An Introduction to Forcing

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Outline

Preliminary

1. Our Logic
   - Soundness and Completeness of first-order predicate logic
     T is consistent if and only if T has a (countable) model.
   - Gödel's incompleteness results
     We can only hope relative consistency results, e.g.

$$\text{Con}(\mathrm{ZFC}) \rightarrow \text{Con}(\mathrm{ZFC} + \mathbf{V} = \mathbf{L})$$

2. Our Theory of Sets
   - Axioms of ZFC
   - Partial orders, boolean algebras, filters, dense sets, chain/antichain, etc.
   - Relativization and absoluteness
   - Others: Δ-system, cardinal arithmetic, etc.

Intention

Prove the independence of *CH*

- We have found a “model” **L** of constructible sets in the ground model and shown that

$$(\mathrm{ZF} + \mathbf{V} = \mathbf{L})^{\mathbf{L}}, \quad \mathbf{V} = \mathbf{L} \rightarrow \mathrm{GCH} \wedge \mathrm{AC}. \tag{1}$$

- No inner model can be found to make $(\mathrm{ZF} + \mathbf{V} \neq \mathbf{L})$ true in it as long as ZFC is consistent.
- We should extend our ground model M to be $M[G]$, the generic extension.

Basic Idea

- Start from M, a countable, transitive model of ZFC
- Design a partial order $\mathbb{P}$ (the set of conditions) in M
- Pick a generic filter $G \subseteq \mathbb{P}$, usually $G \notin M$.
- Make $M[G]$ the smallest transitive model of ZFC containing both M and G.
- The truth in $M[G]$ base mainly on the ground model M and the partial order $\mathbb{P}$.

Foundational Theorem of Forcing

Theorem (Theorem of Generic Model)

Given ground model M, partial order $\mathbb{P} \in M$, and generic filter G, there is a $M[G]$ such that

- *$M[G]$ is a transitive model of ZFC;*
- *$M \subseteq M[G]$ and $G \in M[G]$;*
- *$M[G]$ is the smallest such model.*

Theorem (Forcing Theorem)

Under the hypotheses of the previous theorem. Given formula $\varphi(v_1, \ldots, v_n)$ and $\mathbb{P}$-name $\tau_1, \ldots, \tau_n \in M$.

$$\varphi(\tau_1^G, \ldots, \tau_n^G)^{M[G]} \text{ if and only if } \exists p \in G\big(p \Vdash \varphi(\tau_1, \ldots, \tau_n)\big)^M.$$

Generic Filter

Definition

$G \subseteq \mathbb{P}$ is a generic filter if G is a filter and for each dense $D \subseteq \mathbb{P}$ such that $D \in M$, $G \cap D \neq \emptyset$.

- We can always found a generic filter in a countable ground model.
- We can do forcing from arbitrary partial order $\mathbb{P}$, but only the following case is nontrivial.

For each $p \in \mathbb{P}$, there are $q \leq p$ and $r \leq p$ such that $q \perp r$.

$\mathbb{P}$-name

People in M should think about possible extensions, and denote the objects in them by $\mathbb{P}$-names.

Definition

τ is $\mathbb{P}$-name if τ is a relation, and for all $(\pi, p) \in \tau$, π is $\mathbb{P}$-name, $p \in \mathbb{P}$.

- The definition of $\mathbb{P}$ must be considered as inductive.
- $\mathbb{P}$-name is an absolute notion.
- $\mathbb{P}$-names can be ranked.

The Generic Extension $M[G]$

We define the object τ^G that the name τ denotes, and G assigns to.

Definition

τ is $\mathbb{P}$-name,

$$\tau^G = \{\pi^G \mid (\exists p \in G)(\pi, p) \in \tau\}. \tag{2}$$

Note that the definition is also inductive.
The generic extension is defined as,

Definition

$$M[G] = \{\tau^G \mid \tau \in M^{\mathbb{P}}\}. \tag{3}$$

$M[G]$ is transitive if M is.

The Canonical Names

Definition

For each set x in the ground model, we define

$$\check{x} = \{(\check{y}, p) \mid y \in x, p \in \mathbb{P}\}. \tag{4}$$

We claim that $\check{x}^G = x$. Thus $M \subseteq M[G]$.

Definition

$$\dot{G} = \{(\check{p}, p) \mid p \in \mathbb{P}\}. \tag{5}$$

- $\dot{G}$ has nothing to do with G.
- $G = \dot{G}^G \in M[G]$.
- G is the oracle beyond M and finally decides $M[G]$.

Forcing Relation
Atomic Case

We define the forcing relation $p \Vdash \varphi(\tau_1, \ldots, \tau_n)$, where p is a condition, $\tau_1, \ldots, \tau_n$ are $\mathbb{P}$-name.

Definition

1. For atomic formula, we define by induction on $\big(\text{rank}(\tau_1), \text{rank}(\tau_2)\big)$
   - $p \Vdash \tau_1 = \tau_2$ iff $p \Vdash \tau_1 \subseteq \tau_2$ and $p \Vdash \tau_2 \subseteq \tau_1$, $p \Vdash \tau_1 \subseteq \tau_2$ iff for each $(\pi, r) \in \tau_1$, $\big\{q \bigm| q \leq r \to q \Vdash \pi \in \tau_1\big\}$ is dense below p;
   - $p \Vdash \tau_1 \in \tau_2$ iff $\big\{q \bigm| \exists(\pi, r) \in \tau_2(q \leq r \land q \Vdash \pi = \tau_1)\big\}$ is dense below p.

- The atomic case is absolute.

Forcing Relation
Boolean and Quantifier Cases

We continue the definition by induction on the complexity of formula.

Definition

2. $p \Vdash \varphi \wedge \psi$ if and only if $p \Vdash \varphi$ and $p \Vdash \psi$;
3. $p \Vdash \neg\varphi$ if and only if for each $q \leq p$, $q \nVdash \varphi$;
4. $p \Vdash \exists x \varphi(x)$ if and only if $\{q \mid \exists\pi(\pi \text{ is } \mathbb{P}\text{-name } \wedge q \Vdash \varphi(\pi))\}$ is dense below p.

- Forcing relation is not absolute generally.
- The forcing relation is the “logic” of the people living in M. It decides the outline of every possible $M[G]$.

Some Additional Property of Forcing Relation

Theorem

- *If* $q \leq p$*, then* $p \Vdash \varphi$ *implies* $q \Vdash \varphi$*.*
- $\{p \mid p \Vdash \varphi \vee p \Vdash \neg\varphi\}$ *is dense.*
- *No* $p \in \mathbb{P}$ *forces both* φ *and* $\neg\varphi$*.*

Proof of the Forcing Theorem

We prove that

$$\varphi(\tau_1^G, \ldots, \tau_n^G) \text{ iff } \exists p \in G\big(p \Vdash \varphi(\tau_1, \ldots, \tau_n)\big)^M$$

- Atomic case
- Boolean and quantifier case

Finish the Proof of the Generic Model Theorem

Lemma

$M[G] \models ZFC$.

Proof.

- Extensionality: $M[G]$ is transitive.
- Foundation: holds in each $\in$ model.
- For those axioms that asserts existence of sets, we should design appropriate names.

Lemma

If N is a transitive model of ZFC and that $M \subseteq N$, $G \in N$, then $M[G] \subseteq N$.

Finite Partial Functions

The crucial trick is to design the partial order. Here we give a simple example.

Definition (Finite partial functions)

$$Fn(I, J) = \{p :\mid |p| < \omega \wedge p \text{ is a function } \wedge \text{dom}\, p \subseteq I \wedge \text{ran}\, p \subseteq J\}. \tag{6}$$

The order on $Fn(I, J)$ is defined as

$$p \leq q \leftrightarrow p \supseteq q. \tag{7}$$

Some Example

- Forcing with $Fn(\omega, \omega_1)$

$$\bigcup G \text{ is a total function mapping } \omega \text{ onto } \omega_1 ? \tag{8}$$

- Forcing with $Fn(\kappa \times \omega, 2)$
  - $f = \bigcup G : \kappa \times \omega \mapsto 2$ is total.
  - For $\alpha < \kappa$, Letting

$$f_\alpha : \omega \mapsto 2, \text{ such that } f_\alpha(n) = f(\alpha, n). \tag{9}$$

  - $\langle f_\alpha : \alpha < \kappa \rangle$ is an one-to-one sequence mapping κ into 2^ω in $M[G]$.

Preserving Cardinals

Theorem

$\mathbb{P} \in M$. *If* $(\mathbb{P}$ *is c.c.c.*$)^M$, *then for each generic* G *of* $\mathbb{P}$ *over* M *and ordinal* $\alpha \in M$,

$$(\alpha \text{ is a cardinal})^M \leftrightarrow (\alpha \text{ is a cardinal})^{M[G]}.$$

Lemma

$Fn(\kappa \times \omega, 2)$ *is c.c.c..*

Use the Δ-system theorem.

Further Discussion

Theorem

For each κ with $\operatorname{cf}\kappa > \omega$.

$$\text{Con}(ZFC) \to \text{Con}(ZFC + 2^{\omega} = \kappa).$$

- We have shown that $\text{Con}(\mathrm{ZFC}) \to \text{Con}(\mathrm{ZFC} + 2^{\omega} > \kappa)$ for each κ.
- Forcing with a ground model of $\mathrm{ZFC} + \mathrm{GCH}$.

Summary

- The generic extension $M[G]$ is built from the ground model M, the partial order $\mathbb{P} \in M$, and the generic filter G (usually not in M).
- The The general truth in $M[G]$ is already described by the forcing relation in M, so is decided mainly by $\mathbb{P}$ and M.

- Outlook
  - Proper Forcing.
  - $\mathbb{P}_{max}$ Forcing.

For Further Reading

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