Paraconsistent Frege

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Outline

- Introduction
- *LP* and RM_3
- Non-triviality
- Paraconsistent Hume's Principle
- Peano Arithmetic
- Philosophical Discussion

Logicism

Frege claimed:

- Arithmetic can be reduced to logic
- Arithmetic is a highly developed logic
- Arithmetic is a branch of logic

Grundgesetze

In *Grundgesetze*, Frege derived the axioms of arithmetic from second-order logic and *Basic Law V*; the latter says that the extension of the concept X is the same as the extension of the concept Y if and only if X and Y are equivalent:

$$\varepsilon X = \varepsilon Y \leftrightarrow \forall z (Xz \leftrightarrow Yz)$$

where ε is the extension operator.

Russell's Paradox

Russell's paradox can also be derived from second-order logic and Basic Law V. The origin of Russell's paradox is the inconsistency of Basic Law V and *second-order comprehension*; the latter says that every expressible formula asserts the existence of a concept:

$$\exists X \forall x (Xx \leftrightarrow \varphi(x))$$

where X does not occur free in $\varphi(x)$

Russell's Paradox

- Proof-theoretically, according to comprehension, $\exists X(x = \varepsilon X \wedge \neg Xx)$ can assert the existence of the concept 'not belong to itself', that is, R; then, according to Basic Law V, there exists an extension of that concept, that is, εR; therefore, εR falls under R if and only if εR does not falls under R.
- Model-theoretically, comprehension requires that the set over which concept variables range is the power set of the set over which object variables range, while Basic Law V requires that there exists one-one correspondence between concepts and objects; however, according to *Cantor's Theorem*, these two requirements cannot be satisfied.

The Failure of Frege's Logicism

After Russell's paradox, few people pay attention to Frege's Logicism, because his system implies contradiction, and contradiction implies everything.

Neo-Logicism

Neo-Logicists discover the following facts:

- Frege only makes use of Basic Law V to derive *Hume's Principle*.
- Then he derives the axioms of arithmetic from Hume's Principle, where he makes no essential use of Basic Law V.

Hume's Principle

Hume's Principle says that the number of the concept F is the same as the number of the concept G if and only if F and G are equinumerous:

$$\#F = \#G \leftrightarrow F \approx G$$

where $\#$ is the number operator, and $\approx$ is equinumerosity, which is second-order definable:

$$F \approx G \leftrightarrow \exists R(\forall x(Fx \to \exists y(Gy \wedge Rxy)) \wedge \forall y(Gy \to \exists x(Fx \wedge Rxy)))$$
$$\forall x \forall y \forall z(Rxy \wedge Rxz \to y = z) \wedge \forall x \forall y \forall z(Rxz \wedge Ryz \to x = y)$$

Frege Arithmetic

- Hume's Principle is consistent with second-order comprehension; and Peano arithmetic can be derived from *Frege arithmetic*, the theory consisting of Hume's Principle and second-order logic.
- If Frege could have appealed to Hume's Principle rather than Basic Law V, then, in some sense, his Logicism would be established.
- However, many people argue against Hume's Principle: it is neither analytic nor *a priori*, and it suffers from the so-called *Julius Caesar Objection* and *Bad Company Objection*

Question

By appealing to Basic Law V rather than Hume's Principle, whether we can construct a paraconsistent and non-trivial theory; that is, *whether there is a way to save Frege's Grundgesetze from triviality if contradiction has to be admitted*.

The Development of Logicism in Non-Classical Logic

- Intuitionistic Logic
- Quantum Logic *OML* and *OL*
- Relevant Logic *R* and *E*
- Free Logic
- Modal Logic
- Łukasiewicz/Kleen Logic

LP

The language of LP is the same as that of first-order classical logic. In order to avoid confusion, I make the following convention:

	negation	implication	equivalence
classical	$\neg\varphi$	$\varphi \rightarrow \psi := \neg\varphi \vee \psi$	$\varphi \leftrightarrow \psi$
paraconsistent *LP*	$\sim\varphi$	$\varphi \Rightarrow \psi := \sim\varphi \vee \psi$	$\varphi \Leftrightarrow \psi$
relevant RM_3	$\sim\varphi$	$\varphi \rightsquigarrow \psi$	$\varphi \leftrightsquigarrow \psi$

Interpretation

The interpretation of LP is a pair (D, d), where D is the domain, while d is a function which maps an individual constant c into an element a in D, and maps a predicate constant P into a pair (E_P, A_P), where E_P is the extension of P, and A_P is the anti-extension of P. It is required that $E_P \cup A_P = D$. Note that it is not necessary that $E_P \cap A_P$ is empty.

Atomic Formula

For atomic formulas:

$$1 \in V(Pc) \quad \text{iff} \quad d(c) \in E_P$$
$$0 \in V(Pc) \quad \text{iff} \quad d(c) \in A_P$$

Equality

For equality,

$$1 \in V(c_1 = c_2) \quad \text{iff} \quad V(c_1) \text{ is identical with } V(c_2)$$
$$0 \in V(c_1 = c_2) \quad \text{iff} \quad V(c_1) \text{ is not identical with } V(c_2)$$

Connectives and Quantifiers

For connectives and quantifiers:

$$
\begin{array}{rcl}
1 \in V(\sim \varphi) & \text{iff} & 0 \in V(\varphi) \\
0 \in V(\sim \varphi) & \text{iff} & 1 \in V(\varphi) \\
1 \in V(\varphi \wedge \psi) & \text{iff} & 1 \in V(\varphi) \text{ and } 1 \in V(\psi) \\
0 \in V(\varphi \wedge \psi) & \text{iff} & 0 \in V(\varphi) \text{ or } 0 \in V(\psi) \\
1 \in V(\forall x \varphi) & \text{iff} & \text{for every } a \in D, 1 \in V_{(a/x)}(\varphi) \\
0 \in V(\forall x \varphi) & \text{iff} & \text{for some } a \in D, 0 \in V_{(a/x)}(\varphi)
\end{array}
$$

Many-Valued Interpretation

- LP can be regarded as a three-valued logic.
  The set of truth-values is $\{1, b, 0\}$, while the set of designated values is $\{1, b\}$.
- For any atomic formula φ, $V(\varphi) \in \{1, b, 0\}$.
  If $V(c) = a$ and $V(P) = (E_p, A_p)$, then

$$
\begin{array}{lll}
V(Pc) = 1 & \text{iff} & a \in E_p \text{ and } a \notin E_P \cap A_P \\
V(Pc) = b & \text{iff} & a \in E_P \cap A_P \\
V(Pc) = 0 & \text{iff} & a \in A_P \text{ and } a \notin E_P \cap A_P
\end{array}
$$

Negation and Conjunction

φ	$\sim \varphi$
1	0
b	b
0	1

$\varphi \wedge \psi$		ψ		
		1	b	0
φ	1	1	b	0
	b	b	b	0
	0	0	0	0

Implication and Equivalence

$\varphi \Rightarrow \psi$		ψ = 1	ψ = *b*	ψ = 0
φ	1	1	*b*	0
	b	1	*b*	*b*
	0	1	1	1

$\varphi \Leftrightarrow \psi$		ψ = 1	ψ = *b*	ψ = 0
φ	1	1	*b*	0
	b	*b*	*b*	*b*
	0	0	*b*	1

Conjunction and Disjunction

If truth-values are ordered as $0 \leqslant b \leqslant 1$, then the semantic conditions for conjunction and disjunction are as follows:

$$V(\varphi \wedge \psi) = glb\{V(\varphi), V(\psi)\}$$
$$V(\varphi \vee \psi) = lub\{V(\varphi), V(\psi)\}$$

where, *glb* and *lub* are greatest lower bound and least upper bound respectively.

Universal and Existential Quantifiers

Universal and existential quantifiers can be regarded respectively as infinite conjunction and disjunction:

$$V(\forall x\varphi) = glb\{\varphi(a/x) : a \in D\}$$
$$V(\exists x\varphi) = lub\{\varphi(a/x) : a \in D\}$$

Consequence Relation

The consequence relation is defined as follow:

$$\varphi \models \psi \quad \text{iff} \quad \text{if } V(\varphi) \in \{1, b\}, \text{ then } V(\psi) \in \{1, b\}$$

Paraconsistent Rules

- According to the truth table of paraconsistent implication, *modus ponens* (or implication elimination) does not hold,

$$\varphi \Rightarrow \psi, \varphi \not\models \psi$$

- Further, transitivity of paraconsistent implication does not hold,

$$\varphi \Rightarrow \psi, \psi \Rightarrow \chi \not\models \varphi \Rightarrow \chi$$

- But implication introduction holds,

$$(PII) \quad \text{if } \Phi, \varphi \models \psi, \text{then } \Phi \models \varphi \Rightarrow \psi$$

RM$_3$

- The language of RM$_3$ is the same as that of LP except that it has $\rightsquigarrow$ as a primitive symbol for relevant implication.
- The truth tables for relevant implication and equivalence are as follows:

$\varphi \rightsquigarrow \psi$		ψ: 1	ψ: b	ψ: 0
φ	1	1	0	0
	b	1	b	0
	0	1	1	1

$\varphi \leftrightsquigarrow \psi$		ψ: 1	ψ: b	ψ: 0
φ	1	1	0	0
	b	0	b	0
	0	0	0	1

Relevant Rules

- According to the truth table of relevant implication, implication introduction does not hold.
  It might be the case that $\Phi, \varphi \models \psi$ but $\Phi \not\models \varphi \rightsquigarrow \psi$.
- However, *modus ponens* and transitivity of implication hold,

$$(RMP) \quad \varphi \rightsquigarrow \psi, \varphi \models \psi$$
$$(RTI) \quad \varphi \rightsquigarrow \psi, \psi \rightsquigarrow \chi \models \varphi \rightsquigarrow \chi$$

- Further, the following rules also hold:

$$(R1) \quad \varphi \rightsquigarrow \psi \models \varphi \wedge \chi \rightsquigarrow \psi \wedge \chi$$
$$(R2) \quad \varphi \wedge \psi \rightsquigarrow \chi \models \varphi \rightsquigarrow \psi \rightsquigarrow \chi$$
$$(R3) \quad \models \varphi \rightsquigarrow \varphi$$

Leibniz Law

- Leibniz Law does not hold,

$$\not\models x = y \rightsquigarrow (\varphi(x) \leftrightsquigarrow \varphi(y))$$

  It might be the case that $V(x = y) = 1$ but $V(\varphi(x) \leftrightsquigarrow \varphi(y)) = b$.

- However, the following rule about identity holds:

$$(RID) \;\; x = y \models \varphi(x) \leftrightsquigarrow \varphi(y)$$

Paraconsistent Theory

- Language:
  $x, y, z, \ldots X, Y, Z, \ldots R, S, T, \ldots$
  $\sim, \rightsquigarrow, \wedge, \forall, =, \varepsilon$

- Axioms:

$$\varepsilon X = \varepsilon Y \leftrightsquigarrow \forall z(Xz \leftrightsquigarrow Yz)$$

$$\exists X \forall x(Xx \leftrightsquigarrow \varphi(x)) \quad \textit{where} = \textit{and} \rightsquigarrow \textit{ do not occur in } \varphi(x)$$

$$\exists R \forall x \forall y(Rxy \leftrightsquigarrow \psi(x, y)) \quad \textit{where} = \textit{and} \rightsquigarrow \textit{do not occur in } \psi(x, y)$$

Paraconsistent Theory

- Let D be the set of natural numbers.
- Let $(D, \mathfrak{B})$ be the *co-finite topology* on D,
- Let first-order variables range over D.
- Let second-order concept variables range over $\mathfrak{A} = \{(cl(A), cl(A^c)) | A \subseteq D\}$, where cl is the closure operator; that is, second-order concept variables range over the set of covering pairs of closed sets of the topology D.
- Let second-order relation variables range over $\mathfrak{A} \times \mathfrak{A}$; that is, second-order relation variables range over the set of covering pairs of closed sets of the product topology $D \times D$.
- Then the semantical conditions for atomic formulas are as follows:

$$1 \in V(Xx) \quad \text{iff} \quad a \in cl(A)$$
$$0 \in V(Xx) \quad \text{iff} \quad a \in cl(A^c)$$

Theorem

Theorem

Paraconsistent Basic Law V holds in the above model.

Theorem

Paraconsistent comprehension holds in the above model.

Equality

- If equality can be added into the paraconsistent comprehension, then the relation defined by the formula $x = y$ must be a covering pair of closed sets of the product topology $D \times D$, that is, there must be a U such that the relation defined by $x = y$ is $\{(cl(U), cl(U^c)) | \; U \subseteq D \times D\}$.
- If $\{(x, y)|x, y \in D \text{ and } x = y\}$ is a closed set, then there is such a U.
- The Hausdorff space has an important property: a topology is a Hausdorff space if and only if the diagonal of the product topology $X \times X$, $\{(x, y) \mid x, y \in X \text{ and } x = y\}$, is a closed set.
- Thus, if equality can be added into paraconsistent comprehension, then the above topological model should be a Hausdorff space.

Theorem

Proposition

For complete metric space M, the reflexive equation $M \cong \mathcal{F}(M)$ has a unique solution, where $\cong$ is isometry, and $\mathcal{F}(M) = \wp_{cl}(M) = \{A \subseteq M | A$ is closed and non-empty$\}$, or $\mathcal{F}(M) = M \times M$.

Proposition

There is a compact metric space M together with an homeomorphism from M onto $\mathcal{F}'(M)$, where $\mathcal{F}'(M) = \{(A, B) \mid A, B$ closed in X and $A \cup B = M\}$

Theorem

The resulting theory is non-trivial if equality is added into paraconsistent comprehension.

The definition of number operator

In order to define the number operator in terms of the extension operator, it is required that equinumerosity occur in the right side of paraconsistent comprehension; thus, equinumerosity must be defined in terms of paraconsistent implication rather than relevant implication.

$$F \approx G \leftrightsquigarrow \exists R(\forall x(Fx \Rightarrow \exists y(Gy \wedge Rxy)) \wedge \forall y(Gy \Rightarrow \exists x(Fx \wedge Rxy)))$$
$$\forall x \forall y \forall z(Rxy \wedge Rxz \Rightarrow y = z) \wedge \forall x \forall y \forall z(Rxz \wedge Ryz \Rightarrow x = y)$$

Then number operator can be defined as follow:

$$\#F = \varepsilon[x : \exists X(x = \varepsilon X \wedge X \approx F)]$$

Derivation of Hume's Principle

In order to derive paraconsistent Hume's Principle from paraconsistent Basic Law V, it is required to show that equinumerosity is an equivalence relation, that is, it is reflexive, symmetrical, and transitive. However, the paraconsistent implication is so weak that its transitivity does not hold.

Theory of Equinumerosity

- To solve this problem, I give the following *theory of equinumerosity*, abbreviated as E.
- The language of E is a standard dyadic second-order language with $\approx$ as a primitive symbol for equinumerosity. The additional formation rule is as follows:

$$If\ X\ and\ Y\ are\ second-order\ variables,$$
$$then\ X \approx Y\ is\ a\ well-formed\ formula$$

Axioms of Equinumerosity

The axioms of E are as follows:

$(E1)\ F \approx F$

$(E2)\ F \approx G \to G \approx F$

$(E3)\ F \approx G \wedge G \approx H \to F \approx H$

$(E4)\ \forall x(Fx \leftrightarrow Gx) \to F \approx G$

$(E5)\ F \approx [x : \neg x = x] \leftrightarrow \forall x \neg Fx$

$(E6)\ F \approx G \wedge Fx \wedge Gy \to [z : Fz \wedge \neg z = x] \approx [z : Gz \wedge \neg z = y]$

$(E7)\ [z : Fz \wedge \neg z = x] \approx [z : Gz \wedge \neg z = y] \wedge Fx \wedge Gy \to F \approx G$

where $[x : \varphi(x)] \approx [x : \psi(x)]$ is abbreviation:

$$[x : \varphi(x)] \approx [x : \psi(x)] \leftrightarrow \exists X \exists Y(\forall x(Xx \leftrightarrow \varphi(x)) \wedge \forall x(Yx \leftrightarrow \psi(x)) \wedge X \approx Y)$$

Axioms of Equinumerosity in *LP*

$(RE1)\ F \approx F$

$(RE2)\ F \approx G \rightsquigarrow G \approx F$

$(RE3)\ F \approx G \wedge G \approx H \rightsquigarrow F \approx H$

$(RE4)\ \forall x(Fx \leftrightsquigarrow Gx) \rightsquigarrow F \approx G$

$(RE5)\ F \approx [x :\sim x = x] \leftrightsquigarrow \forall x \sim Fx$

$(RE6)\ F \approx G \wedge Fx \wedge Gy \rightsquigarrow [z : Fz \wedge \sim z = x] \approx [z : Gz \wedge \sim z = y]$

$(RE7)\ [z : Fz \wedge \sim z = x] \approx [z : Gz \wedge \sim z = y] \wedge Fx \wedge Gy \rightsquigarrow F \approx G$

Theorem

Theorem

The resulting theory is still non-trivial if equinumerosity as a primitive symbol is added into the paraconsistent comprehension.

Paraconsistent Hume's Principle

Then, the following *paraconsistent Hume's Principle* can be derived from paraconsistent Basic Law V.

Theorem

$$\#F = \#G \Leftrightarrow F \approx G$$

From Paraconsistent Hume's Principle to Peano Arithmetic

The proof of Frege's Theorem relies heavily on *modus ponens*; however, when we are reasoning with paraconsistent implication, *modus ponens* must be given up.

Classical Inference and Paraconsistent Inference

Priest makes a distinction between valid inference and quasi-valid inference.

$$\varphi \models_p \psi \quad \text{iff} \quad \text{if } V(\varphi) \in \{1, b\}, \text{ then } V(\psi) \in \{1, b\}$$
$$\varphi \models_c \psi \quad \text{iff} \quad \text{if } V(\varphi) \in \{1\}, \text{ then } V(\psi) \in \{1\}$$

That is, when we make use of classical inference, all classical rules can be recaptured.

Methodological Maxim

Unless we have specific grounds for believing that paradoxical sentences are occurring in our argument, we can allow ourselves to use both valid and quasi-valid inferences.

Paraconsistent Logicism

If it is permissible to transform from paraconsistent inference to classical inference and, in particular, to transform from paraconsistent Hume's Principle to Hume's Principle, then the proof of Frege's Theorem can be reconstructed.

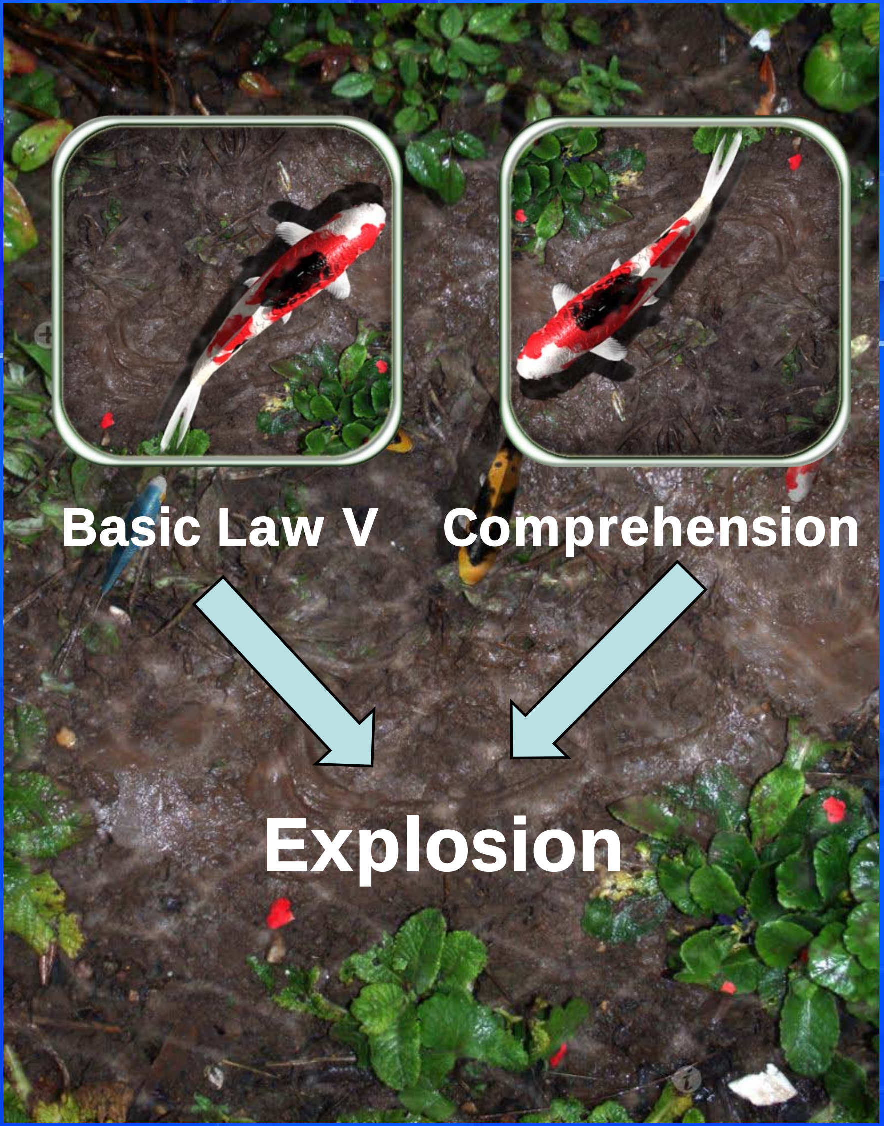
Paraconsistent Logicism

- First, we present Frege's *Grundgesetze*, the theory consisting of second-order logic and Basic Law V.
- Second, we try to show the consistency of Frege's *Grundgesetze*: if it is consistent, then we make use of classical inference; otherwise, we make use of paraconsistent inference. Since Basic Law V is inconsistent with second-order logic, we have to make use of paraconsistent inference to derive what we needs, Hume's Principle, from Basic Law V.
- Third, we try to show the consistency of second-order logic and Hume's Principle: if they are consistent, and we no longer make use of the inconsistent Basic Law V, then we transform to classical inference; otherwise, we maintain paraconsistent inference. Since Hume's Principle is consistent with second-order logic, we can make use of classical inference to derive the axioms of arithmetic from Hume's Principle.

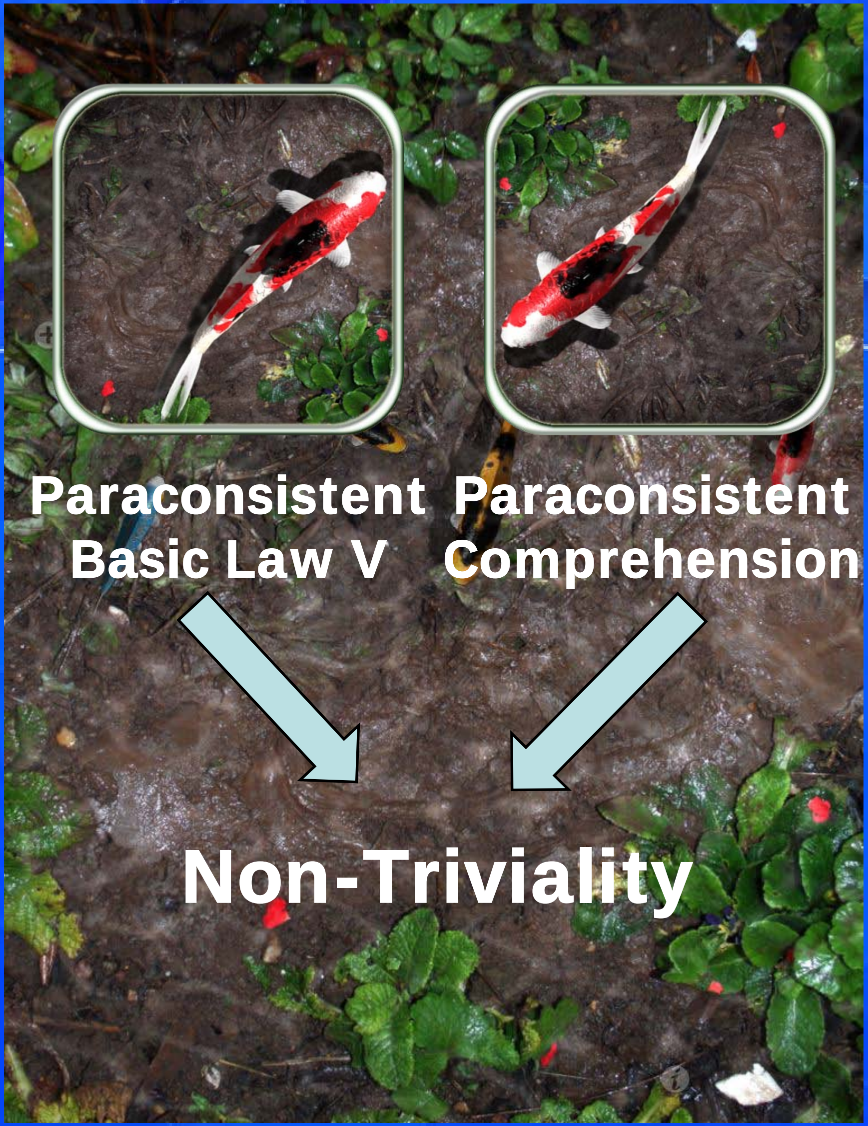
Summary

1. The theory consisting of paraconsistent comprehension and paraconsistent Basic Law V does not lead to triviality.
2. Paraconsistent Hume's Principle can be derived from the theory consisting of paraconsistent comprehension, paraconsistent Basic Law V and axioms of equinumerosity.
3. If it is permissible to transform from paraconsistent logic to classical logic, in particular, to transform from paraconsistent Hume's Principle to Hume's Principle, then the proof of Frege's Theorem can be reconstructed.

Classical Pond

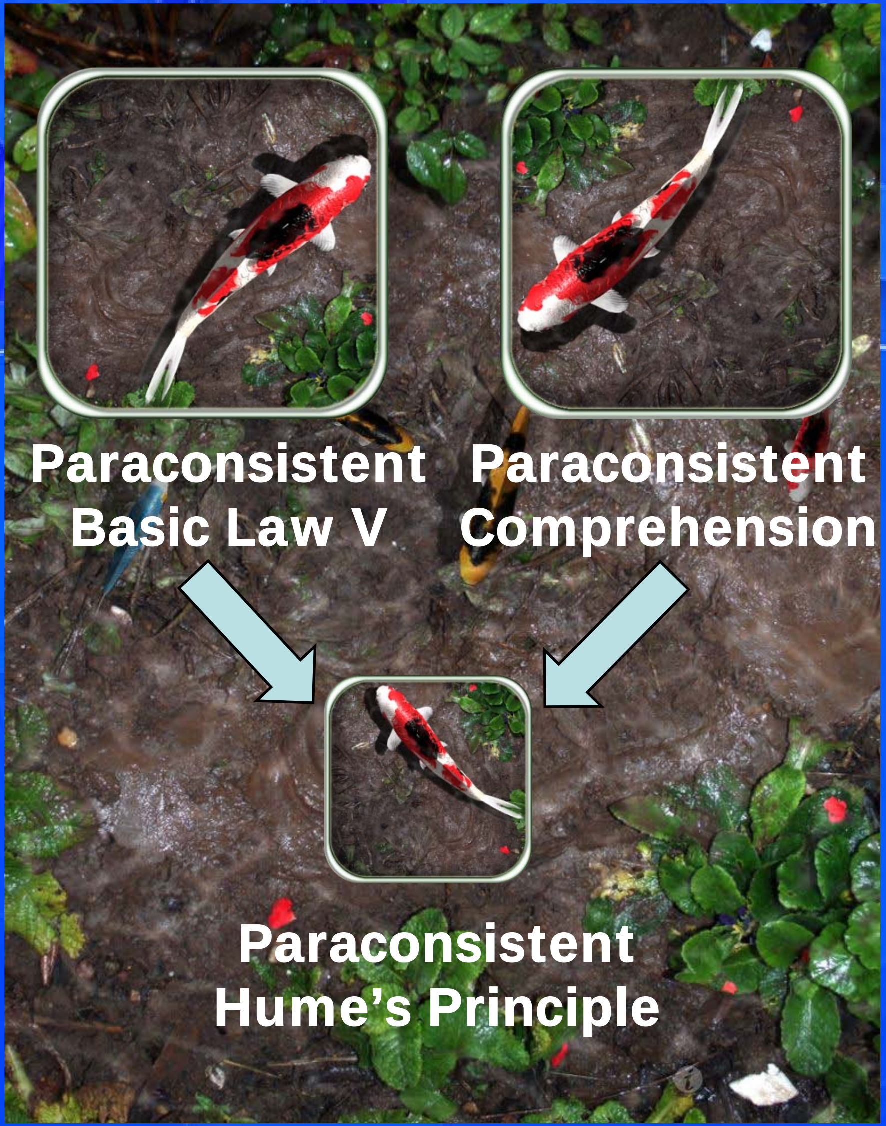


Paraconsistent Pond



Breeding

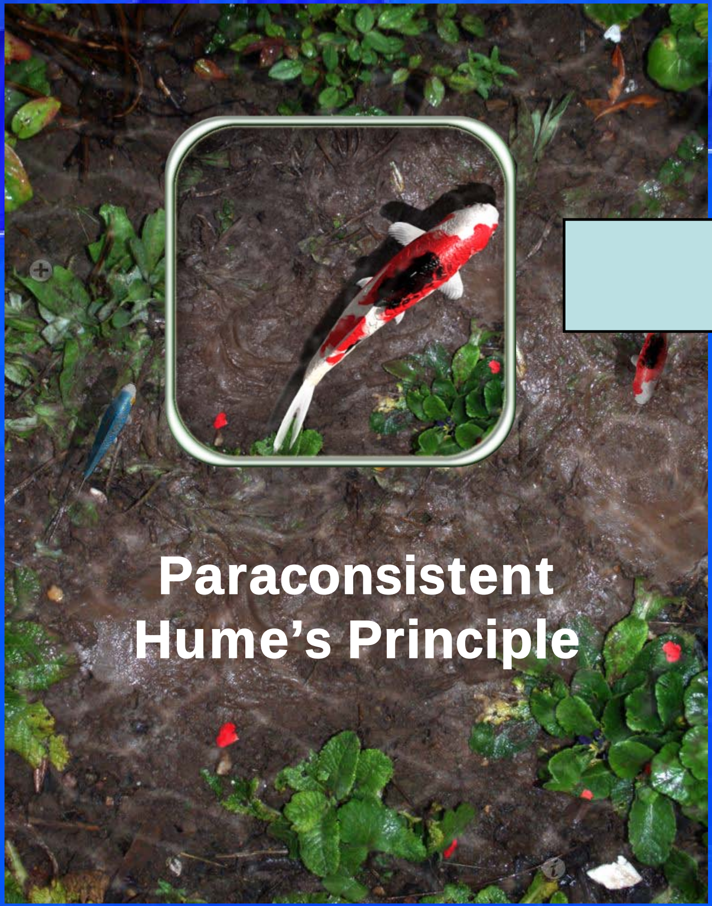
Paraconsistent Pond

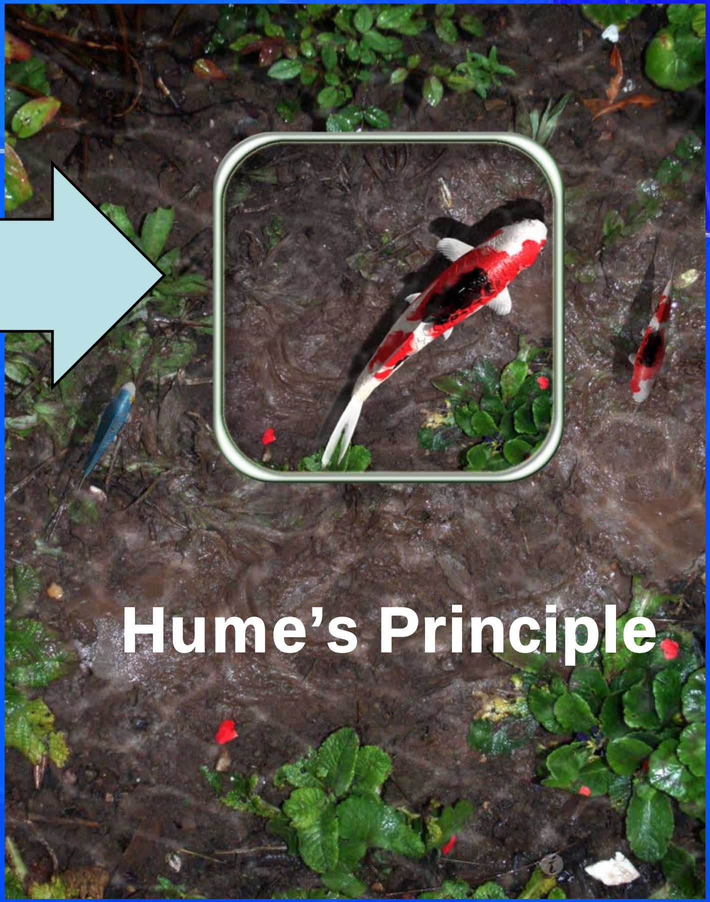


Paraconsistent Pond

Moving

Paraconsistent Pond **Classical Pond**





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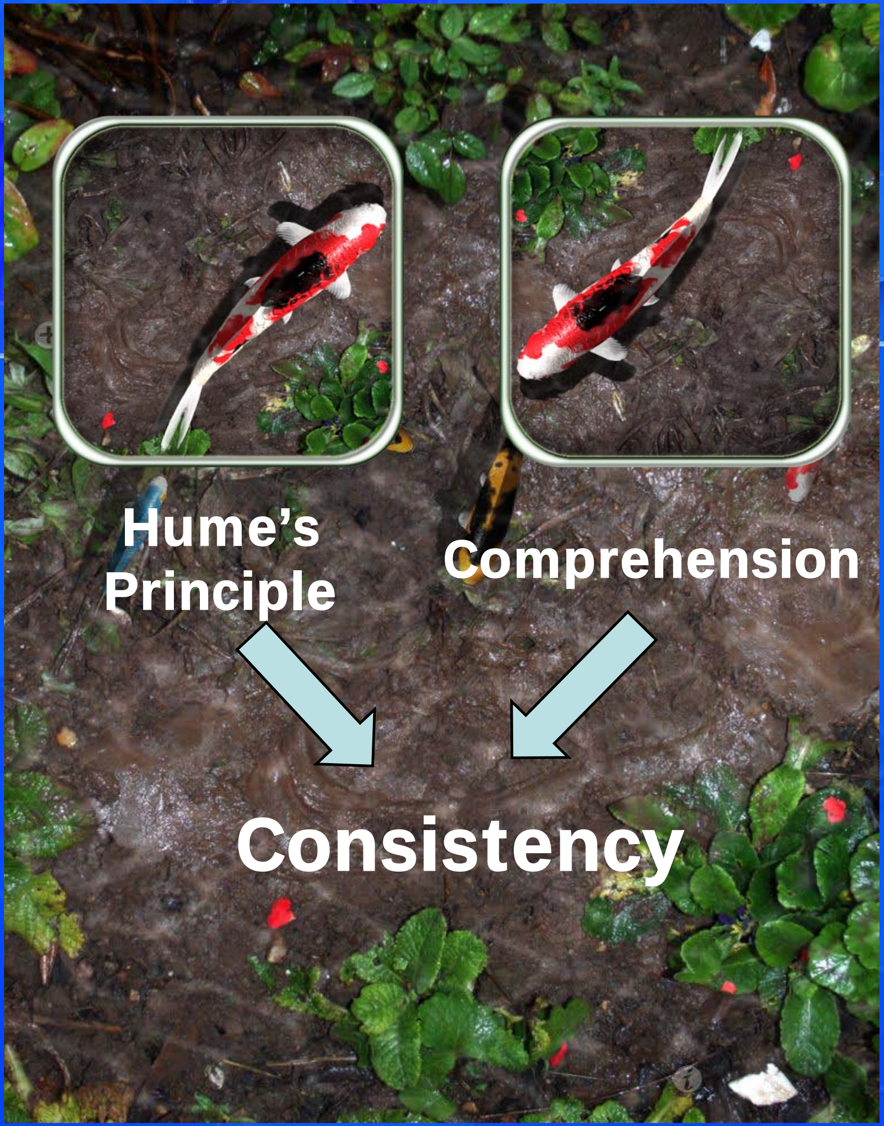
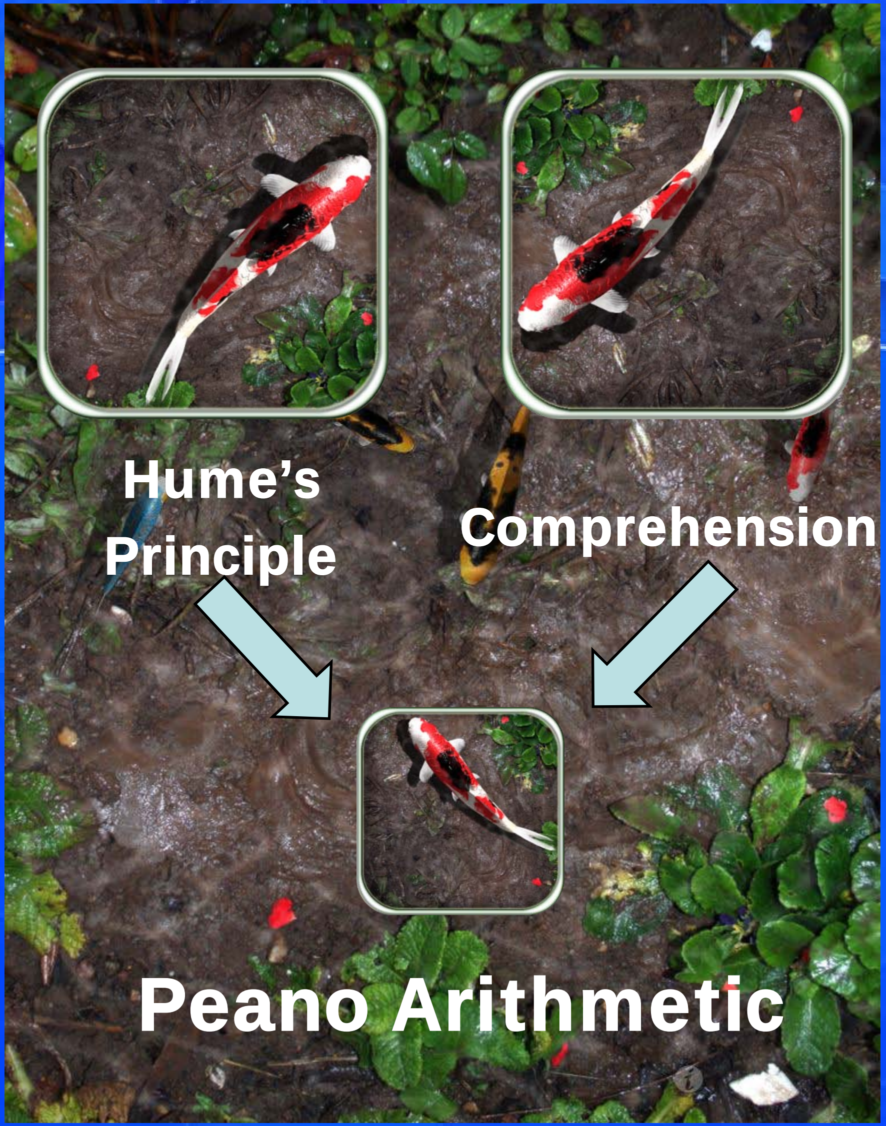
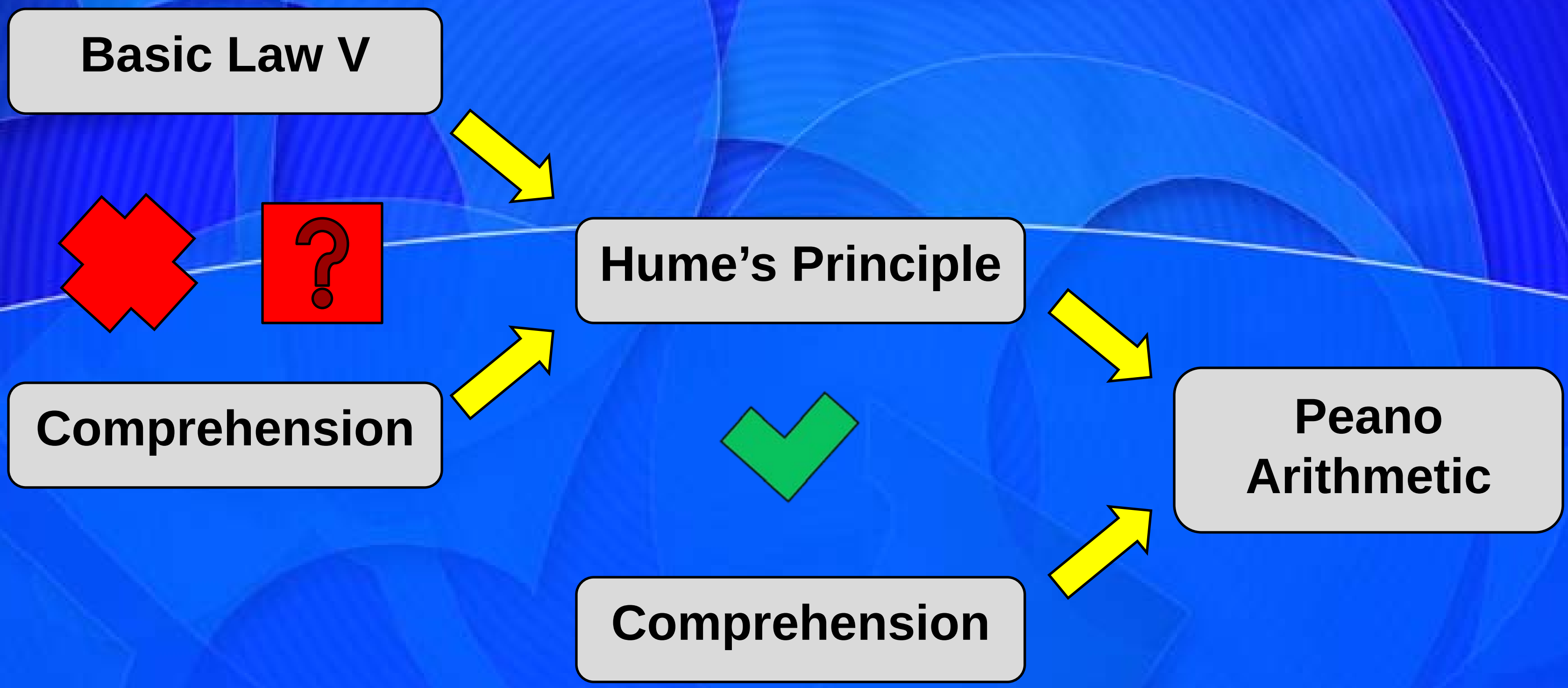


Breeding

Classical Pond

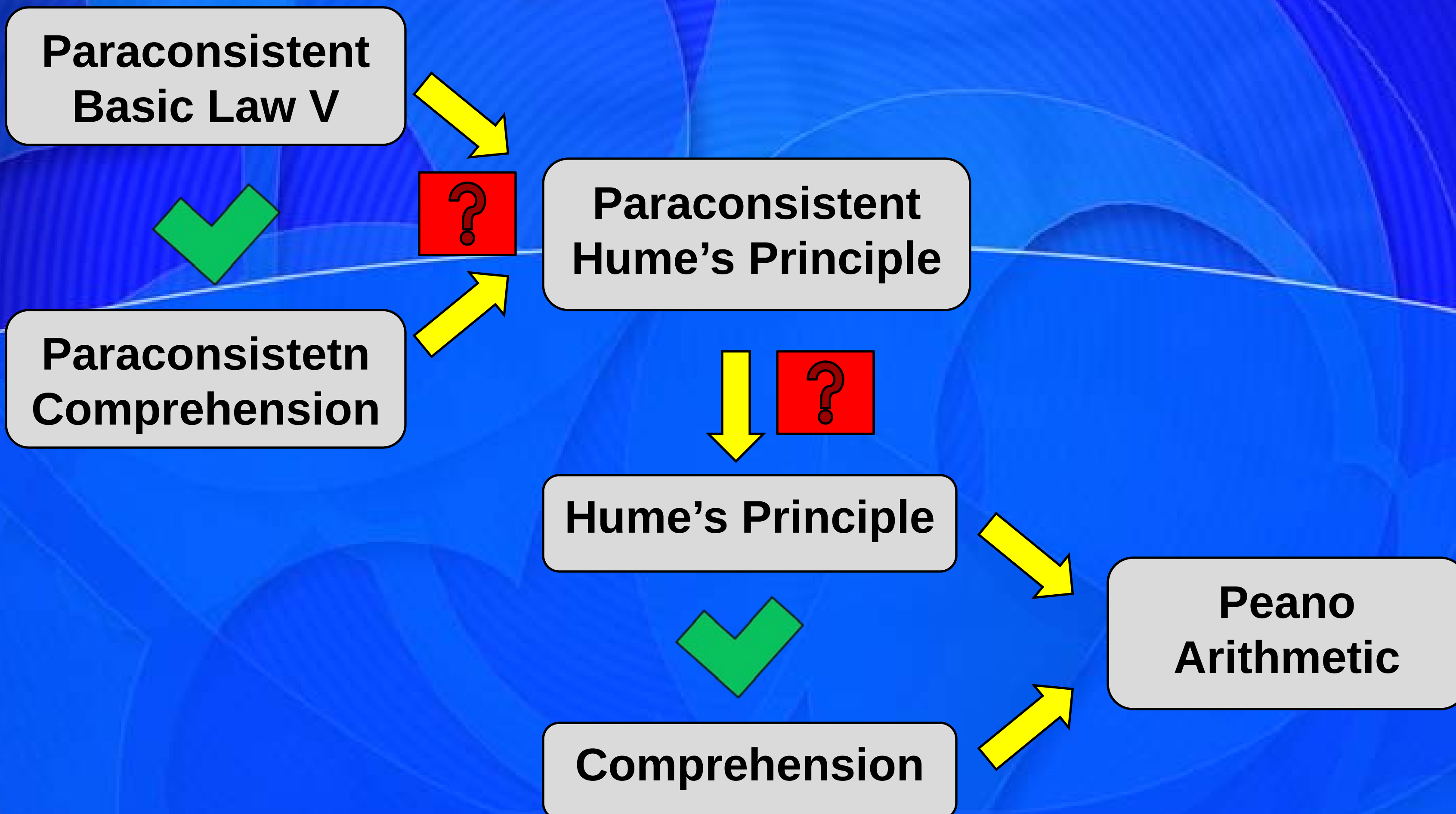


Frege



Basic Law V
Comprehension
Hume's Principle
Comprehension
Peano Arithmetic

Paraconsistent Frege



Problem

1′ Does paraconsistent Basic Law V have any epistemological virtue? Is it analytical or *a priori*?

2′ Can the axioms of equinumerosity formulized in LP be regarded as laws of logic?

3′ Is paraconsistent comprehension a law of logic? Can the epistemological status of paraconsistent Hume's Principle be reduced to that of paraconsistent Basic Law V by means of second-order paraconsistent logic.

4′ To what extent the transformation from paraconsistent logic to classical logic is plausible? In particular, Is it plausible to transform from paraconsistent Hume's Principle to Hume's Principle? Can the epistemological status of Hume's Principle be reduced to that of paraconsistent Hume's Principle by means of such transformation?

Logicism

The aim of Frege's Logicism is to derive the laws of arithmetic from the laws of logic; then he can reduce the question of epistemological status of arithmetic to that of epistemological status of logic, that is, to guarantee the analyticity and apriority of arithmetic by the analyticity and apriority of logic.

Truth-Preserving and Information-Preserving

- Classical inference (from truth to truth) as truth-preserving inference can preserve epistemological status.
- However, classical inference is not the only way to preserve epistemological status. Paraconsistent inference (from truth or dialetheias to truth or dialetheias) as information-preserving inference should also be regarded as one way to preserve epistemological status.
- Thus, the epistemological status of paraconsistent Basic Law V can be reduced to that of paraconsistent Hume's Principle by means of information-preserving inference.

Open Problems

- One way for permitting the transformation from paraconsistent logic to classical logic is to find out certain syntactic criteria to sort out consistent and acceptable theorems from inconsistent or unacceptable ones that are derived from the paraconsistent theory.
- Another strategy for the further development is to reject such transformation, and show directly what content of inconsistent arithmetic is interpretable in the paraconsistent theory. But paraconsistent implication is so weak that *modus ponens* does not hold; thus, I think the second strategy seems unfeasible.

Thank You !!!