Polyadic modal logic

From ”Knowing value” logic to WAL

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March 5, 2017

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Introduction

Polyadic modal logics are logics with n-ary modalities other than the usual unary ones. They differ from the standard modal logic in various aspects, e.g., Sahlqvist-like fragments in correspondence theory, detailed techniques for proving completeness, and issues about decidability.

Recently Wang and Gu found in [4] that the “Knowing value” logic (KvL) can be treated as a normal modal logic with binary modalities. This led them to consider some basic properties about binary modal logic, in order to prove the decidability of KvL. However, there is not much previous research on this topic in general.

Introduction

Actually KvL is a special kind of polyadic logic since we only need special binary modalities where the two arguments are the same. It is also called (binary) “diagonal modalities”. In general, it is also natural to consider the n-ary diagonal modalities where the arguments are all the same. There is indeed some research on this topic in the literature of modal logic. The logic with such modalities are called weak aggregative modal logics (WAL), which was first introduced by Schotch and Jennings in [7], and the completeness proofs for these logics are highly non-trivial (e.g., [2]).

Recently, Yanjing Wang and I show that the WAL do have a natural bisimulation notion which can be used to give a van Benthem-like characterization theorem, and we can simplify the completeness proof based on some new ideas.

Polyadic modal logic

Definition[1]

A modal similarity type is a pair $\tau = (O, \rho)$ where O is a non-empty set, and ρ is a function $O \to \mathbb{N}$. The elements of O are called modal operators; we use $\triangle, \triangle_0, \dots \triangle_n$ to denote elements of O. The function ρ assigns to each $\triangle \in O$ a finite arity, indicating the number of arguments $\triangle$ can be applied to.

A modal language $ML(\tau, \Phi)$ is built up using a modal similarity type $\tau = (O, \rho)$ and a set of propositional letters Φ. Formulas is given by the rule

$$\phi := p \mid \bot \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \triangle(\phi_1, \dots, \phi_{\rho(\triangle)})$$

where $p \in \Phi$.

[1] We use the definition in [3].

Polyadic frame

Definition

A frame is a tuple F consisting of the following ingredients:

(i) a non-empty set W,

(ii) for each $n \geq 1$, and each n-ary modal operator $\triangle$, an $(n+1)$-ary relation $R_\triangle$.

As for the modal case, when $\rho(\triangle) > 0$ we define

$M, w \models \triangle(\phi_1, \ldots, \phi_n)$ iff for some $v_1, \ldots, v_n \in W$ with $R_\triangle w v_1, \ldots, v_n$ we have, for each i, $M, v_i \models \phi_i$.

Definition[2]

A modal logic is a set of formulas containing all tautologies that is closed under modus ponens and uniform substitution. A modal logic Λ is normal if for every operator $\triangledown$ it contains: the axiom $K^i_\triangledown$(for all i such that $1 \leq i \leq \rho(\triangledown)$); the axiom $Dual_\triangledown$ and is closed under $Nec^*_\triangledown$.

The required axioms are obvious polyadic analogs of the K and $Dual$ axioms:

$$
\begin{array}{cc}
(K^i_\triangledown) & \triangledown(r_1, \dots p \to q, \dots, r_{\rho(\triangle)}) \to (\triangledown(r_1, \dots p, \dots, r_{\rho(\triangle)}) \\
 & \to \triangledown(r_1, \dots q, \dots, r_{\rho(\triangle)}))^3 \\
Dual_\triangledown & \triangle(r_1, \dots, r_{\rho(\triangle)}) \leftrightarrow \neg\triangledown(\neg r_1, \dots \neg r_{\rho(\triangledown)}) \\
Nec^*_\triangledown & \vdash_\Lambda \phi \Longrightarrow \vdash_\Lambda \triangledown(\bot, \dots \phi, \dots \bot)
\end{array}
$$

[2]This one is from [3] and we will show it's not the right version leter.
[3]where p and q occur in the i-th argument place

A Remark for the above definition

The right version of the normal logic should be subsititute Nec^*_∇ by Nec_∇, which is as follows.

$Nec_\nabla \quad \vdash_\wedge \phi \quad \Longrightarrow \quad \vdash_\wedge \nabla(\phi_0, \ldots \phi, \ldots \phi_n)$, where $\phi_0, \ldots \phi_n$ are arbitrary formulas.

Now we will show that we cannot derive the rule *Nec* in the above system K^*.

Remark

Nec **is independent in the logic** K^***.**

Proof. We define a new semantics $\Vdash$ w.r.t. the Kripke model to show the independence.

The truth definitions for the propositional letters and boolean cases are the same as $\models$. For the modal case: $w \Vdash \nabla(\phi_1, \ldots, \phi_n)$ iff one of the followings hold:

1. w is a dead end, i.e. there is no $v_1, \ldots, v_n$ s.t. $Rwv_1, \ldots, v_n$.

2. There are some $v_1, \ldots, v_n$ s.t. $Rwv_1, \ldots, v_n$ and

$\exists k \in [1, n](\forall w_1, \ldots, w_n(Rww_1, \ldots, w_n \rightarrow$

$(w_k \Vdash \phi_k \wedge \forall m \neq k \exists w_1', \ldots, w_n'(Rww_1', \ldots, w_n' \wedge w_m' \Vdash \neg\phi_m)$.

(The above statement says that there is a unique argument which is true at the corresponding position of every sequence of successors, and we call this argument the unique truth.)

Remark

continue the proof

Now we varify that $\Vdash$ is valid w.r.t. K^*. Since we don't change any definition of the propositional connectives, each tautology is still valid. The case for dual and US are trivial and it is also easy to show that $\Vdash$ preserves truth under Nec^*. For the axioms K, suppose that M, w is a pointed model and the two premises are satisfied at M, w. Then we know that p_i is the unique true argument for some i, and if $i = n+1$, q_{n+1} must be the unique, which means $\triangledown(p_1, \ldots q_{n+1}, \ldots, p_m)$ is true at w, since other p_j must be wrong at some successors of w. If $i \neq n+1$, it follows that q_{n+1} must be wrong somewhere and hence we also have $\triangledown(p_1, \ldots q_{n+1}, \ldots, p_m)$ is true at w. As a result, $\Vdash K$.

Let M, w be a point model where w is not a dead end. Trivially, $\Vdash \top$, but $w \Vdash \neg(\top, \ldots, \top, \ldots, \top)$, which means Nec cannot preserve truth. Thus we know that Nec is independent in the logic K^*.

Remark

Actually we don't need to add the full version of *Nec* to our system to get what we need. If we want to make the necessitation simple as *Nec**, we only need to add a series of axioms as follows:

$C_2 : \nabla(p, \bot) \to (\nabla(\bot, q) \to \nabla(p, q))$

$C_n : \nabla(p_1, \bot, \dots, \bot) \wedge \nabla(\bot, p_2, \bot, \dots, \bot) \wedge \cdots \wedge \nabla(\bot, \dots, \bot, p_n) \to \nabla(p_1, \dots, p_n)$

For n-ary modal logic, we need the C_n, and one can easily check that with the help of C_n, we can derive *Nec* in our system.

Weak aggregative logic

Preliminaries

Definition(language)

A weak aggregative language is built up using a set of propositional letters Φ and a single unary modality $\Diamond$. Formulas is given by the rule

$$\phi := p \mid \bot \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \Diamond\phi$$

where $p \in \Phi$.

So the language here is as same as the basic modal language.

Preliminaries

Definition(Frames)

A frame is a tuple F consisting of the following ingredients:

(i) a non-empty set W,

(ii) an $(n+1)$-ary relation R, where n is a fixed positive natural number.

For WAL, the modal truth is just a special case of polyadic modal logic, which is defined as follows:

$M, w \models \Diamond\phi$ iff for some $v_1, \ldots, v_n \in W$ with $Rwv_1, \ldots, v_n$ we have, for each i, $M, v_i \models \phi$.

Definition(Logic)

The weakly aggregative modal logic K_n is axiomatized as follows[4]:

Axiom: $\Box\phi_0 \wedge \cdots \Box\phi_n \to \Box \bigvee_{(0\leq i<j\leq n)} \phi_i \wedge \phi_j$ (we also call this formula K_n)

Rules: $\vdash \phi \Longrightarrow \vdash \Box\phi$ (N);

$\vdash \phi \to \psi \Longrightarrow \vdash \Box\phi \to \Box\psi$ (RM).

It is easy to check that K_1 is just the normal modal logic K. In the following completeness proof we assume that the set of proposition letters is countable to avoid the use of the axiom of choice. (actually for the uncountable situation, we need the fact that the power set of our language is well-orderable.)

[4] This is first introduced in [7]

Completeness

- We will use the canonical model method to give a direct proof for the completeness theorem of the weakly aggregative modal logic.
  The definition of the canonical model is the same as in [2].

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- We will use the canonical model method to give a direct proof for the completeness theorem of the weakly aggregative modal logic.
  The definition of the canonical model is the same as in [2].
- Let $n \geq 1$ and Γ be the logic K_n. Define the canonical $(n+1)$-ary model of Γ to be the model $M^\Gamma = \langle W^\Gamma, R^\Gamma, V^\Gamma \rangle$, where
  $W^\Gamma =: \{\Sigma \subseteq Form \mid \Sigma$ is maximally Γ-consistent$\}$,
  $R^\Gamma =: \{(\Sigma_0, \Sigma_1, \ldots \Sigma_n) \mid \Box(\Sigma_0) \subseteq \cup\{\Sigma_1, \ldots \Sigma_n\}\}$,
  $V^\Gamma(p) = \{\Sigma \in W^\Gamma \mid p \in \Sigma\}$
  (here $\Box(\Sigma_0) = \{\alpha \mid \Box\alpha \in \Sigma_0\}$).
  So we have: $xRy_0y_1 \ldots y_n$ iff for each $\Box\phi \in x$, $\phi \in \Sigma_i$ for some $i \leq n$.

Completeness

In [2] and [1], Apostoli prove the completeness for these K_n by reduce the n-forcing relation which introduce by jennings(1980). In [5], Jennings with his two partners give a direct proof which based on syntax. Here we will use the canonical model method to give a proof just like we ordinarily did for normal modal logic, and in this proof we will use a syntax proof in [5].

Completeness

proposition[Truth lemma]

$M^\Gamma, x \models \phi$ iff $\phi \in x$.

Completeness

proposition[Truth lemma]

$M^\Gamma, x \models \phi$ iff $\phi \in x$.

proof

By induction on ϕ and the key step is the one for $\Diamond$. So we need the existential lemma.

Completeness

proposition[Existential lemma]

$\Diamond\phi \in x$ iff there are $y_1 \ldots y_n$ such that $xRy_0y_1 \ldots y_n$ and $\phi \in y_i$ for each $i \leq n$.

Completeness

proposition[Existential lemma]

$\Diamond\phi \in x$ iff there are $y_1 \ldots y_n$ such that $xRy_0y_1 \ldots y_n$ and $\phi \in y_i$ for each $i \leq n$.

proof

Since the language is countable, we can assume that $\Box(\Sigma_0) = \{\psi_i \mid i \in \omega\}$. What we need to show is that there is an n-partition(here we admit $\varnothing$ part.) $(\Sigma_1, \ldots, \Sigma_n)$ of $\Box(\Sigma_0)$ such that each $\Sigma_i \cup \{\phi\}$ is consistent. From that we can use the traditional lindenbaum method to get n MCSs $y_1 \ldots y_n$ s.t. $xRy_0y_1 \ldots y_n$ and $\phi \in y_i$ for each $i \leq n$. Notice that ϕ itself is consistent since we have the rule N, so if $\Sigma_i = \varnothing$, $\Sigma_i \cup \{\phi\}$ is consistent.

Completeness

proof

First we show that if Σ_i is a singleton, $\Sigma_i \cup \{\phi\}$ is consistent. Suppose that $\{\psi_i, \phi\}$ is not consistent. So we have $\vdash \psi_i \to \neg\phi$, and by RM we know that $\vdash \Box\psi_i \to \Box\neg\phi$. Since $\Box\psi_i \in x$, $\Box\neg\phi \in x$, which contradicts the fact that x is a MCS and $\Diamond\phi \in x$.

We define a sequence of sets of n-tuple as follows:for each i,

$B_0 = \{(\sigma_1, \ldots \sigma_n) \mid \sigma_i = \{\phi\}$ for each $i\}$;

$B_m = \{(\sigma_1^{s_1}, \ldots \sigma_n^{s_n}) \mid \sigma_i^{s_i} = \sigma_i \cup s_i$ where $(s_1, \ldots, s_n)$ is an n-partition of $A_m = \{\psi_i \mid i < m\}$ s.t. each $\sigma_i \cup s_i$ is consistent.$\}$ for $m > 0$.

Completeness

proof

Notice that $|B_m| \leq ! \left|(\rho(A_m))^{\leq n}\right|$, which means each B_m is finite.

Claim 1: each B_i is not empty.

Claim 2: there are $\langle \sigma_1^j \mid j < \omega \rangle, \ldots, \langle \sigma_n^j \mid j < \omega \rangle$ s.t. for each $i \leq n$, $\sigma_i^j \subseteq \sigma_i^k$ if $j < k$, where $(\sigma_1^j, \ldots \sigma_n^j) \in B_j$.

If Claim 2 is true, let $\Sigma_i = \bigcup_{j \in \omega} \sigma_i^j$ for each $i < n$. From the definition and claims above, It's easy to show that each Σ_i is consistent and each $\psi_j \in \Sigma_i$ for some i. It follows that $\Sigma_1, \ldots, \Sigma_n$ is an n-partition of $\Box(\Sigma_0)$ such that each $\Sigma_i \cup \{\phi\}$ is consistent.

proof

Next we prove that Claim 1 implies Claim 2. Suppose Claim 1 holds.

First we show that for each $j \in \omega$, there must be some $(\sigma_1^j, \dots \sigma_n^j) \in B_j$ s.t.

$$\forall k > j \exists j' > k \exists (\sigma_1^{j'}, \dots \sigma_n^{j'}) \in B_{j'} (\bigwedge_{i \leq n} \sigma_i^j \subseteq \sigma_i^{j'}) \tag{1}$$

Here we call such a $(\sigma_1^{j'}, \dots \sigma_n^{j'})$ an extension of $(\sigma_1^j, \dots \sigma_n^j)$.

So $(\sigma_1^j, \dots \sigma_n^j)$ satisfies 1 means that its extensions are unbounded.

Completeness

proof

If the assertion above is not true, we will find j and k s.t. for each $x \in B_j$, there are no extension of x in B_k since B_j is finite. So B_k must be empty, contradicts claim 1.(Since if $y = (\sigma_1, \ldots \sigma_n) \in B_k$, we can delete all the ψ_h s.t. $h \geq j$ from these σ to get a $y' = (\sigma'_1, \ldots \sigma'_n) \in B_j$ and it's clear that y is an extension of $y^{'}$.)

Now we know that in each B_m there are some x whose extensions are unbounded, which means that in B_{m+1} there must be some $x^{'}$ extended x s.t. $x^{'}$ has unbounded many extensions.

Completeness

proof

Here we define a well-ordering $<_w$on $\bigcup_{m\in\omega} C_m$, where $C_m = \{f \mid f$ is from m to $n\}$.(one can see that each member of C_m decides a partition of A_m and hence decides a member of B_m.) If $f \in C_m$ and $g \in C_k$ where $m < k$, then $f <_w g$. If f_0,$f_1 \in C_m$, we order f_0,f_1 by their left-lexicographic order on n^m. Obviously $<_w$is a well-ordering on $\bigcup_{m\in\omega} C_m$.

It follows that for $x \in B_m$ we can choose the least extension $x' \in B_{m+1}$ s.t. x' has unbounded many extensions. As a result we find what we need for claim 2.

Completeness

proof

Now we come to prove claim 1. Suppose that B_m is empty for some $m \in \omega$. First it's clear that $m > n$ by our observation above. It follows from the assumption that each partition P of A_m would introduce an inconsistent $s(P)$, which means for each partition P, we have $\vdash \bigwedge_{i \in s(P)} \psi_i \to \neg\phi$. It follows that $\vdash \bigvee_P \bigwedge_{i \in s(P)} \psi_i \to \neg\phi$. By RM, we have $\vdash \Box \bigvee_P \bigwedge_{i \in s(P)} \psi_i \to \Box\neg\phi$.

Completeness

Claim 3

$\vdash \bigwedge_{i<m} \Box\psi_i \to \Box(\bigvee_P \bigwedge_{i\in s(P)} \psi_i)$

Suppose that claim 3 holds, then by our original assumption, $\Box\psi_i \in x$ for each $i \in \omega$, so we have $\Box(\bigvee_P \bigwedge_{i\in s(P)} \psi_i) \in x$.

But $\vdash \Box \bigvee_P \bigwedge_{i\in s(P)} \psi_i \to \Box\neg\phi$, so $\Box\neg\phi \in x$, contradicts that $\Diamond\phi \in x$.

We will give a sketch of proof for claim 3 later by using the method in [5].

Completeness

First we give some basic definition.

Definition

A non-empty family A of non-empty sets is an n-trace on a non-empty set B iff for any n-partition $(\sigma_1, \ldots \sigma_n)$ of B, there is some $a \in A$ s.t. $a \subseteq \sigma_i$ for some $i \leq n$.

Completeness

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If the above holds for $B = \cup A$, we call A is an n-trace. For any n-trace A on some finite subset of Form, we define $F(A) = \bigvee_{a \in A} \bigwedge_{\phi \in a} \phi$.

Completeness

Definition

The χ-product of a family A of sets is defined as follows:

$$\chi(A) = \{b \mid \forall a \in A(a \cap b \neq \varnothing) \wedge \forall b' \subset b \exists a \in A(b' \cap a = \varnothing)\}$$

[5] One can find a proof in [5]

Completeness

Definition

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Proposition

[5] Let A be an arbitrary n-trace on a finite subset of $\Sigma = \{\psi_i \mid i \in \omega\}$, then $\vdash \bigwedge_{\phi \in \cup A} \Box\phi \to \Box F(A)$.

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Completeness

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Proposition

[5] Let A be an arbitrary n-trace on a finite subset of $\Sigma = \{\psi_i \mid i \in \omega\}$, then $\vdash \bigwedge_{\phi \in \cup A} \Box\phi \to \Box F(A)$.

Since the Claim 3 is just a special case of the above proposition, we can get the final result.

[5] One can find a proof in [5]

Completeness

Theorem

K_2 is complete w.r.t. all the ternary frame.

Definition

[bisimulation for WAL]Let $M = (W, R_\triangle, V)$ and $M' = (W', R'_\triangle, V')$ be two models. A non-empty binary relation $Z \subseteq W \times W'$ is called a bisimulation between M and M' if the following conditions are satisfied:

(i) If wZw', then w and w' satisfy the same propositional letters.

(ii) If wZw' and $R_\triangle wv_1, \ldots, v_n$ then there are $v'_1, \ldots, v'_n$ in W' s.t. $R_\triangle w' v'_1, \ldots, v'_n$ and $\exists f \subseteq Z^-$(f is a function from $\{v'_i \mid 1 \leq i \leq n\}$ to $\{v_i \mid 1 \leq i \leq n\}$) (the forth condition).

(iii) If wZw' and $R_\triangle w' v'_1, \ldots, v'_n$ then there are $v_1, \ldots, v_n$ in W s.t. $R_\triangle wv_1, \ldots, v_n$ and $\exists f \subseteq Z$(f is a function from $\{v_i \mid 1 \leq i \leq n\}$ to $\{v'_i \mid 1 \leq i \leq n\}$) (the back condition)

Bisimulation

When Z is a bisimulation linking two states w in M and $w^{'}$ in $M^{'}$ we say that w and $w^{'}$ are bisimilar, and we write $Z : M, w \underline{\leftrightarrow} M^{'}, w^{'}$. If there is a bisimulation Z such that $Z : M, w \underline{\leftrightarrow} M^{'}, w^{'}$, we sometimes write $M, w \underline{\leftrightarrow} M^{'}, w^{'}$; likewise, if there is some bisimulation between M and $M^{'}$, we write $M \underline{\leftrightarrow} M^{'}$,saying M and $M^{'}$are bisimilar.

From the definition above we can see that if w and $w^{'}$ are modal bisimilar, they are clearly WAL-bisimilar. Since we know that WAL is a fragment of polyadic modal logic, we will show that it is exactly the fragment closed under the bisimulation above. First we show the bisimulation is indeed sound w.r.t. the WAL-equivalence.

Proposition

Let $M = (W, R_{\triangle}, V)$ and $M^{'} = (W^{'}, R^{'}_{\triangle}, V^{'})$ be two models. Then for every $w \in W$ and $w^{'} \in W^{'}$, $w \leftrightarroweq w^{'}$ implies $w \leftrightsquigarrow w^{'}$. In words, WAL formulas are invariant under bisimulation.

Proof.

We use induction on formulars, and we focus on the modality case, since others are trivial. Suppose that $w \leftrightarroweq w^{'}$ and $w \models \Diamond\phi$. Then we know that there are $v_1, \ldots, v_n$ s.t. $R_{\triangle}wv_1, \ldots, v_n$, and each $v_i \models \phi$. By the forth condition, there are $v^{'}_1, \ldots, v^{'}_n$ in $W^{'}$ s.t. $R_{\triangle}w^{'}v^{'}_1, \ldots, v^{'}_n$ and $\exists f \subseteq Z^{-}(f$ is a function from $\{v^{'}_i \mid 1 \leq i \leq n\}$ to $\{v_i \mid 1 \leq i \leq n\})$. From the I.H. we have each $v^{'}_i \models \phi$ since for each $v^{'}_i$ there is a v_j s.t. $v_jZv^{'}_i$. As a result, $w^{'} \models \Diamond\phi$. For the converse direction just use the back condition. $\square$

The bisimulation notion for WAL is similar with which for modal logic, since we can generalize some modal theorem to WAL ones.

Theorem (Hennessy-Milner Theorem)

Let $M = (W, R_\triangle, V)$ *and* $M' = (W', R'_\triangle, V')$ *be two image-finite models. Then for every* $w \in W$ *and* $w' \in W'$, $w \leftrightarrows w'$ *iff* $w \leftrightsquigarrow w'$.

Proof.

Similar to the proof for basic modal logic. $\square$

Bisimulation

It is obvious that we have a standard translation from WAL to PML. We can just translate $\Diamond\phi$ intio $\triangle(\phi, \ldots, \phi)$, where the number of arguments is depended on which polyadic language we use. Like the Van Benthem Characterization Theorem for modal logic, we will show a similar theorem for WAL: a PML formula is equivalent to the translation of a WAL formula if and only if it is invariant under WAL-bisimulations. The proof is based on the way of proving Van Benthem Characterization Theorem for modal logic–by a Detour Lemma.

Definition (Hennessy-Milner Classes)

K is a Hennessy-Milner class, or has the Hennessy-Milner property, if for every two pointed models M, w and M', w' in K, $w \leftrightarrows w'$ iff $w \leftrightsquigarrow w'$.

Here we give a alternative definition of modal saturated model.

Definition (wa-saturated)

Let $M = (W, R_\triangle, V)$ be an model where $R_\triangle$ is $n+1$–ary. M is called wa-saturated if for every state $w \in W$ and every sequence $\Sigma_1, \ldots, \Sigma_n$ of sets of WAL formulas we have the following.

If for every sequence of finite subsets $\Delta_1 \subseteq \Sigma_1, \ldots, \Delta_n \subseteq \Sigma_n$ there are states $v_1, \ldots, v_n$ s.t. $R_\triangle w v_1, \ldots, v_n$ and for each i there is a j s.t. $v_i \models \Delta_j$. then there are $w_1, \ldots, w_n$ s.t. $R_\triangle w w_1, \ldots, w_n$ and for each i there is a j s.t. $w_i \models \Sigma_j$.

Proposition

The class of wa-saturated models has the Hennessy-Milner property.

proof

Let $M = (W, R_\triangle, V)$ and $M' = (W', R'_\triangle, V')$ be two wa-saturated models. It suffices to prove that the relation $\leftrightsquigarrow$ of modal equivalence between states in M and states in M' is a bisimulation. We focus on the forth condition, since the case for propositional letters is trivially satisfied, and the back condition is completely analogous to the case we prove.

proof

Assume that $w, v_1, \ldots, v_n \in W$ and $w' \in W'$ satisfy $Rwv_1, \ldots, v_n$ and $w \leftrightsquigarrow w'$. Let Σ_i be the set of formulas true at v_i for each $i \leq n$. obviously, for every sequence of finite subset $\Delta_1 \subseteq \Sigma_1, \ldots, \Delta_n \subseteq \Sigma_n$, we have for each i, $M, v_i \models \bigvee_{j \leq n} \bigwedge \Delta_j$, hence $M, w \models \Diamond \bigvee_{j \leq n} \bigwedge \Delta_j$. It follows that $M', w' \models \Diamond \bigvee_{j \leq n} \bigwedge \Delta_j$ since $w \leftrightsquigarrow w'$, which means w' has a sequence of sueccessors $v'_1, \ldots, v'_n$ s.t. for each i, $v'_i \models \bigvee_{j \leq n} \bigwedge \Delta_j$ i.e. $v'_i \models \bigwedge \Delta_j$ for some j. By wa-saturation, for each i there is a j s.t. $v'_i \models \Sigma_j$, which means for each v'_i there is a v_j s.t. $v'_i \leftrightsquigarrow v_j$.

Proposition

Any countably saturated model is wa-saturated. It follows that the class of countably saturated models has the Hennessy-Milner property.

proof

Suppose that $M = (W, R_\triangle, V)$ is a countably saturated model. Let $w \in W$ and $\Sigma_1, \ldots, \Sigma_n$ be a sequence of sets of WAL formulas s.t. for every sequence of finite subsets $\Delta_1 \subseteq \Sigma_1, \ldots, \Delta_n \subseteq \Sigma_n$ there are states $v_1, \ldots, v_n$ s.t. $R_\triangle wv_1, \ldots, v_n$ and for each i there is a j s.t. $v_i \models \Delta_j$. Define $\Sigma = \{Rwx_1, \ldots, x_n\} \cup \bigcup_{j \leq n} \{ \bigvee_{i \leq n} ST_{x_j}(\phi_i) \mid \phi_i \in \Sigma_i\}$.

Claim: Σ is consistent with $th((M, w))$, the first-order theory of (M, w).

Proof

If we proves the Claim, we know that Σ itself is realized in some $v_1, \ldots, v_n \in W$ since Σ is a n-type with just one parameter and M is countably saturated. By $(M, w) \models Rwx_1, \ldots, x_n[v_1, \ldots, v_n]$ it follows that $Rwv_1, \ldots, v_n$ and by $(M, w) \models \bigcup_{j \leq n} \{ \bigvee_{i \leq n} ST_{x_j}(\phi_i) \mid \phi_i \in \Sigma_i \}[v_1, \ldots, v_n]$, $v_j \models \{ \bigvee_{i \leq n} ST_{x_j}(\phi_i) \mid \phi_i \in \Sigma_i \}$. Thus, $v_j \models \Sigma_i$ for some i: if not, we will know that for each i, there is some $\phi_i \in \Sigma_i$ s.t. $v_j \models \neg ST_{x_j}(\phi_i)$, which means $v_j \models \bigwedge_{i \leq n} \neg ST_{x_j}(\phi_i)$, contradicts that $v_j \models \bigvee_{i \leq n} ST_{x_j}(\phi_i)$.

Proof

If we proves the Claim, we know that Σ itself is realized in some $v_1, \ldots, v_n \in W$ since Σ is a n-type with just one parameter and M is countably saturated. By $(M, w) \models Rwx_1, \ldots, x_n[v_1, \ldots, v_n]$ it follows that $Rwv_1, \ldots, v_n$ and by $(M, w) \models \bigcup_{j \leq n} \{ \bigvee_{i \leq n} ST_{x_j}(\phi_i) \mid \phi_i \in \Sigma_i \}[v_1, \ldots, v_n]$, $v_j \models \{ \bigvee_{i \leq n} ST_{x_j}(\phi_i) \mid \phi_i \in \Sigma_i \}$. Thus, $v_j \models \Sigma_i$ for some i: if not, we will know that for each i, there is some $\phi_i \in \Sigma_i$ s.t. $v_j \models \neg ST_{x_j}(\phi_i)$, which means $v_j \models \bigwedge_{i \leq n} \neg ST_{x_j}(\phi_i)$, contradicts that $v_j \models \bigvee_{i \leq n} ST_{x_j}(\phi_i)$.

Now we prove the Claim: Suppose that Σ is not consistent with the first-order theory of (M, w). It follows that there are a sequence of finite subsets $\Delta_1 \subseteq \Sigma_1, \ldots, \Delta_n \subseteq \Sigma_n$ s.t. $\bigcup_{i \leq n} \{ \bigvee_{i \leq n} ST_{x_j}(\phi_i) \mid \phi_i \in \Delta_i \} \cup \{Rwx_1, \ldots, x_n\} \cup th((M, w))$ is inconsistent. But that is impossible since we already know that for every sequence of finite subsets $\Delta_1 \subseteq \Sigma_1, \ldots, \Delta_n \subseteq \Sigma_n$ there are states $v_1, \ldots, v_n$ s.t. $R_{\triangle} wv_1, \ldots, v_n$ and for each i there is a j s.t. $v_i \models \Delta_j$.

Theorem (Van Benthem Characterization Theorem)

Let ϕ be a polyadic modal formula. Then ϕ is invariant for WAL-bisimulations iff it is (equivalent to) the standard translation of a WAL formula.

Proof.

Just as the one for modal logic.

□

Some syntax remarks

Introduction

In this section we will give some results about the relation between KvL, PML and WAL, and basically we consider some syntax properties.

Recall that the syntax Gu and Wang give for KvL in [4] is as follows.

System $\mathbb{MLKV}^r$

System $\mathbb{MLKV}^r$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
DISTKv^r	$\Box_i(p \to q) \to (\Box_i^c p \to \Box_i^c q)$
Kv$^r\vee$	$\Diamond_i(p \wedge q) \wedge \Diamond_i^c(p \vee q) \to (\Diamond_i^c p \vee \Diamond_i^c q)$

Rules

MP	$\dfrac{\phi, \phi \to \psi}{\psi}$
NECK	$\dfrac{\phi}{\Box_i \phi}$
NECKv^r	$\dfrac{\phi}{\Box_i^c \phi}$
SUB	$\dfrac{\phi}{\phi[p/\psi]}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\phi \leftrightarrow \phi[\psi/\chi]}$

Extended language $\mathbf{MLKv}^{r+}$

The extended language $\mathbf{MLKv}^{r+}$ is:

$$\phi ::= \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box_i \phi \mid \Box_i^c(\phi, \phi)$$

The difference is that we no longer assume the two arguments to be the same here.

The semantics are similar (easier to understand as $\Diamond_i^c(\phi, \psi)$).

System $\mathbb{MLKV}^{r+}$

System $\mathbb{MLKV}^{r+}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
SYM	$\Box_i^c(p,q) \to \Box_i^c(q,p)$
DISTBK	$\Box_i^c(p \to q, r) \to (\Box_i^c(p,r) \to \Box_i^c(q,r))$
INC	$\Diamond_i^c(p,q) \to \Diamond_i p$
ATEUC	$\Diamond_i^c(p,q) \wedge \Diamond_i r \to \Diamond_i^c(p,r) \vee \Diamond_i^c(q,r)$

Rules

MP	$\dfrac{\phi, \phi \to \psi}{\psi}$
NECK	$\dfrac{\phi}{\Box_i \phi}$
NECKv^r	$\dfrac{\phi}{\Box_i^c(\phi,\psi)}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\phi \leftrightarrow \phi[\psi/\chi]}$
SUB	$\dfrac{\phi}{\phi[p/\psi]}$

Remark

First we show that in KvL, one can derive a formula similar with K_2, and we call that formula K_2^*.

K_1: $\dot{\Box}p \wedge \dot{\Box}q \rightarrow \dot{\Box}(p \wedge q)$

K_2^*: $\dot{\Box}p \wedge \dot{\Box}q \wedge \dot{\Box}(\neg p \wedge \neg q) \rightarrow \dot{\Box}(p \wedge q)$

K_2: $\dot{\Box}p \wedge \dot{\Box}q \wedge \dot{\Box}r \rightarrow \dot{\Box}((p \wedge q) \vee (p \wedge r) \vee (q \wedge r))$

K_{n+1}^*: $\phi_n \wedge \dot{\Box}(\neg(p_0 \vee \cdots \vee p_n)) \rightarrow \psi_n$

K_n^*: $\bigwedge_i \dot{\Box}p_i \wedge \dot{\Box} \bigwedge_i \neg p_i \rightarrow \dot{\Box} \bigwedge_i p_i$

Proposition

$\vdash_{MLKV^r} K_2^*$

Proof

1		$\Diamond(\neg p \wedge \neg q) \wedge \dot{\Box} p \wedge \dot{\Box} q \to \dot{\Box}(p \wedge q)$	$\mathsf{Kv}^r\mathsf{V}$
2		$\dot{\Box}\bot \to \dot{\Box}(p \wedge q)$	
	2.1	$\bot \to p \wedge q$	taut
	2.2	$\Box(\bot \to p \wedge q)$	neck
	2.3	$\Box(\bot \to p \wedge q) \to (\dot{\Box}\bot \to \dot{\Box}(p \wedge q))$	Kv^r
	2.4	$\dot{\Box}\bot \to \dot{\Box}(p \wedge q)$	mp2,3
3		$\dot{\Box}(\neg p \wedge \neg q) \wedge \neg\dot{\Box}\bot \to \Diamond(\neg p \wedge \neg q)$	
	3.1	$\Box(\top \to (p \vee q))) \leftrightarrow \Box(\neg(p \vee q) \to \bot)$	RE
	3.2	$\Box(\neg(p \vee q) \to \bot) \to (\dot{\Box}\neg(p \vee q) \to \dot{\Box}\bot)$	Kv^r
	3.3	$\Box(\neg(p \vee q) \to \bot) \to (\dot{\Diamond}\top \to \dot{\Diamond}(p \vee q))$	taut2
	3.4	$\Box(\top \to (p \vee q)) \to (\dot{\Diamond}\top \to \dot{\Diamond}(p \vee q))$	taut1,3
	3.5	$\Box(p \vee q) \to \Box(\top \to (p \vee q))$	K
	3.6	$\Box(p \vee q) \wedge \neg\dot{\Box}\bot \to \dot{\Diamond}(p \vee q)$	mp4,5
	3.7	$\Box(p \vee q) \to \dot{\Diamond}(p \vee q) \vee \dot{\Box}\bot$	taut6
	3.8	$\dot{\Box}(\neg p \wedge \neg q) \wedge \neg\dot{\Box}\bot \to \Diamond(\neg p \wedge \neg q)$	taut7
4		$\dot{\Box}(\neg p \wedge \neg q) \wedge \neg\dot{\Box}\bot \wedge \dot{\Box} p \wedge \dot{\Box} q \to \dot{\Box}(p \wedge q)$	taut1,3
5		$\dot{\Box} p \wedge \dot{\Box} q \wedge \dot{\Box}(\neg p \wedge \neg q) \to \dot{\Box}(p \wedge q)$	taut2,4

Remarks

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- However, we cannot replace K_2 by K_2^* in WAL, which can be showed by the following independence proof.
- First we need to introduce neighborhood semantics in [6] for modal logic.
- A pair $F = \langle W, N \rangle$ is a called a neighborhood frame, or a neighborhood system, if W is a non-empty set and N is a neighborhood function from W to $\mathcal{P}(\mathcal{P}(W))$. $M = \langle F, V \rangle$ is a model if $V : prop \to 2^W$ is a valuation function.
  The truth for modality is different from which in Kripke frame, and we introduce it here:
  $M, w \models \Box\phi$ iff $(\phi)^M \in N(w)$
  $M, w \models \Diamond\phi$ iff $W - (\phi)^M \notin N(w)$
  where $(\phi)^M = \{w \mid M, w \models \phi\}$, i.e. the truth set of ϕ.

Remarks

Lemma (universality)

M preserves truth under Nec iff $\forall w \in W(W \in N(w))$.

Lemma (monotonicity)

M preserves truth under RM iff $\forall w \in W \forall S \subseteq S' \subseteq W(S \in N(w) \rightarrow S' \in N(w))$.

Proposition

$K_2^* \not\models K_2$

Proof

Define $F = \langle W, N \rangle$ be the frame s.t. $W = \{a_n \mid 0 \leq n < 5\}$, $N(a_0) = \{S \mid S \supseteq \{a_1\} \vee S \supseteq \{a_2\} \vee S \supseteq \{a_3, a_4\}\}$ and for $n > 0$, $N(a_n) = \{W\}$. Let V be the assignment s.t. $V(p_0) = \{a_1, a_4\}$, $V(p_1) = \{a_2\}$ and $V(p_2) = \{a_3, a_4\}$. Of course $M = \langle F, V \rangle$ has the above two properties, so it preserves *Nec* and *RM*. It is easy to check that $\Box(\neg p_0 \wedge \neg p_1)$ is wrong at a_0 since $\{a_0, a_3\} \notin N(a_0)$, so $a_0 \models K_2^*$. By the definition, we know $a_0 \models \Box p_0 \wedge \Box p_1 \wedge \Box p_2$. So we only need to show that $a_0 \models \neg\Box((p_0 \wedge p_1) \vee (p_1 \wedge p_2) \vee (p_0 \wedge p_2))$, but this is obvious: only a_4 satisfies two of p_i but $\{a_4\} \notin N(a_0)$.

Remarks

- However, we can prove K_2 in the normal polyadic modal logic, and hence we know that WAL(K_2) is a subsystem of KvL, since KvL is normal.

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Remarks

- However, we can prove K_2 in the normal polyadic modal logic, and hence we know that WAL(K_2) is a subsystem of KvL, since KvL is normal.

- Proposition

  $\vdash_K K_2$, *where K means the normal polyatic modal logic, and* K_2 *stands for its translation in this language.*

- proof

  It is easy to show that in the normal polyatic modal logic K, the rule RM(from $\vdash \phi \to \psi$ we can get $\vdash \nabla(\phi, \phi) \to \nabla(\psi, \psi)$) is derivable, and actually we can prove a alternative version: $\vdash \phi \to \psi \& \vdash \phi' \to \psi' \Longrightarrow \vdash \nabla(\phi, \phi') \to \nabla(\psi, \psi')$, so we will just use them directly. Recall that the formula K_2 is just $\nabla(\phi_0, \phi_0) \wedge \nabla(\phi_1, \phi_1) \wedge \nabla(\phi_2, \phi_2) \to \nabla((\phi_0 \wedge \phi_1) \vee (\phi_0 \wedge \phi_2) \vee (\phi_1 \wedge \phi_2), (\phi_0 \wedge \phi_1) \vee (\phi_0 \wedge \phi_2) \vee (\phi_1 \wedge \phi_2))$ in polyatic modal logic. For convenience, we still use $\Box\phi$ standing for $\nabla(\phi, \phi)$.

1	$\phi_0 \to \phi_0 \vee \phi_1, \phi_0 \to \phi_0 \vee \phi_2, \ldots, \phi_2 \to \phi_1 \vee \phi_2$	taut
2	$\phi_0 \vee \phi_1 \to ((\phi_0 \vee \phi_2 \to ((\phi_1 \vee \phi_2) \to (\phi_0 \vee \phi_1) \wedge (\phi_0 \vee \phi_2) \wedge (\phi_1 \vee \phi_2)))$	taut
3	$(\phi_0 \vee \phi_1) \wedge (\phi_0 \vee \phi_2) \wedge (\phi_1 \vee \phi_2) \to (\phi_0 \wedge \phi_1) \vee (\phi_0 \wedge \phi_2) \vee (\phi_1 \wedge \phi_2)$	taut
4	$\Box((\phi_0 \vee \phi_1) \wedge (\phi_0 \vee \phi_2) \wedge (\phi_1 \vee \phi_2)) \to \Box((\phi_0 \wedge \phi_1) \vee (\phi_0 \wedge \phi_2) \vee (\phi_1 \wedge \phi_2))$	$RM3$
5	$\nabla(\phi_0 \vee \phi_1, \phi_0 \vee \phi_1 \to ((\phi_0 \vee \phi_2 \to ((\phi_1 \vee \phi_2) \to (\phi_0 \vee \phi_1) \wedge (\phi_0 \vee \phi_2) \wedge (\phi_1 \vee \phi_2))))$	$Nec2$
6	$\nabla(p, q \to r) \to (\nabla(p, q) \to \nabla(p, r))$	K
7	$\nabla(\phi_0, \phi_0) \to \nabla(\phi_0 \vee \phi_1, \phi_0 \vee \phi_1)$	$RM1$
8	$\nabla(\phi_0 \vee \phi_1, \phi_0 \vee \phi_1)$	$mp7, hyp$
9	$\nabla(\phi_0 \vee \phi_1, \phi_0 \vee \phi_2 \to ((\phi_1 \vee \phi_2) \to (\phi_0 \vee \phi_1) \wedge (\phi_0 \vee \phi_2) \wedge (\phi_1 \vee \phi_2)))$	$mp6, 8, 5$
10	$\nabla(\phi_0 \vee \phi_1, \phi_0 \vee \phi_2)$	$RM1, hyp$
11	$\nabla(\phi_0 \vee \phi_1, (\phi_1 \vee \phi_2 \to (\phi_0 \vee \phi_1) \wedge (\phi_0 \vee \phi_2) \wedge (\phi_1 \vee \phi_2))$	$mp6, 9, 10$
12	$\nabla(\phi_0 \vee \phi_1, \phi_1 \vee \phi_2)$	$RM1, hyp$
13	$\nabla(\phi_0 \vee \phi_1, (\phi_0 \vee \phi_1) \wedge (\phi_0 \vee \phi_2) \wedge (\phi_1 \vee \phi_2))$	$mp6, 11, 12$
14	$\nabla(\phi_0 \vee \phi_2, (\phi_0 \vee \phi_1) \wedge (\phi_0 \vee \phi_2) \wedge (\phi_1 \vee \phi_2))$	similar with the proof of 13
15	$\nabla(\phi_1 \vee \phi_2, (\phi_0 \vee \phi_1) \wedge (\phi_0 \vee \phi_2) \wedge (\phi_1 \vee \phi_2))$	similar with the proof of 13
	Let $\psi = (\phi_0 \vee \phi_1) \wedge (\phi_0 \vee \phi_2) \wedge (\phi_1 \vee \phi_2)$	
16	$\nabla(\phi_0 \vee \phi_1 \to ((\phi_0 \vee \phi_2 \to ((\phi_1 \vee \phi_2) \to (\phi_0 \vee \phi_1) \wedge (\phi_0 \vee \phi_2) \wedge (\phi_1 \vee \phi_2))), \psi)$	$Nec2$
17	$\nabla((\phi_0 \vee \phi_2 \to ((\phi_1 \vee \phi_2) \to (\phi_0 \vee \phi_1) \wedge (\phi_0 \vee \phi_2) \wedge (\phi_1 \vee \phi_2)), \psi)$	$mp6, 13, 16$
18	$\nabla((\phi_1 \vee \phi_2 \to (\phi_0 \vee \phi_1) \wedge (\phi_0 \vee \phi_2) \wedge (\phi_1 \vee \phi_2)), \psi)$	$mp6, 14, 17$
19	$\nabla((\phi_0 \vee \phi_1) \wedge (\phi_0 \vee \phi_2) \wedge (\phi_1 \vee \phi_2), \psi)$, i.e. $\Box((\phi_0 \vee \phi_1) \wedge (\phi_0 \vee \phi_2) \wedge (\phi_1 \vee \phi_2))$	$mp6, 15, 18$
20	$\Box((\phi_0 \wedge \phi_1) \vee (\phi_0 \wedge \phi_2) \vee (\phi_1 \wedge \phi_2))$	$mp4, 19$

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Thank you