The World below RT_2^2 in Reverse Mathematics

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Outline

Introduction

Global Looking

The theorems of interest in our investigations are typically of the form $\forall A(\Theta(A) \to \exists B\Phi(A, B))$ where Θ and Φ are arithmetic and $A, B \in 2^\omega$. Thus,

- from the computability theoretic point of view,we want to bound or characterize the computational complexity of B given an A satisfying Θ ;
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- from the reverse mathematics point of view, we want to determine the axiom systems in which the theorem is provable.

Unlike the case of WKL_0, there do not seem to be many principles equivalent to RT_2^2 over RCA_0. There is, however, a whole world of principles, particularly combinatorial ones, below RT_2^2 (i.e., provable in $RCA_0 + RT_2^2$) in the reverse mathematical universe.

Introduction

Language

The language L_2 of second-order arithmetic is a two-sorted language with number variables x,y,z,... intended to range over natural numbers and set variables X,Y,Z,... intended to range over sets of natural numbers. In addition, the language include $+, \cdot$ as operation symbols, 0, 1 as constants and $<$ as a relation symbol, with adding a binary relation $\in$ to relate the two sorts.

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Axioms

- Basic axioms ;
- Induction axiom :

$$(0 \in X \wedge \forall n(n \in X) \to n+1 \in X)) \to \forall n(n \in X);$$

- Comprehension scheme :

$$\exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where $\varphi(n)$ is any formula in which X does not occur freely.

Introduction

Models

A model for L_2 or an L_2-structure is an ordered 7-tuple

$$\mathfrak{M} = (M, \mathcal{S}, +, \cdot, 0, 1, <)$$

where M is a set; $\mathcal{S}$ is a collection of subsets of M; $+$ and $\cdot$ are function from $M \times M$ into M; $<$ is a subset of $M \times M$; and 0 and 1 are distinguished elements of M.

Introduction

Definition 1.1

The class of partial recursion functions, $\mathcal{C}$, is the least class of functions closed under the following schemata.

(a) *The successor function; the constant functions; the identity functions (also called projections);*

(b) *(Composition) If $g_1,...,g_m,h \in \mathcal{C}$, then*

$$f(x_1, \dots, x_n) = h(g_1(x_1, \dots, x_n), \dots g_m(x_n))$$

is in $\mathcal{C}$ where $g_1, \dots, g_m$ are functions of n variables and h is a function of m variables;

(c) *(Primitive Recursion) If $g, h \in \mathcal{C}$ and $n \geq 1$ then $f \in \mathcal{C}$ where*

$$f(0, x_2, \dots, x_n) = g(x_2, \dots, x_n),$$
$$f(x_1 + 1, x_2, \dots, x_n) = h(x_1, f(x_1, x_2, \dots, x_n), x_2, \dots, x_n),$$

assuming g and h are functions of $n - 1$ and $n + 1$ variables respectively;

(d) *(Minimalization) If $\theta(x_1, \dots, x_n, y)$ is a function of $n + 1$ variables in $\mathcal{C}$ and*

$$\psi(x_1, \dots, x_n) = \mu y[\theta(x_1, \dots, x_n, y) \downarrow = 0 \text{ and } \forall z \leq y[\theta(x_1, \dots, x_n, z) \downarrow]].$$

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Definition 1.2

- *A partial function ψ is Turing computable in A, written $\psi \leq_T A$, if there a program P_e such that if the machine has χ_A written on the oracle tape, then for all x and y, $\psi(x) = y$ iff P_e on input x halts and yields output y. In the case we write $\psi = \{e\}^A$ or Φ_e^A.*

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- *A degree $\mathbf{a} \leq \mathbf{0}'$ is low $\mathbf{a}' = \mathbf{0}'$, and high if $\mathbf{a}' = \mathbf{0}''$.*
- *A set $A \leq_T 0'$ is low (high) if $deg(A)$ is low (high).*

Standard systems

Recursive Comprehension Axioms (RCA_0)

In addition to basic axioms, its axioms include the schemes of Δ^0_1 comprehension and Σ^0_1 induction. This is a system just strong enough to prove the existence of the computable sets but not of $0'$ nor indeed of any noncomputable set.

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Weak König's Lemma (WKL_0)

This system consists of RCA_0 plus the statement that every infinite subtree of $2^{<\omega}$ has an infinite path.
It is connected to the Low Basis Theorem (Jockush and Sore[1972]) of computability theory, which says that every such tree has an infinite path whose jump is computable in that of the tree itself.

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Arithmetic Comprehension Axioms (ACA_0)

This system consists of RCA_0 plus the axioms $\exists X \forall n(n \in X \leftrightarrow \varphi(n))$ for every arithmetic formula φ in which X is not free.
In computability theoretic terms, ACA_0 proves the existence of $0'$ and by relativization it proves, and in fact is equivalent to, the existence of X' for every set X.

Ramsey's theorem

Definition 1.3

For a map f and a subset X of its domain, we denote the image of X under f by $f``X$. An n-coloring of $[\omega]^k$ is a map $f : [\omega]^k \to n$. A subset H of $\omega$ is homogeneous for the coloring f if H is infinite and $|f``[H]^k|=1$.

Ramsey's Theorem for pairs (RT^2_2)

Every 2-coloring of $[\omega]^2$ has a homogeneous set.

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Ramsey's Theorem for pairs (RT^2_2)

Every 2-coloring of $[\omega]^2$ has a homogeneous set.

It is easy to show that the number of color does not matter, in the sense that for each $n > 2$, Ramsey's theorem for 2-colorings of pairs is equivalent to Ramsey's Theorem for n-coloring of pairs.

Some result about RT^2_2

- Specker[1971] constructed a computable coloring of $[\omega]^2$ with no computable homogeneous set. Thus $RCA_0 \nvdash RT^2_2$.

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- Jockusch[1972] constructed a computable coloring of $[\omega]^3$ such that every homogeneous set computes $0'$. This construction can be carried out in RCA_0, and thus shows that Ramsey's Theorem for triples implies ACA_0. In fact, one of the standard proofs of Ramsey's Theorem for k-tuples is equivalent to ACA_0.

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- Seetapum and Slaman[1993] proved a degree theoretic cone avoiding theorem that implies that RT^2_2 does not imply ACA_0 (even over WKL_0).

Some result about RT_2^2

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- Seetapum and Slaman[1993] proved a degree theoretic cone avoiding theorem that implies that RT_2^2 does not imply ACA_0 (even over WKL_0).
- Liu[2012] prove WKL_0 is independent of RT_2^2.

Stability and cohesiveness

In their landmark paper on Ramsey's Theorem for Pairs, Cholak, Jockusch and Slaman [2001] introduced the important idea of splitting RT_2^2 into a stable part and a cohesive part.

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Definition 2.1

A coloring of $[\omega]^2$ *is stable if* $\forall x \exists y \forall z > y (f(x,y) = f(x,z))$.

Stable Ramsey's Theorem for pairs (SRT_2^2)

Every stable coloring of $[\omega]^2$ has a homogeneous set.

Stability and cohesiveness

Definition 2.2

If $\overrightarrow{R} = \langle R_i | i \in \omega \rangle$ is a sequence of sets, an infinite set S is $\overrightarrow{R}$-cohesive if

$$\forall i \exists s (\forall j > s (j \in S \rightarrow j \in R_i) \vee \forall j > s (j \in S \rightarrow j \notin R_i)).$$

Cohesive Principle (COH)

For every sequence $\overrightarrow{R} = \langle R_i | i \in \omega \rangle$ there is an $\overrightarrow{R}$-cohesive set.

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Note that COH easily implies the principle, which asserts that for every coloring of $[\omega]^2$ there is a set such that the coloring restricted to that set is stable.

Stability and cohesiveness

Cohesive Ramsey's theorem for pairs (CRT^2_2)

For every coloring f of $[\omega]^2$ there is an infinite set S such that

$$\forall x \in S \exists y \forall z \in S (z > y \rightarrow f(x,y) = f(x,z)).$$

Proposition 2.3

$RCA_0 \vdash COH \rightarrow CRT^2_2$.

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$RCA_0 \vdash COH \rightarrow CRT^2_2$.

By the theorem and system RT^2_2 splitting into SRT^2_2 and COH we mean the fact that $RCA_0 \vdash RT^2_2 \leftrightarrow COH \wedge SRT^2_2$.

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- It is immediate from the above proposition that $COH \wedge SRT_2^2$ implies RT_2^2, and RT_2^2 obviously implies SRT_2^2.

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The primary combinatorial/reverse mathematical question left here is whether SRT^2_2 implies COH (and so RT^2_2).

Cooper [1972] showed that every uniformly computable sequence $\overrightarrow{R}$ has a Δ^0_2 cohesive set, but for some such $\overrightarrow{R}$ the only Δ^0_2 cohesive set are high, i.e. $S' \equiv_T 0''$. Consequently, WKL_0 can not imply COH.

Stability and cohesiveness

Principle D_2^2 (D_2^2)

Given any Δ_2^0-set A, either A or its complement contains an infinite subset.

Σ_2-bounding collection ($B\Sigma_2^0$)

$\forall x < n \exists y \varphi(x, y) \to \exists m \forall x < n \exists y < m \varphi(x, y)$, where φ is Σ_2^0.

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Proposition 2.4 (Cholak, Jockusch and Slaman 2001)

$RCA_0 + B\Sigma_2 \vdash SRT^2_2 \leftrightarrow D^2_2$.

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Theorem 2.5 (Chong, Lempp and Yang 2010)

$RCA_0 \vdash D^2_2 \to B\Sigma^0_2$.

Stability and cohesiveness

Recently, Chong, Slaman and Yang [2014] use nonstandard model of arithmetic to construct a model separating RT^2_2 and SRT^2_2. The main technical theorem they establish is following.

Theorem 2.6

There is a model $\mathfrak{M} = \langle M, \mathcal{S}, +, \cdot, 0, 1, < \rangle$ of $RCA_0 + B\Sigma^0_2$ but not $I\Sigma^0_2$ such that every $G \in \mathcal{S}$ is low and $\mathfrak{M} \models D^2_2$.

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Open problems

- Does CRT^2_2 implie COH ?
- Does SRT^2_2 implies RT^2_2 in every ω-model of arithmetic ? More generally, does $\mathrm{RCA}_0 + \mathrm{I}\Sigma_2$ separate these two principles ?

RT^2_2 and ordering principles

There are many natural principles that can be seen as special cases of the idea, embodied in Ramsey's Theorem, that large structures must contain large ordered substructures.

RT^2_2 and ordering principles

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The Ascending/Descending Sequence Principle (ADS)

Every infinite linear order has an infinite ascending or descending sequence.

The Chain/Antichain Principle (CAC)

Every infinite partial order has an infinite chain or antichain.

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Every infinite partial order has an infinite chain or antichain.

Proposition 3.1

$RCA_0 \vdash CAC \to ADS.$

Ascending and descending sequence

Given a partial order, if we give the pair $\{x, y\}$ the color 0 if x and y are comparable and the color 1 otherwise, then any homogeneous set for this coloring is a chain or antichain of our partial order. Thus RT^2_2 implies CAC over RCA_0.

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Definition 3.2

- ω *if every element has finitely many predecessors ;*
- ω^* *if every element has finitely many successors ;*
- $\omega + \omega^*$ *if it is not of type* ω *or* $\omega + \omega^*$ *and every element has either finitely many predecessors or finitely many successors.*

Proposition 3.3

ADS is equivalent over RCA_0 to the statement that every infinite linear order has a suborder of type ω *or* ω^**.*

Ascending and descending sequence

Every linear order $\mathcal{L}$ of type $\omega + \omega^*$ is stable in a sense analogous to that for colorings, i.e. $\forall x \exists y \forall z > y (x <_L y \leftrightarrow x <_L z)$. Thus the analog of SRT^2_2 is SADS.

Stable ADS (SADS)

Every linear order of type $\omega + \omega^*$ has a subset of order type ω or ω^*.

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Stable ADS (SADS)

Every linear order of type $\omega + \omega^*$ has a subset of order type ω or ω^*.

Similarly, reducing to an order $\mathcal{L}$ of type $\omega + \omega^*$ corresponds to producing a cohesive set, as in such an order S, we have

$$\forall i \in S \exists s (\forall j > s (j \in S \rightarrow i <_L j) \vee \forall j > s (j \in S \rightarrow i >_L j)).$$

This gives us CADS as the analog of COH.

Cohesive ADS (CADS)

Every linear order has a subset S of order type ω, ω^* or $\omega + \omega^*$.

Ascending and descending sequence

Proposition 3.4

$RCA_0 \vdash ADS \leftrightarrow SADS \wedge CADS.$

Proposition 3.5

$RCA_0 \vdash SRT^2_2 \to SADS.$

Proposition 3.6

$RCA_0 \vdash CRT^2_2 \to CADS$ *and so* $RCA_0 \vdash COH \to CADS.$

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Theorem 3.7 (Hirschfeldt and Shore 2007)

Every X-computable linear order of type $\omega + \omega^*$ contains a suborder A of type ω or ω^* that is low over X, i.e. $(A \oplus X)' \leq_T X'$.

Corollary 3.8

There is an ω-model of $RCA_0 + SADS$ consisting entirely of low sets.

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There is an ω-model of $RCA_0 + SADS$ consisting entirely of low sets.

Downey, Hirschfeldt, Lempp and Solomon [2001] have constructed a computable stable coloring of $[\omega]^2$ with no low homogeneous set, and so the model of SADS constructed in corollary above cannot be a model of SRT^2_2.

Ascending and descending sequence

Proposition 3.9

There is a computable linear order with no low suborder of type ω, ω^* or $\omega + \omega^*$.

Corollary 3.10

SADS $\nvdash$ CADS, and SADS $\nvdash$ WKL_0.

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SADS $\nvdash$ CADS, and SADS $\nvdash$ WKL_0.

Theorem 3.11 (Hirschfeldt and Shore 2007)

None of the following principles are implied by COH (nor CADS, which follows from it) : RT^2_2, SRT^2_2 , WKL_0, CAC, ADS, SADS.

Partial orders and CAC

Definition 3.12

A partial order $\mathcal{P}$ is stable if either

(1) $\forall i \in P \exists s(\forall j > s(j \in P \rightarrow i <_P j) \vee \forall j > s(j \in P \rightarrow i|_P j))$, *or*

(2) $\forall i \in P \exists s(\forall j > s(j \in P \rightarrow i >_P j) \vee \forall j > s(j \in P \rightarrow i|_P j))$.

Stable CAC (SCAC)

Every infinite stable partial order has an infinite chain or antichain.

Partial orders and CAC

Definition 3.12

A partial order $\mathcal{P}$ is stable if either

(1) $\forall i \in P \exists s(\forall j > s(j \in P \to i <_P j) \vee \forall j > s(j \in P \to i|_P j))$, *or*

(2) $\forall i \in P \exists s(\forall j > s(j \in P \to i >_P j) \vee \forall j > s(j \in P \to i|_P j))$.

Stable CAC (SCAC)

Every infinite stable partial order has an infinite chain or antichain.

Proposition 3.13

$RCA_0 \vdash SRT^2_2 \to SCAC \to SADS$.

Partial orders and CAC

Theorem 3.14 (Hirschfeldt and Shore 2007)

Every stable partial order $\mathcal{P}$ computable in X has a chain or antichain S that is low over X, i.e. $(A \oplus X)' \leq_T X'$.

Corollary 3.15

$SCAC \nvdash SRT^2_2$; $SCAC \nvdash CADS$ and so $SCAC \nvdash COH$; $SCAC \nvdash ADS$.

Partial orders and CAC

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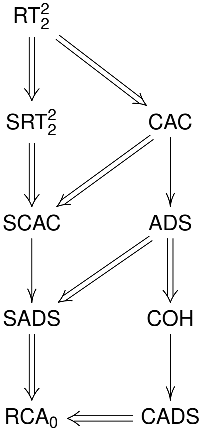
Cohesive CAC(CCAC)

Every partial order has a stable suborder.

Proposition 3.16

$RCA_0 \vdash CCAC \leftrightarrow ADS$.

Summary



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THANKS !