} \wedge \bigwedge V) \in s$, we can use Proposition 18, and obtain the following:

$$(vii) \quad Kv(\chi_{k+1} \vee \bigvee V, d) \in s.$$

Now let us change the from of (v) to the following:

$$(v') \quad K(\phi \to (\bigvee V \vee \chi_{k+1}) \in s,$$

Based on $(v'), (vii)$ and $\texttt{DISTKv}^r$, we have $Kv(\phi, d) \in s$, contradiction.
In sum, one of $B_k \cup \{\neg\chi_{k+1}\}$ and $C_k \cup \{\neg\chi_{k+1}\}$ is consistent. □

Based on Lemma 25, using Lindenbaum-like argument, we have the following:

Lemma 26. *There are X_0, X_1 such that $X_0 \cap X_1 = \emptyset$, $X_0 \cup X_1 = X$, and both $B = Z \cup X_0$ and $C = Z \cup X_1$ are consistent. Therefore B and C can be extended into two maximal consistent sets.*

Note that the above lemma does not rule out the possibility that the two maximal consistent sets being the same. We will handle this by using different copies in the canonical model.

Now we will prove the completeness of $\mathbb{ELKV}_1^r$, i.e., every valid $\mathbf{ELKv}_1^r$ formula is $\mathbb{ELKV}_1^r$-provable. As usual, we construct the canonical model which can satisfy each consistent set of $\mathbf{ELKv}_1^r$ formulas.

Lemma 27 (Truth Lemma)**.** *For any $\mathbf{ELKv}_1^r$ formula ϕ, $\phi \in s$ iff $\mathcal{M}^c, s \vDash \phi$.*

Proof. By induction on ϕ. We will only show the case of $Kv(\phi, d)$ in detail since other cases are standard exercises as in normal modal logic. The direction from $Kv(\phi, d) \in s$ to $\mathcal{M}^c, s \vDash Kv(\phi, d)$ is straightforward based on the induction hypothesis and our definition of R.

Now for the converse, suppose that $Kv(\phi, d) \notin s$, we need to show $\mathcal{M}^c, s \nvDash Kv(\phi, d)$. Recall that $Z = \{\psi \mid K\psi \in s\} \cup \{\phi\}$ and $X = \{\neg\chi \mid Kv(\chi, d) \in s\}$. Lemma 26 guarantees that we can find two (possibly identical) maximal consistent sets B' and C' such that $Z \cup X_0 \subseteq B'$ and $Z \cup X_1 \subseteq C'$ for some X_0, X_1 satisfying $X = X_0 \cup X_1$.

Now we can construct two different worlds $t_0 = (B', 0)$ and $t_1 = (C', 1)$. It is clear that $t_0, t_1 \in S^c$. Since $Z \subseteq t_0 \cap t_1$ we have $t_0 \sim^c s \sim^c t_1$, and $\mathcal{M}^c, t_0 \vDash \phi$ and $\mathcal{M}^c, t_1 \vDash \phi$ due to the induction hypothesis. Now we claim $((d, t_0), (d, t_1)) \notin R$. To see this, consider any $Kv(\chi, d) \in s$ (equivalently, any $\neg\chi \in X$), we have either $\chi \notin B'$ or $\chi \notin C'$ since $X_0 \cup X_1 = X$ and B', C' are consistent. □

Based on the above lemma it is routine to show:

Theorem 28. *$\mathbb{ELKV}_1^r$ is sound and complete.*

Based on this theorem and Theorem 15, $\mathbb{ELKV}_1^r$+$\texttt{!ATOM}$+$\texttt{!NEG}$+$\texttt{!CON}$+$\texttt{!K}$+$\texttt{!Kv}^r$ is sound and complete for $\mathbf{PALKv}_1^r$.

6 Expressivity

In this section we compare the expressivity of the various logical languages that we discussed in this paper. The results are summarized in the following (transitive) diagram:

$$\begin{array}{ccc} \textbf{ELKv}^r & \longleftrightarrow & \textbf{PALKv}^r \\ \uparrow & & \updownarrow \\ \textbf{ELKv} & \longrightarrow & \textbf{PALKv} \end{array}$$

Note that, in contrast to the axiomatization result, in this section all the results hold in multi-agent cases.

Theorem 14 already shows that **ELKv**^{r} and **PALKv**^{r} are equally expressive. In the sequel, to complete the diagram, we first show that **ELKv** is strictly less expressive than **PALKv** then prove that **ELKv**^{r} is equally expressive as **PALKv**. To compare **ELKv** and **PALKv**, we give two models that cannot be distinguished by any **ELKv** formula but can be distinguished by a **PALKv** formula. First we need a notion of bisimulation for **ELKv** models.

Definition 29. *Let* $\mathcal{M}_1 = \langle S_1, \{\sim_i^1 \mid i \in \boldsymbol{I}\}, V_1, V_{\boldsymbol{D}}^1\rangle$ *and* $\mathcal{M}_2 = \langle S_2, \{\sim_i^2 \mid i \in \boldsymbol{I}\}, V_2, V_{\boldsymbol{D}}^2\rangle$ *be two models. A* d*-bisimulation between* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *is a non-empty relation* $\mathcal{R} \subseteq S_1 \times S_2$ *such that if* $s_1 \mathcal{R} s_2$ *then the following requirements hold for all* $i \in \boldsymbol{I}$*:*

Inv: $V_1(s_1) = V_2(s_2)$;

Zig: *if* $s_1 \sim_i^1 t_1$*, then there exists* $t_2 \in S_2$ *such that* $s_2 \sim_i^2 t_2$ *and* $t_1 \mathcal{R} t_2$

Zag: *if* $s_2 \sim_i^2 t_2$*, then there exists* $t_1 \in S_1$ *such that* $s_1 \sim_i^1 t_1$ *and* $t_1 \mathcal{R} t_2$

Kv-Zig: *if* $t_1 \sim_i^1 s_1 \sim_i^1 t_1'$*, and* $V_{\boldsymbol{D}}^1(d, t_1) \neq V_{\boldsymbol{D}}^1(d, t_1')$ *for some* d *then there exist* $t_2, t_2' \in S_2$ *such that* $t_2 \sim_i^2 s_2 \sim_i^2 t_2'$*, and* $V_{\boldsymbol{D}}^2(d, t_2) \neq V_{\boldsymbol{D}}^2(d, t_2')$

Kv-Zag: *if* $t_2 \sim_i^2 s_2 \sim_i^2 t_2'$*, and* $V_{\boldsymbol{D}}^2(d, t_2) \neq V_{\boldsymbol{D}}^2(d, t_2')$ *for some* d *then there exist* $t_1, t_1' \in S_1$ *such that* $t_1 \sim_i^1 s_1 \sim_i^1 t_1'$*, and* $V_{\boldsymbol{D}}^1(d, t_1) \neq V_{\boldsymbol{D}}^1(d, t_1')$

We write $\mathcal{M}_1, s_1 \leftrightarroweq_d \mathcal{M}_2, s_2$ *iff there is a* d*-bisimulation between* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *linking* s_1 *and* s_2*.*

We now show d-bisimulation preserves **ELKv** formulas:

Theorem 30. *If* $\mathcal{M}_1, s_1 \leftrightarroweq_d \mathcal{M}_2, s_2$*, then* $\mathcal{M}_1, s_1 \equiv_{\textbf{\textit{ELKv}}} \mathcal{M}_2, s_2$*.*

Proof. Suppose that $\mathcal{M}_1, s_1 \leftrightarroweq_d \mathcal{M}_2, s_2$, we proceed by induction on the structure of **ELKv** formula ϕ. Here we only show the non-trivial case for $Kv_i d$.

Suppose $\mathcal{M}_1, s_1 \nvDash Kv_i d$, then there exists t_1, t_1' with $t_1 \sim_i^1 s_1 \sim_i^1 t_1'$ and $V_{\mathbf{D}}^1(t_1, d) \neq V_{\mathbf{D}}^1(t_1', d)$. By *Kv*-Zig, there exist $t_2, t_2' \in S_2$ such that $t_2 \sim_i^2 s_2 \sim_i^2 t_2'$, and $V_{\mathbf{D}}^2(d, t_2) \neq V_{\mathbf{D}}^2(d, t_2')$. Therefore $\mathcal{M}_2, s_2 \nvDash Kv_i d$.

The other direction can be proved similarly by using *Kv*-Zag. □

Now consider the following two models (using ∘ and • for the objects assigned to d):

$$s : p \circ \overset{1}{—} \neg p \circ \overset{1}{—} p \bullet \qquad s' : p \circ \overset{1}{—} \neg p \bullet$$

It is not hard to see that these two models are d-bisimiliar linking s and s'. However, we can distinguish s and s' easily by a **PALKv** formula $[p]Kv_1 d$.

Based on this example and the fact that **PALKv** is an extension of **ELKv**, we have the following result:

Theorem 31. ***PALKv*** *is strictly more expressive than* ***ELKv***.

Some readers may recall a similar comparison in the context of (relativised) common knowledge between **ELC**, **ELC**^{r}, **PALC** and **PALC**^{r} in [vBvEK06] where **PALC** is strictly less expressive than **PALC**^{r}. However, in our case the situation is different.

Theorem 32. ***ELKv***^{r} *is no more expressive than* ***PALKv***.

Proof. Define a translation function t : **ELKv**^{r} $\to$ **PALKv** as follows:

$$
\begin{array}{lcl}
t(\top) & = & \top \\
t(p) & = & p \\
t(\neg\phi) & = & \neg t(\phi) \\
t(\phi \wedge \psi) & = & t(\phi) \wedge t(\psi) \\
t(K_i\phi) & = & K_i t(\phi) \\
t(Kv_i(\phi, d)) & = & K_i \neg t(\phi) \vee \hat{K}_i \langle t(\phi) \rangle Kv_i d
\end{array}
$$

It is not hard to see that t can eliminate the $Kv_i(\phi, \cdot)$ operators.

We can show by induction on ϕ that for any pointed model $\mathcal{M}, s$: $\mathcal{M}, s \vDash \phi \iff \mathcal{M}, s \vDash t(\phi)$ for every $\phi \in$ **ELKv**^{r}. □

Now based on Theorem 32, the fact that **ELKv**^{r} and **PALKv**^{r} are equally expressive, and the fact that **PALKv**^{r} is clearly no less expressive than **PALKv**, the following corollary is immediate:

Corollary 33. ***PALKv***^{r}*,* ***PALKv*** *and* ***ELKv***^{r} *are equally expressive.*

7 Conclusion and Future work

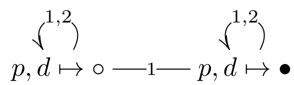
In this paper, we proved that the system $\mathbb{PALKV}_p$ is not complete. On the other hand, $\mathbb{ELKV}_1^r$ is complete for single agent **ELKv**^{r}. Based on this and Theorem 15, $\mathbb{ELKV}_1^r$+!ATOM+!NEG+!CON+!K+!Kv^r is complete for **PALKv**$_1^r$. We conjecture that **PALKv**$_1^r$ is decidable, since a version of filtration technique should work to show a small model property as in normal modal logic. We leave it to a future occasion.

It is not very clear whether multi-agent **ELKv**^{r} can be completely axiomatized by the multi-agent version of $\mathbb{ELKV}_1^r$. Two main difficulties in the completeness proof are as follows:

1. The definition of R in the canonical model is to be revised: we may need to take the reflexive transitive closure of R to make it an equivalence relation.

2. The definition of $\sim_i^c$ also needs to be revised.

To see the second point, consider the following model:



It is clear that these two worlds satisfy exactly the same set of $\mathbf{PALKv}^r$ formulas. However, it is impossible to embed this model in our previously defined canonical model, since these two worlds must be connected by $\sim_2$ if they have the same maximal consistent set. We leave the multi-agent axiomatization and the applications of this new relativized *Kv* operator for future work.

References

[BMS98] A. Baltag, L. Moss, and S Solecki. The logic of public announcements, common knowledge, and private suspicions. In *Proceedings of TARK '98*, pages 43–56. Morgan Kaufmann Publishers Inc., 1998.

[Car79] D. Carr. The logic of knowing how and ability. *Mind*, 78:394–409, 1979.

[DW10] Francien Dechesne and Yanjing Wang. To know or not to know: epistemic approaches to security protocol verification. *Synthese*, 177(Supplement-1):51–76, 2010.

[FHMV95] R. Fagin, J. Halpern, Y. Moses, and M. Vardi. *Reasoning about knowledge*. MIT Press, Cambridge, MA, USA, 1995.

[Ger99] Jelle Gerbrandy. *Bisimulations on Planet Kripke*. PhD thesis, University of Amsterdam, 1999.

[GG97] J. Gerbrandy and W. Groeneveld. Reasoning about information change. *Journal of Logic, Language and Information*, 6(2):147–169, April 1997.

[Hin62] J. Hintikka. *Knowledge and Belief: An Introduction to the Logic of the Two Notions*. Cornell University Press, Ithaca N.Y., 1962.

[HP03] J. Y. Halpern and R. Pucella. Modeling adversaries in a logic for security protocol analysis. In *Formal Aspects of Security*, pages 87–100, 2003.

[McC79] J. McCarthy. First-Order theories of individual concepts and propositions. *Machine Intelligence*, 9.:129–147, 1979.

[Pla89] J. A. Plaza. Logics of public communications. In M. L. Emrich, M. S. Pfeifer, M. Hadzikadic, and Z. W. Ras, editors, *Proceedings of the 4th International Symposium on Methodologies for Intelligent Systems*, pages 201–216, 1989.

[PR85] R. Parikh and R. Ramanujam. Distributed processes and the logic of knowledge. In *Proceedings of Conference on Logic of Programs*, pages 256–268, London, UK, 1985. Springer-Verlag.

[RS05] R. Ramanujam and S. P. Suresh. Decidability of secrecy for context-explicit security protocols. *Journal of Computer Security*, 13(1):135–165, 2005.

[Ryl49] Gilbert Ryle. *The Concept of Mind*. Penguin, 1949.

[SW01] J. Stanley and T. Williamson. Knowing how. *Journal of Philosophy*, 98:411–444, 2001.

[vBvEK06] J. van Benthem, J. van Eijck, and B. Kooi. Logics of communication and change. *Information and Computation*, 204(11):1620–1662, November 2006.

[vD07] H. van Ditmarsch. Comments to 'logics of public communications'. *Synthese*, 158(2):181–187, 2007.

[vDRV08] H. van Ditmarsch, J. Ruan, and R. Verbrugge. Sum and product in dynamic epistemic logic. *Journal of Logic and Computation*, 18(4):563–588, 2008.

[vDvdHK07] H. van Ditmarsch, W. van der Hoek, and B. Kooi. *Dynamic Epistemic Logic*. (Synthese Library). Springer, 1st edition, November 2007.

[VW51] G. H. Von Wright. *An Essay in Modal Logic*. North Holland, Amsterdam, 1951.

[WC13] Yanjing Wang and Qinxiang Cao. On axiomatizations of public announcement logic. *Synthese*, 2013. `http://dx.doi.org/10.1007/s11229-012-0233-5`.