

Proof-theoretic approach to logic an introduction

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- 1 Historical notes on proof theory
- 2 Syntactic approach to logic
- 3 Analysis of structures of proofs
- 4 What are hidden behind sequent formulation

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- 2 But, it was found that various paradoxes will occur if mathematics is to be developed freely. For example, axiom of comprehension admits the existence of a set $\{x|P(x)\}$ for any given "property" $P(x)$.
- 3 Then, **Russell's paradox** happens, if we take $x \notin x$ for $P(x)$. In fact, if the set a is defined by $a = \{x|x \notin x\}$, then the equality $a \in a \iff a \notin a$ must hold. Obviously this is a contradiction.

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Formalism (D. Hilbert), Intuitionism (L.E.J. Brouwer), Logicism (B. Russell)

Hilbert's program — formalist point of view

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- On the other hand, contradictions may sneak into the theory if we allow unlimited abstraction and unrestricted use of notions, as examples of paradoxes in set theory have shown.
- Therefore, the consistency of each mathematical theory, like natural number theory, real number theory and ultimately set theory, should be guaranteed.

David Hilbert (1862 - 1943)



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- To show that a proposition β is provable in T , it is enough to give a concrete proof of β .
- But, how can we assure that “ β is not provable in T ”? We need to check infinitely many “possible proofs” and show that any of them never be a proof of β .

♣ Formalizing theories

Moreover, we must state explicitly which statement is an axiom of T and which is not, and also which inference or reasoning is allowed. To eliminate any ambiguity, it will be necessary to represent them by using formal or symbolic expressions.

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- 2 Formalizing mathematical theories (number theory, set theory) ... G. Peano, E. Zermelo, A. Fraenkel ...

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D. Hilbert & W. Ackermann, Grundzüge der theoretischen Logik, 1928, 1938.

Proof theory of Hilbert-style systems (with epsilon symbol)

Axiomatization of number theory (Peano arithmetic **PA**)

- $s(x) \neq 0, \quad s(x) = s(y) \rightarrow x = y,$
- $x + 0 = x, \quad x + s(y) = s(x + y),$
- $x \times 0 = 0, \quad x \times s(y) = (x \times y) + x,$
- (mathematical induction) for any formula $\varphi(x),$
 $[\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(s(x)))] \rightarrow \forall x\varphi(x).$

▶ Is Peano arithmetic consistent? If so, how can we show this?

Here, we say that Peano arithmetic is consistent if the formula $0 = 1$ is not provable in it.

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▶ What assumptions can we assume in the proof of the consistency? For example, can we accept the following argument?

- It suffices to give a concrete model of **PA**, in order to show the consistency. So, take the set \mathbb{N} of natural numbers, and interpret function symbols $+$, \times etc. in a natural way. Surely, \mathbb{N} gives a model of **PA**, and hence **PA** is consistent.

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But, how we can assure the existence of \mathbb{N} ? By set theory? But we don't know whether set theory is consistent. Moreover, if it is consistent, the consistency of **PA** becomes trivial.

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This shows a limitation of Hilbert's program in its original form. Nevertheless, by analyzing structures of proofs in the sequent system for Peano arithmetic, Gentzen succeeded to show that:

♣ If a certain **transfinite induction** is added to (a weaker) Peano arithmetic then the consistency of Peano arithmetic is provable in this system.

Gentzen and sequent systems

Gentzen introduced both natural deduction systems and sequent systems for classical and intuitionistic logics in his thesis (1935). His main result was to show cut elimination theorem.

Gerhard Gentzen: Untersuchungen über das logische Schliessen I, II, Mathematische Zeitschrift (1934, 1935)

Then, he formalized Peano arithmetic over the sequent system **LK** for classical logic, and obtained the aforementioned consistency result, through deep analysis of proofs of his system.

Gerhard Gentzen (1909 - 1945)



♣ Sequent system **LK** for classical logic

A sequent of **LK** is an expression of the following:

$$\alpha_1, \dots, \alpha_m \Rightarrow \beta_1, \dots, \beta_n$$

Intuitively, it expresses that $\beta_1 \vee \dots \vee \beta_n$ follows from the assumptions $\alpha_1, \dots, \alpha_m$, or equivalently from the assumption $\alpha_1 \wedge \dots \wedge \alpha_m$. Note that ‘the meaning of “commas” in the left-hand side is different from that in the right-hand side.

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Each sequent system consists of **initial sequents** (axioms) and **rules**. They determine “true” sequents in the system.

An **initial sequent** of the sequent system **LK** is a sequent of the form $\alpha \Rightarrow \alpha$ for any formula α .

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Rules of **LK** can be divided into the following three.

- Rules for logical connectives
- Cut rule
- Structural rules

♣ Gentzen's system **LK**

Capital Greek letters denote **finite sequences of formulas** in the following.

Rules for implication

$$\frac{\Gamma \Rightarrow \Lambda, \alpha \quad \beta, \Delta \Rightarrow \Pi}{\alpha \rightarrow \beta, \Gamma, \Delta \Rightarrow \Lambda, \Pi} (\rightarrow \Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \Lambda, \beta}{\Gamma \Rightarrow \Lambda, \alpha \rightarrow \beta} (\Rightarrow \rightarrow)$$

Rules for negation

$$\frac{\Gamma \Rightarrow \Lambda, \alpha}{\neg \alpha, \Gamma \Rightarrow \Lambda} (\neg \Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \Lambda}{\Gamma \Rightarrow \Lambda, \neg \alpha} (\Rightarrow \neg)$$

Rules for \vee and \wedge :

$$\frac{\Gamma \Rightarrow \Lambda, \alpha}{\Gamma \Rightarrow \Lambda, \alpha \vee \beta} (\Rightarrow \vee 1) \qquad \frac{\Gamma \Rightarrow \Lambda, \beta}{\Gamma \Rightarrow \Lambda, \alpha \vee \beta} (\Rightarrow \vee 2)$$
$$\frac{\alpha, \Gamma \Rightarrow \Pi \quad \beta, \Gamma \Rightarrow \Pi}{\alpha \vee \beta, \Gamma \Rightarrow \Pi} (\vee \Rightarrow)$$

$$\frac{\alpha, \Gamma \Rightarrow \Pi}{\alpha \wedge \beta, \Gamma \Rightarrow \Pi} (\wedge 1 \Rightarrow) \qquad \frac{\beta, \Gamma \Rightarrow \Pi}{\alpha \wedge \beta, \Gamma \Rightarrow \Pi} (\wedge 2 \Rightarrow)$$
$$\frac{\Gamma \Rightarrow \Lambda, \alpha \quad \Gamma \Rightarrow \Lambda, \beta}{\Gamma \Rightarrow \Lambda, \alpha \wedge \beta} (\Rightarrow \wedge)$$

Cut rule

$$\frac{\Gamma \Rightarrow \Lambda, \alpha \quad \alpha, \Delta \Rightarrow \Pi}{\Gamma, \Delta \Rightarrow \Lambda, \Pi} \text{ (cut)}$$

Structural rules

- exchange rules:

$$\frac{\Theta, \alpha, \beta, \Gamma \Rightarrow \Pi}{\Theta, \beta, \alpha, \Gamma \Rightarrow \Pi}$$

$$\frac{\Gamma \Rightarrow \Theta, \alpha, \beta, \Sigma}{\Gamma \Rightarrow \Theta, \beta, \alpha, \Sigma}$$

- contraction rules:

$$\frac{\alpha, \alpha, \Gamma \Rightarrow \Pi}{\alpha, \Gamma \Rightarrow \Pi}$$

$$\frac{\Gamma \Rightarrow \Sigma, \alpha, \alpha}{\Gamma \Rightarrow \Sigma, \alpha}$$

- weakening rules:

$$\frac{\Gamma \Rightarrow \Pi}{\alpha, \Gamma \Rightarrow \Pi}$$

$$\frac{\Gamma \Rightarrow \Lambda}{\Gamma \Rightarrow \Lambda, \alpha}$$

A sequent S is *provable* in **LK** (**LJ**) if it can be derived from initial sequents by applying rules of **LK** (**LJ**) repeatedly. Tree-like figures which express how S can be derived are called **proofs** of S .

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We can show the following:

Lemma

A sequent $\alpha_1, \dots, \alpha_m \Rightarrow \beta_1, \dots, \beta_n$ is provable in **LK** iff $\alpha_1 \wedge \dots \wedge \alpha_m \Rightarrow \beta_1 \vee \dots \vee \beta_n$ is provable in **LK**.

- A proof of distributive law in **LK** (we omitted applications of exchange rule below)

$$\begin{array}{c}
 \frac{\frac{\alpha \Rightarrow \alpha}{\alpha, \beta \Rightarrow \alpha} \text{ (weak)} \quad \frac{\beta \Rightarrow \beta}{\alpha, \beta \Rightarrow \beta} \text{ (weak)}}{\alpha, \beta \Rightarrow \alpha \wedge \beta} \quad \frac{\frac{\alpha \Rightarrow \alpha}{\alpha, \gamma \Rightarrow \alpha} \text{ (weak)} \quad \frac{\gamma \Rightarrow \gamma}{\alpha, \gamma \Rightarrow \gamma} \text{ (weak)}}{\alpha, \gamma \Rightarrow \alpha \wedge \gamma} \\
 \frac{\alpha, \beta \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \quad \alpha, \gamma \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}{\alpha, \beta \vee \gamma \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)} \\
 \frac{\alpha \wedge (\beta \vee \gamma), \beta \vee \gamma \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}{\alpha \wedge (\beta \vee \gamma), \alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)} \\
 \frac{\alpha \wedge (\beta \vee \gamma), \alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)}{\alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)} \text{ (cont)}
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 \end{array}$$

- ▶ Give a proof of the following dual form of the distributive law.

$$(\alpha \vee \beta) \wedge (\alpha \vee \gamma) \Rightarrow \alpha \vee (\beta \wedge \gamma)$$

Cut elimination — the existence of “normal” proofs

The aim of proof theory is to analyze structures of proofs and to extract logical information or logical properties from them. In this respect, the following result must be fundamental, which says that each sequent has a “normal proof” as long as it is provable. In fact, “normal” proofs are proofs without detour (or without taking any roundabout way).

- Here, the word “normal” means “canonical”, rather than “ordinary”.

Theorem (Cut elimination for **LK**)

*If a given sequent is provable in **LK**, it is provable in **LK** without using cut rule.*

But what are benefits of this result?

Theorem (subformula property)

*If a given sequent $\Gamma \Rightarrow \Delta$ is provable in **LK**, it has such a proof \mathcal{P} that every formula appearing in \mathcal{P} is a subformula of a formula either in Γ or in Δ . In fact, any cut-free proof has this property.*

This follows from cut elimination and the following observation: In each rule except cut, every formula in upper sequents appears also in the lower sequent as a subformula of a formula in the lower sequent.

▶ Note:

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An elementary fact of classical logic says that if both $\alpha \rightarrow \beta$ and $\beta \rightarrow \gamma$ hold then obviously $\alpha \rightarrow \gamma$ holds. If \mathbf{LK}^- were to be a system for classical logic, then $\alpha \Rightarrow \gamma$ must be derived from $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \gamma$ in \mathbf{LK}^- . To show this, i.e. to show the admissibility of cut rule in \mathbf{LK}^- is nothing but to show cut elimination in \mathbf{LK} .

Many important logical properties follow from cut elimination.
Here are a few examples.

- 1 decidability, and often tractable proof search algorithms,
- 2 disjunction property
- 3 Maksimova's variable separation property,
- 4 Craig's interpolation property.

- Comparison with natural deduction systems
 - ① Left rules for logical connectives in sequent systems correspond to *elimination rules* in natural deduction systems, while right rules correspond to *introduction rules*.
 - ② normalization theorem vs cut elimination theorem
 - ③ nice correspondence between natural deduction for intuitionistic logic and typed λ -calculus — Curry-Howard isomorphism

■ Comparison with Hilbert-style systems

Modus ponens

- from α and $\alpha \rightarrow \beta$, infer β .

Axioms

- $\alpha \rightarrow (\beta \rightarrow \alpha)$ (weakening),
- $(\alpha \rightarrow (\alpha \rightarrow \gamma)) \rightarrow (\alpha \rightarrow \gamma)$ (contraction),
- $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\beta \rightarrow (\alpha \rightarrow \gamma))$ (exchange),
- $0 \rightarrow \alpha$ and $(\alpha \rightarrow \beta) \rightarrow ((\gamma \rightarrow \alpha) \rightarrow (\gamma \rightarrow \beta))$,
- $(\alpha \rightarrow \gamma) \rightarrow ((\beta \rightarrow \gamma) \rightarrow ((\alpha \vee \beta) \rightarrow \gamma))$,
- $\alpha \rightarrow (\alpha \vee \beta)$ and $\beta \rightarrow (\alpha \vee \beta)$,
- $(\gamma \rightarrow \alpha) \rightarrow ((\gamma \rightarrow \beta) \rightarrow (\gamma \rightarrow (\alpha \wedge \beta)))$,
- $(\alpha \wedge \beta) \rightarrow \alpha$ and $(\alpha \wedge \beta) \rightarrow \beta$.
- $\neg\neg\alpha \rightarrow \alpha$ (Here, $\neg\alpha$ is defined by $\alpha \rightarrow 0$.)

- 1 In Hilbert-style systems, the implication \rightarrow takes all jobs of implications, commas and arrows in sequent systems.
- 2 Axioms for each logical connective are always expressed in combination with implication. Thus, it can hardly have subformula property.
- 3 It has a few rules of inference. Thus, for describing logics, it has high “universality” or “modularity”.
- 4 By the same reason., it lacks “sensitivity” to logical properties of a given logic. On the other hand, a cut-free system (i.e. a sequent system for which cut elimination holds) is often highly sensitive to these properties, while only a limited number of logics have cut-free systems.

Uniform framework provided by sequent formulation

It turned out recently that sequent formulation can provide us a fresh perspective of nonclassical logics, through developments of the study of [substructural logics](#). In the rest of my talk, we will make a brief introduction to substructural logics, and explain the idea.

N. Galatos, P. Jipsen, T. Kowalski, HO: [Residuated Lattices: an algebraic glimpse at substructural logics](#), *Studies in Logic and Foundations of Mathematics*, vol.151, Elsevier, April, 2007



Nick, Peter and Tomasz

♣ Our starting point:

- H. Ono and Y. Komori, Logics without the contraction rule, Journal of Symbolic Logic (1985).

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♣ Development of the idea of grasping various nonclassical logics as logics lacking some or all of structural rules:

- Suggested also by works on Lambek calculus (J. van Benthem and W. Buszkowski), and on linear logic (J.-Y. Girard)
- H. Ono, Structural rules and a logical hierarchy, presented at the Heyting '88 conference in Bulgaria.

Nonclassical logics from substructural viewpoint

- Lambek calculus — logic without structural rules

Calculus for categorial grammar introduced by Ajdukiewicz and Bar-Hillel (J. Lambek, 1958), which was rediscovered in early 80s by J. van Benthem and W. Buszkowski.

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- Logic of residuated lattices

In 1974, S. Tamura introduced a sequent calculus for residuated lattices and proved cut elimination theorem. As a result, he showed that the equational theory of residuated lattices is decidable.

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In 1974, S. Tamura introduced a sequent calculus for residuated lattices and proved cut elimination theorem. As a result, he showed that the equational theory of residuated lattices is decidable.
- Relevant logics — logics without weakening rules
Logics of relevant implication mainly developed by A. Anderson, N. Belnap Jr., R.K. Meyer and M. Dunn. A. Urquhart showed the undecidability of relevant propositional logic **R** in 1984. Sometimes, contraction-free relevant logics are also studied.

- Logics without contraction rule (BCK logics)
In his book in 1963, H. Wang mentioned that classical predicate logic *without contraction rules* is decidable. In the 1970s, V. Grishin pointed out that contraction rules are essentially used in deriving Russell's paradox. H.O. & Y. Komori in 1985 developed syntactic and semantical study of logics without contraction rules (also Došen 88, 89).
- Łukasiewicz's many-valued logics and fuzzy logics by P. Hájek — logics without contraction rules.

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- Linear logic — logic only with exchange rule
J.-Y. Girard introduced linear logic in his influential paper published in 1987. Due to his exposition, substructural logics are sometimes regarded as *resource sensitive logics*.
- Johansson's minimal logic — logic without right-weakening

★ Substructural logics are logics lacking some or all of structural rules when they are formalized in sequent calculi.

♣ Fusion — new logical connective

If a sequent calculus lacks either contraction or weakening, each comma in a sequent can no more express **conjunction**. To represent a comma explicitly as a logical symbol, we introduce a new logical connective **fusion** (\cdot , in symbol).

Rules for fusion

$$\frac{\Gamma \Rightarrow \alpha \quad \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \alpha \cdot \beta} (\Rightarrow \cdot) \qquad \frac{\Sigma, \alpha, \beta, \Delta \Rightarrow \varphi}{\Sigma, \alpha \cdot \beta, \Delta \Rightarrow \varphi} (\cdot \Rightarrow)$$

Suppose that both $\alpha \Rightarrow \beta$ and $\alpha \Rightarrow \gamma$ are provable. Then;

- (1) $\alpha, \alpha \Rightarrow \beta \cdot \gamma$ is provable,
- (2) $\alpha \Rightarrow \beta \cdot \gamma$ is not always provable,
- (3) $\alpha \Rightarrow \beta \wedge \gamma$ is provable.

► What does this mean, and what are the differences?

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Assumptions will be “consumed”.

- $\alpha \wedge \beta$ is equivalent to $\alpha \cdot \beta$ for all α and β iff both weakening rule and contraction rule hold.

★ Substructural logics are resource sensitive logics

♠ As we have shown, the class of substructural logics include many important nonclassical logics as its subclasses.

▶ But, why "substructural"? Sequent calculi will be one of options in formalizing nonclassical logics, and structural rules are usually regarded as auxiliary rules.

Is there any particular reason to be substructural, i.e. to take sequent formulation?

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Is there any particular reason to be substructural, i.e. to take sequent formulation?

- There must be some reasons. As a matter of fact, if we try to formulate standard substructural logics by using either a Hilbert-style formal system or a natural deduction system, we will end up with a non-transparent and complicated formulation.

The following three conditions are mutually equivalent.

① $\vdash \alpha \cdot \beta \Rightarrow \varphi,$

② $\vdash \alpha, \beta \Rightarrow \varphi,$

③ $\vdash \beta \Rightarrow \alpha \rightarrow \varphi.$

◆ Implication is the **residual** of fusion (or comma), or has the Galois connection with fusion (in the dual order).

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cf. Division is a residual of multiplication in arithmetic: $a \times b \leq c$ iff $b \leq c/a$.

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- 1 Differences of logics come mostly from differences of their implication.
- 2 Through the residuation relation, a change of fusion affects a change of implication, and vice versa,
- 3 Fusion is usually hidden, or unnoticed in a logic, but **its sequent formulation will explicate the role of fusion in the form of commas in sequents.**
- 4 Structural rules represent basic conditions of fusion, and hence are essential for the behavior of implication.

★ Substructural logics are logics of residuated structures

- H. Ono, Substructural logics and residuated lattices — an introduction, 50 Years of Studia Logica, Trends in Logic 21, 2003