Philosophy of Set Theory

A Pragmatistic View

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Outline

1. A Pragmatistic View on the Philosophy of Mathematics
   - The Natural of Mathematical Achievements
   - How Philosophy Impact on Mathematics
2. Thinking about Multiverse
   - Universe and Multiverse
   - Multiverse Axioms

Pragmatism

Pragmatism is introduced to solve dilemmas.

Basic Rule of Pragmatism

The meaning of concept, statement or opinion should be clarified by considering their practical consequences.

Remark:

- Epistemological puzzle: experience
- The way to 'Ethical' judgement

Mathematics Provide Knowledge

Question

- If mathematical truths are analytic, does mathematics still gives knowledge?
- If mathematics gives knowledge, what form the knowledge is presented in?

My Answer:
The knowledge is given in the following form:

There exists (constructively) a sequence witnessing the fact:

$$\Lambda \vdash_L \varphi$$

Absoluteness of Mathematical Achievements

A mathematical achievement will never fade away

- even the axioms are no longer recognized,
  Russell's Paradox
- even the underground logic is discarded,
  Intuitionistic Logic
- even the subject itself is out of the stage.
  Gödel's coding

The structure of practical consequences is atomic.

Philosophy gives Programs

- Frege's Program
  Motivation: Logicism; Consequences: formalization of predicate logic and axiomatization of set theory
- Hilbert's Program
  Motivation: Formalism and Finitism; Consequences: proof theory, Gödel's Incompleteness theorem
- Other program based on constructivism
  intuitionistic analysis, finitistic mathematics

Gödel's Program

- The phenomenons of incompleteness
- Realism
- Large cardinals
  - The well-ordering of large cardinals
  - What large cardinals can do: consistency, $V \neq L$, PD, etc.
  - What large cardinals cannot do: CH
- Inner model program
  - Bad news: It is extremely hard to move up
  - Good news: Ultimate L

Friedman's Program

Find simple (Π^0_2 or even Π^0_1) and natural (non-metamathematical, e.g. consistency) arithmetical statements that require strong systems to settle.

Philosophy Predicts

Definition (Reinhardt 1967)

κ is a Reinhardt cardinal if there exists an non-trivial elementary embedding $j : V \mapsto V$ such that the critical point of j is κ.

Theorem (Kunen 1971)

Assuming AC. There is no Reinhardt cardinal.

Open Problem

Whether ZF $+$ Reinhardt cardinal is consistent.

Realism predicts it is inconsistent.

Philosophy Sets Barriers

Russell's ramified type theory and Gödel's L.

What if the $V =$ Ultimate L ?

The Universe View

The universe view is the traditional realistic view on set theory. It holds that the universe consists of all sets, and we are supposed to explore in the universe and uncover the the truths of it. ZF, AC, and even large cardinals are considered as our discovery of the universe.

Mathematicians' observation of the universe can be wrong as physicists can be wrong of the physical universe.

The Multiverse View

The Multiverse View

There are many set-theoretic universes.

In other words, there are numerous distinct concepts of set, no just one absolute concept of set.

The Multiverse view is claimed to be a realism, a second order realism about universes.

Arguments Against Universe

Mathematical History:

- Irrational number
- Complex number
- Non-Euclidean geometry

The fact that we are so familiar with living in different universes of set theory makes it impossible to settle problems like CH as the universe view hopes.

Multiverse Axioms (Hamkins, Gitman)

- Inner models are exists as a universe.
- Forcing extensions are exists as a universe.
- For each universe V, there exists a taller universe W such that V is an initial segment of W.
- Every universe is countable from the perspective of another universe.
- Every universe is ill-founded from the perspective of another universe.
- For every universe V and every embedding $j : V \to M$, there exists a universe W and an embedding h such that j is the iterate of h:

$$W \xrightarrow{h} V \xrightarrow{j} M$$

- Every universe V is a countable transitive model in another universe satisfying $V = L$.

The Consistency of Multiverse Axioms

Theorem

Gitman, Hamkins If ZFC is consistent, then the Multiverse Axioms are consistent.

Actually, if there exists a model of ZFC, then the collection of countable computably-saturated models of ZFC satisfies all the multiverse axioms.

Trick in the proof