GL in normal modal logic and tense logic

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Basic Definition

- *GL* is a normal modal logic axiomatized by the schema

$$\Box(\Box\phi \to \phi) \to \Box\phi$$

.

$$wid_n : \bigwedge_{i \leq n} \Diamond p_i \to \bigvee_{0 \leq i \neq j \leq n} \Diamond(p_i \wedge (p_j \vee \Diamond p_j)) \; [n \geq 1]$$

Background

- *GL* has some interesting explanation：
  Solovay's theorem： $GL \vdash A$ if and only if for all realizations f, $PA \vdash f(A)$.
  Lőb's theorem： Read $\Box\phi$ as ϕ is provable.

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- The finite model property(f.m.p.) has some good consequences： decidability.
- Researches before on f.m.p. of GL always need to restrict the width.
- Proofs on GL are related to the axiom of choice.

Goals

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- For each n, $GL \oplus Wid_n$ has the f.m.p.
- In $F, P-$tense logic, all normal extensions of $L \supseteq G^+ \oplus G^- \oplus .3^- \oplus wid_n^+$ have the f.m.p if the weak canonical models for L has no infinite chains.

Related results

- Kit Fine's completeness theorem in [2]:
  [AC]Each finite width logic L is complete.
  Let L be a logic that is complete for a condition that is closed under subframes. Then L has f.m.p.

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  [AC]Each finite width logic L is complete.
  Let L be a logic that is complete for a condition that is closed under subframes. Then L has f.m.p.
- Frank wolter's results in [5]:
  [AC]All subframe logics above $G^+ \oplus G^- \oplus wid_n^+ \oplus wid_n^-$ have the finite model property.
  [AC]All subframe logics above $G^+ \oplus G.3^-$ have the f.m.p. and are finitely axiomatizable.

Part one

Definition

- A frame $\mathfrak{F} = \langle W, R \rangle$ is an upward well-founded order if R is transitive and satisfied

$$\forall X \subseteq W(X \neq \varnothing \rightarrow \exists x \in X(\forall y \in X(\neg xRy))) \qquad (1)$$

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$$\forall X \subseteq W(X \neq \varnothing \rightarrow \exists x \in X(\forall y \in X(\neg xRy))) \qquad (1)$$

- For a frame $\mathfrak{F} = \langle W, R \rangle$, if R is transitive, then an increasing infinite chain in $\mathfrak{F}$ is a sequence $\langle z_n \mid n \in \omega \rangle \in W^\omega$ such that $z_n R z_{n+1}$ for all $n \in \omega$.

Here we denote by K_1 the class of upwards well-founded orders, and K_2 the class of orders without any increasing infinite chain. Hence we have:

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Lemma (*ZF*)

$K_1 \subseteq K_2$

Dependent choice

Notice that 1 implies R is irreflexive and if $\mathfrak{F} = \langle W, R \rangle$ has no increasing infinite chain, R is also irreflexive.

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Recall that the Principle of Dependent Choices (DC) is the following weak version of the Axiom of Choice:

let R be a binary relation on a nonempty set A such that $\forall x \in A \exists y \in A(xRy)$, then there is an infinite sequence $\langle z_n \mid n \in \omega \rangle \in A^\omega$ such that $z_n R z_{n+1}$ for all $n \in \omega$.

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From this definition the following is obvious.

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[ZF]DC implies that $K_1 = K_2$.

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Proof.

We only need to show (ii) implies (i). Assume $K_1 = K_2$ and R be a binary relation on a nonempty set A such that $\forall x \in A \exists y \in A(xRy)$. Then $\langle A, R \rangle$ is not upward well-founded, so by (ii), there is an increasing infinite chain $\langle z_n \mid n \in \omega \rangle \in A^\omega$ such that $z_n R z_{n+1}$ for all $n \in \omega$. Thus, DC holds. □

Go back to *GL*

Theorem[ZF] ★

$\mathfrak{F} = \langle W, < \rangle$ is a frame of GL iff $\mathfrak{F} \in K_1$.

Some definitions about tree

definition

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Properties of tree

(a) If $x \in W$, the *height* of x in T, or $ht(T, x)$, is $ot(\{y \in W \mid y < x\})$; ($ot$ means order type)
(b) For each ordinal α, the α-th *level* of T, or $Lev_\alpha(T)$, is $\{x \in W \mid ht(T, x) = \alpha\}$
(c) The *height* of T, or $ht(T)$, is the least α such that $Lev_\alpha(T) = 0$.

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The following property holds in The Second Fraenkel Model V_{F_2}: there exists an infinite binary tree $T = \langle W, < \rangle$ with $ht(T) = \omega$ which does not have an infinite branch. (The Second Fraenkel Model is a model of ZFA, the set theory with atoms, but we can transfer this result into ZF, using the Jech-Sochor Embedding Theorem.) If T has any increasing infinite chain $\langle z_n \mid n \in \omega \rangle$, let $B = \{x \in T \mid x < z_n \text{ for some } n\}$. We will show that B is an infinite branch of T, which will be a contradiction.

First theorem

Proof.

First, for any $x, y \in W$, $x < z_i$ and $y < z_j$ for some $i, j \in \omega$. Let $k = \max\{x, y\}$. If x and y are incomparable, z_k will have two incomparable predecessors, a contradiction. Second, since $\langle z_n \mid n \in \omega \rangle$ is infinite and $ht(T) = \omega$, there is no ordinal α s.t. $Lev_\alpha(T) \cap B = \varnothing$. Finally, $\{z_n \mid n \in \omega\} \subseteq B$, so B is infinite.As a result, $T \in K_2$.

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Let $A = \{x \in W \mid x \text{ has infinitely many successors.}\}$. First we know A is not empty because T is a infinite binary tree. For any $x \in A$, one of the two immediate successors of x must belong to A. Then we have for any $x \in A$, xRy for some $y \in A$, which follows that $T \notin K_1$.[4]and[3] □

End of Part one

Corollary

It is relatively consistent with *ZF* that there is a transitive frame without any increasing infinite chain which is not a *GL*-frame.

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It is relatively consistent with ZF that there is a transitive frame without any increasing infinite chain which is not a GL-frame.

The above show that in ZF, K_2 is the frame correspondent of GL iff DC holds.

Part two

GL in modal logic

we call a transitive frame $\mathfrak{F}$ has width n if each point in $\mathfrak{F}$ has at most n incomparable successors, i.e.

$$\mathfrak{F} \models \forall x \forall y_0 y_1 \dots y_n (xRy_0 \wedge xRy_1 \wedge \cdots \wedge xRy_n) \rightarrow \bigvee_{0 \leq i \neq j \leq n} (y_i R y_j \vee y_j R y_i \vee y_i = y_j)).$$

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lemma

$K4 \oplus wid_n$ is characterized by F_n, where F_n is the class of all transitive frames which have width n.

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proof

Using selective filtration as in [1].
Let $\mathfrak{M}_{L_n} = \langle W_{L_n}, R_{L_n}, V_{L_n} \rangle$ be the canonical model of L_n and $\phi \notin GL \oplus wid_n$. To use filtration we need to choose a formula set, so let $\Sigma = sub(\phi) = \{\psi \mid \psi \text{ is a subformula of } \phi\}$. We know that there is $x_0 \in W_{L_n}$ s.t. $\mathfrak{M}_{L_n}, x_0 \models \neg\phi$. If $x_0 R_{L_n} x_0$, $\Diamond\neg\phi \in x$ by the definition of R_{L_n} and hence $\neg\Box(\Box\phi \to \phi) \in x_0$, which means that $x_0 R y$ for some $y \models \Box\phi \wedge \neg\phi$. So y is an irreflexive point refutes ϕ. So we can treat x_0 as an irreflexive point. We define $\mathfrak{G} \subseteq W_{L_n}$ by induction as follows:
$G_0 = \{x_0\}$ and $\Phi_{x_0} = \{\Box\psi \in \Sigma \mid x_0 \not\models \Box\psi\}$;

GL in modal logic

proof

Suppose that G_n has been defined. If $\Phi_a = \varnothing$ for all $a \in G_n$, stop the construction and let $G = \bigcup_{n}^{0} G_n$. If not, for each $a \in G_n$, if $\Phi_a \neq \varnothing$, there must be an irreflexive $y_{\psi,a} \models \neg\psi \wedge \Box\psi$ and $aR_{L_n}y_{\psi,a}$ for each $\Box\psi \in \Phi_a$. (Just like the above.) So we select for each $\Box\psi$ a point $y_{\psi,a}$ and let G_{n+1} be the set of all these points, i.e. $G^a_{n+1} = \{y_{\psi,a} \mid \Box\psi \in \Phi_a\}$ and $G_{n+1} = \bigcup_{a \in G_n \wedge \Phi_a \neq \varnothing} G^a_{n+1}$. Here we don't need the axiom of choice since Σ is finite. Notice that in every step we add only finitely many points to G_n, so G_i is finite for $i \leq n$. Moreover, $\Phi_{y_{\psi,a}} \subset \Phi_a \subseteq \Sigma$:1. For any χ, if $y_{\psi,a} \not\models \Box\chi$, $a \not\models \Box\chi$ by transitivity; 2. $y_{\psi,a} \models \Box\psi \wedge a \not\models \Box\psi$. It follows that there must be k s.t. $\Phi_a = \varnothing$ for all $a \in G_k$ because Σ is finite.

GL in modal logic

proof

Thus, our construction will finally stopped and G will be a finite subset of W_{L_n}. Let $\mathfrak{F} = \langle G, R \rangle$, where $R = R_{L_n}|_G$.
Claim: $\mathfrak{F}$ is a finite $GL \oplus wid_n$-frame.
proof of the claim: By Lemma**??** and Proposition**??**, R is transitive with width n. By our construction of G, R is irreflexive and G is finite. So R is upward well-founded. Therefore $\mathfrak{F} \models GL \oplus wid_n$.
Let V' be the restriction of V_{L_n} on G.
Claim: $\mathfrak{M}_{L_n}, x \models \phi$ iff $\mathfrak{F}, V', x \models \phi$ for any $x \in G$ and any $\phi \in \Sigma$.
If the above claim holds, $\mathfrak{F}$ will also refute ϕ and we will reach our goal.
proof of the claim: We use induction on ϕ.
$\phi = p$ or $\bot$: obvious because the two model have the same valuation.

proof

$\phi = \psi \to \chi$ or $\phi = \neg\psi$ is also trivial.
$\phi = \Box\psi$: ($\Rightarrow$)Suppose that $\mathfrak{M}_{L_n}, x \models \Box\psi$. If $\mathfrak{F}, V', x \not\models \Box\psi$, there is some $y \in G$ s.t. $xRy \wedge \mathfrak{F}, V', y \not\models \psi$. By induction hypothesis $\mathfrak{M}_{L_n}, y \not\models \psi$ and $xR_{L_n}y$ since $R = R_{L_n}|_G$, a contradiction.
($\Leftarrow$)Suppose that $\mathfrak{F}, V', x \models \Box\psi$. If $\mathfrak{M}_{L_n}$, $x \not\models \Box\psi$, $\Phi_x \neq \varnothing$ and hence there is some $y_{\psi,x} \in G$ s.t. $xR_{L_n}y_{\psi,x} \wedge \mathfrak{M}_{L_n}, y_{\psi,x} \not\models \psi$, by our construction of G. Since $R = R_{L_n}|_G$, xRy and by I.H. we have $\mathfrak{F}, V', y_{\psi,x} \not\models \psi$. Therefore $\mathfrak{F}, V', x \not\models \Box\psi$, a contradiction.

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This theorem is a consequence of Fine's two theorems, but our proof is within ZF.

GL in modal logic

Unfortunately, some extension of $GL \oplus wid_n$ lacks the f.m.p. and the finite axiomatizability.The instance can be found in [1]. From the counter-example we find that there exist some $R-$desending chains in some frame of $GL \oplus wid_n$, which make some logic lack the f.m.p. So a natural idea is to avoid these chains. We know that in some sense GL says there are no ascending chain, so if we use the bimodal language of GL, we can get a logic whose frame has no infinite chains. Hence we will consider GL in tense logic later.

Part three

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$K4_t \oplus wid_n^{+} \oplus wid_n^{-}$ is characterized by F_n^{*}, where F_n^{*} is the class of all transitive frames which have width n for R and R^{-} .

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A tense logic L has the finite ascending chain property(f.a.p.) iff for any weak canonical model $\mathfrak{M}$ of L, the frame of $\mathfrak{M}$ has no infinite ascending chain for R and no infinite ascending chain for R^{-}.

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We call a model $\mathfrak{M} = \langle W, R, R^{-}, V\rangle$ differentiated if $\forall x, y \in W(x \neq y \rightarrow \exists \phi(\mathfrak{M}, x \models \phi \wedge \mathfrak{M}, y \models \neg\phi)$.

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proposition

Suppose $\mathfrak{M} = \langle W, R, V \rangle$ is a weak, transitive and differentiated model of finite width. Then it contains no infinite R-ascending chain, i.e., no distinct points $\langle v_i \mid i < \omega \rangle$ such that $v_i R v_{i+1}$ for $i < \omega$.

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But the tense version of this proposition is wrong: consider the frame of the natural number, i.e. $\langle \omega, \leq, > \rangle$.
From the above observation, we need to add the f.a.p. condition to our proof.

Finite cover property

Given a relation R, we say $w\bar{R}v$ iff wRv & $\neg vRw$. Given a frame $\mathfrak{F} = \langle W, R, R^{-} \rangle$, we say U is an R-cover for $V \subseteq W$ if $\forall v \in V \exists u \in U(v = u \vee v\bar{R}u)$. $\mathfrak{F}$ itself has the R-finite cover property (R-fcp) if
for each $V \subseteq W$ there is a finite cover U for V s.t. $U \subseteq V$. $v \in V$ is R-maximal in V if $\neg \exists u \in V(v\bar{R}u)$.

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Theorem

[AC]Suppose L is a logic of finite width for both directions with

f.a.p. and that $\mathfrak{F} = \langle W, R, R^{-} \rangle$ is generated from a *WCM* of L. Then $\mathfrak{F}$ has R-fcp and R^{-}-fcp.

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Proof.

Just like the modal version as in Fine 1974. □

Eliminable points

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w is R (R^-)-eliminable in a model $\mathfrak{M} = \langle W, R, R^-, V\rangle$ if $w \in W$ and $\forall\phi\exists v(\ \mathfrak{M}, w \models \phi \to w\bar{R}v(v\bar{R}w) \wedge \mathfrak{M}, v \models \phi)$.

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Lemma

Suppose $L \supseteq K4_t$, w is R_L (R_L^-)-eliminable in $\mathfrak{M}_L = \langle W_L, R_L, R_L^-, V_L\rangle$, a weak canonical model defined on Γ of L and $\phi \in w$. Then $\exists v \in W_L(wR_Lv(vR_Lw) \wedge \phi \in v \wedge v$ is noneliminable in $\mathfrak{M}_L)$.

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Suppose $L \supseteq K4_t$, w is R_L (R_L^-)-eliminable in $\mathfrak{M}_L{=}\langle W_L, R_L, R_L^-, V_L\rangle$, a weak canonical model defined on Γ of L and $\phi \in w$. Then $\exists v \in W_L(wR_Lv(vR_Lw) \land \phi \in v \land v$ is noneliminable in $\mathfrak{M}_L)$.

Eliminable points

Now let $\mathfrak{M}_L = \langle \mathfrak{F}_L, V_L \rangle$ be a WCM for L defined on Γ. Let $U_L = \{w \in W_L \mid w$ is R_L-noneliminable in $\mathfrak{M}_L\} \cup \{w \in W_L \mid w$ is R_L^--noneliminable in $\mathfrak{M}_L\}$. Let $\mathfrak{G}_L$ be the restriction of $\mathfrak{F}_L$ to U_L, and $\mathfrak{N}_L$ be the restriction of $\mathfrak{M}_L$ to U_L.

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Lemma

Suppose that $L \supseteq K4_t$ and that $\mathfrak{N}_L \subseteq \mathfrak{A} \subseteq \mathfrak{M}_L$. Then for all w in $\mathfrak{A}$ and formulas $\phi \in Fml_\Gamma$, $\mathfrak{A}, w \models \phi$ iff $\phi \in w$.

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A model $\mathfrak{A}$ is reduced if it contains no eliminable points. By the theorem above, $\mathfrak{N}_L$ is reduced. So we call $\mathfrak{N}_L$ the reduced weak canonical model.

Definability

Let $\mathfrak{M} = \langle W, R, R^{-}, V \rangle$ be any model, let ϕ be any formula, and let $X \subseteq W$. ϕ defines X in $\mathfrak{M}$ if $X = \{w \in W \mid \mathfrak{M}, w \models \phi\} = ||\phi||^{\mathfrak{M}}$. X is definable in $\mathfrak{M}$ if some formula defines X.

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Definable variant

Let $\mathfrak{M} = \langle W, R, R^-, V\rangle$ and $\mathfrak{M}' = \langle W, R, R^-, V'\rangle$ be two models based on the same frame. $\mathfrak{M}'$ is a definable variant of $\mathfrak{M}$ if for each variable p, $V'(p)$ is definable in $\mathfrak{M}$.

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proposition

Suppose Γ is closed under substitution and is true in $\mathfrak{M}$. Then Γ is true in each definable variant of $\mathfrak{M}$.

Eliminable points

Following Fine, we call a model $\mathfrak{M} = \langle W, R, R^-, V \rangle$ natural iff $\mathfrak{M}$ is differentiated and satisfies:
$\forall x, y \in W(\forall \phi(\mathfrak{M}, x \models G\phi \to \mathfrak{M}, y \models \phi) \to wRv) \land \forall x, y \in W(\forall \phi(\mathfrak{M}, x \models H\phi \to \mathfrak{M}, y \models \phi) \to wR^- v)$

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Theorem

Suppose that $\mathfrak{M}$ is natural and transitive with *fcp* for both direction. Then each $w \in \{x \mid x \text{ is } R\text{-noneliminable or } R^{-}\text{-noneliminable}\}$ is definable in $\mathfrak{M}$.

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Theorem

Suppose that $\mathfrak{M}$ is natural and transitive with *fcp* for both direction, that V is a finite subset of $\{x \mid x$ is R-noneliminable or R^{-}noneliminable in $\mathfrak{M}.\}$ and that U, $U^{'} \subseteq V$. Then $T = \{w \in W - V \mid \{v \in V \mid wRv\} = U \wedge \{v \in V \mid wR^{-}v\} = U^{'}\}$ is definable in $\mathfrak{M}$.

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Theorem

[AC]The above two theorems hold for any $\mathfrak{M}$ which is generated from the reduced weak canonical model $\mathfrak{N}_L$ for a finite width logic L which has f.a.p.

Completeness

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definition

$\mathfrak{M} = \langle W, R, R^-, V\rangle$ is n-simple if there is a finite $V \subseteq W$ s.t.
(i) V R, R^--covers $|w|_n = \{v \in W \mid w \leftrightsquigarrow_n v\}$ for each $w \in W$;
(ii) $\forall x, y \in W - V(\{v \in V \mid xRv\} = \{v \in V \mid yRv\} \wedge \{v \in V \mid xR^- v\} = \{v \in V \mid yR^- v\}) \to x \leftrightsquigarrow_0 y$.

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$\mathfrak{M} = \langle W, R, R^-, V\rangle$ is *n*-simple if there is a finite $V \subseteq W$ s.t.
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$\mathfrak{M}$ is simple if $\mathfrak{M}$ is *n*-simple for some $n \in \omega$.

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$\mathfrak{M}$ is simple if $\mathfrak{M}$ is n-simple for some $n \in \omega$.

Lemma

Suppose $\mathfrak{F} = \langle W, R, R^- \rangle$ is a transitive frame with *fcp* for both directions. Then ϕ is valid in $\mathfrak{F}$ if ϕ is true in all weak simple models $\mathfrak{M} = \langle W, R, R^-, V \rangle$.

Completeness

Completeness

Lemma

Suppose that $\mathfrak{M} = \langle W, R, R^{-}, V \rangle$ is natural, reduced, and transitive with *fcp* for both direction. Then any weak simple model $\mathfrak{A} = \langle W, R, R^{-}, V' \rangle$ is a definable variant of $\mathfrak{M}$.

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Suppose that $\mathfrak{M} = \langle W, R, R^{-}, V \rangle$ is natural, reduced, and transitive with *fcp* for both direction. Then any weak simple model $\mathfrak{A} = \langle W, R, R^{-}, V' \rangle$ is a definable variant of $\mathfrak{M}$.

Theorem

Let $\mathfrak{N}_L$ be a reduced weak canonical model for a finite width logic L which has f.a.p. Then the frame $\mathfrak{F}_L$ of $\mathfrak{N}_L$ is an L-frame.

Completeness

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Theorem

Each finite width tense logic L is complete if L has f.a.p.

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Theorem

Each finite width tense logic L is complete if L has f.a.p.

Corollary

Every tense logic $L \supseteq G^{+} \oplus G^{-} \oplus .3^{-} \oplus wid_n^{+}$ is complete if L has f.a.p.

f.m.p.

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Theorem

Every complete tense logic $L \supseteq G^+ \oplus G^- \oplus .3^- \oplus wid_n^+$ has f.m.p.

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Theorem

Every complete tense logic $L \supseteq G^{+} \oplus G^{-} \oplus .3^{-} \oplus wid_n^{+}$ has f.m.p.

proof

For any $\phi \notin L$, there is a L-frame $\mathfrak{F}$, x in $\mathfrak{F}$ and a valuation V s.t. $\mathfrak{F}, V, x \models \neg\phi$. Let $\mathfrak{F}_x = \langle W, R, R^{-} \rangle$ be the generated submodel of $\mathfrak{F}$ by x. We will show that $\mathfrak{F}_x$ is finite and hence L has f.m.p. since $\mathfrak{F}_x$ is also an L-frame.
Since $L \supseteq G^{-}$, every nonempty subset A of W has an R-minimal element, and since $L \supseteq .3^{-}$, for any $y \in W$, $\{x \in W \mid x < y\}$ is well-ordered by $<$. $L \supseteq G^{+} \oplus wid_n^{+}$ so $\mathfrak{F}_x$ must be a $\leq n$-branch tree by the definition of tree.

f.m.p.

proof

Proof within ZFC: If we admit the axiom of choice, it's not hard to see that $\mathfrak{F}_x$ has no infinite increasing chain by Lemma2 and Proposition10, and hence $\mathfrak{F}_x$ must be finite: By König lemma, if $\mathfrak{F}_x$ is infinite, there must be an infinite increasing chain since $\mathfrak{F}_x$ is a $\leq n$-branch tree.

f.m.p.

Proof.

Proof without AC: We define a relation R_d as follows: xR_dy iff $xRy \wedge \forall z(zRy \rightarrow \neg xRz)$; xR_dy means that y is a immediate successor of x. Suppose that $\mathfrak{F}_x$ is infinite. Let $A = \{x \in W \mid x$ has infinitely many $R-$successors.$\}$. First we know A is not empty because $x \in A$. For any $a \in A$, a has at most n different R_d successors, say $\{a_0, \ldots, a_m\}$, so one of its R_d successors must have infinitely many R-successors. For any R-successor $b \notin \{a_0, \ldots a_m\}$ of a is an R-successors of $c \in \{a_0, \ldots a_m\}$. It follows that $a_j \in A$ for some $j \leq m$. Thus A is a nonempty subset of W without $R-$maximal element, which contradicts that $\mathfrak{F}_x \models GL$. □

Final theorem

Theorem

Every tense logic $L \supseteq G^+ \oplus G^- \oplus .3^- \oplus wid_n^+$ has f.m.p if L has f.a.p.

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Every tense logic $L \supseteq G^+ \oplus G^- \oplus .3^- \oplus wid_n^+$ has f.m.p if L has f.a.p.

Corollary

Every finite axiomatizable f.a.p. tense logic $L \supseteq G^+ \oplus G^- \oplus .3^- \oplus wid_n^+$ is decidable.

Further work

Conjectures

- Each finite width tense logic L is complete.

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- All tense logic $L \supseteq G^{+} \oplus G^{-} \oplus wid_n^{-} \oplus wid_n^{+}$ have f.m.p.

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- Each finite width tense logic L is complete.
- All tense logic $L \supseteq G^{+} \oplus G^{-} \oplus wid_n^{-} \oplus wid_n^{+}$ have f.m.p.
- If ZF is consistent, then It's consistent with ZF that there is an incomplete finite width modal logic.

Thank you!

Alexander Chagrov and Michael Zakharyaschev.
Modal logic, volume 35 of oxford logic guides, 1997.

Kit Fine.
Logics containing k4. part i.
Journal of Symbolic Logic, 39(1):31–42, 1974.

Lorenz Halbeisen.
Combinatorial set theory.
Springer, Berlin, 2012.

Thomas Jech.
Set theory, volume 79.
Springer, 1978.

Frank Wolter.
The finite model property in tense logic.
Journal of Symbolic Logic, 60(3):757–774, 1995.