Proving Arrow's Theorem in Modal Logic

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Decision problems

Decision-making is the action/ progress/ result of making a choice with goals on any given occasion.

Decision theory is concerned with the reasoning underlying an agent's choices.
Individual decision-making (against nature)
E.g., Gambling
Individual decision making in interaction
E.g., Playing chess
Collective decision making (pursue a goal in a coherent way)
E.g., Voting in an election

Decision problems

Social choice theory : the formal analysis of collective decision-making.

Voting theory : the study of how to get a ranking of many candidates taking "fairness" into account.
Note that the fairness in the voting procedure requires that the "will and preference of the group" should be reflected adequately.

Voting methods and paradoxes

- Let A, B, C be three distinct candidates and the numbers of voters is 15.
- Plurality rule :

voters	6	5	4
best	A	C	B
	B	B	C
worst	C	A	A

A>C>B

Voting methods and paradoxes

- Condorcet rule :

voters	6	5	4
best	A	C	B
	B	B	C
worst	C	A	A

A	6
B	5+4=9
B	6+4=10
C	5
A	6
C	5+4=9

B>A>C

Voting methods and paradoxes

- Condorcet rule :

voters	6	5	4
best	A	C	B
	B	A	C
worst	C	B	A

A	5+6=11
B	4
B	6+4=10
C	5
A	6
C	5+4=9

A>B>C>A

Voting methods and paradoxes

- Borda count :

voters	6	5	4
best	A	C	B
	B	B	C
worst	C	A	A

(2,1,0)

A	6*2+5*0+4*0=12
B	6*1+5*1+4*2=19
C	6*0+5*2+4*1=14

B>C>A

(20,10,9)

A	6*20+5*9+4*9=201
B	6*10+5*10+4*20=190
C	6*9+5*20+4*10=194

A>C>B

Voting methods and paradoxes

- Saari, 1995

voters	6	5	4
best	A	C	B
	B	A	C
worst	C	B	A

A	11	A	6
B	4		
		C	9

C>A>B

B	10	B	4
C	5		
		A	11

A>B>C

A	6		
C	9	C	5
		B	10

B>C>A

Voting methods and paradoxes

- Many different electoral methods : Plurality, Borda Count, Antiplurality/Veto, and k-approval ; Plurality with Runoff ; Single Transferable Vote (STV)/Hare ; Approval Voting ; Cup Rule/Voting Trees ; Copeland ; Banks ; Slater Rule ; Schwartz Rule ; the Condorcet rule ; Maximin/Simpson, Kemeny ; Ranked Pairs/Tideman ; Bucklin Method ; Dodgson Method ; Young's Method ; Majority Judgment ; Cumulative Voting ; Range/Score Voting ; . . .

Arrow's Impossibility Theorem

- Arrow's Impossibility Theorem
  Suppose that there are at least three candidates and finitely many voters. Any social welfare function that satisfies independence of irrelevant alternatives and Pareto condition is a dictatorship.
  K. Arrow. *Social Choice and Individual Values*. John Wiley & Sons, 1951.

Arrow's Impossibility Theorem

Theorem (Arrow,1951) Suppose that there are at least three candidates and finitely many voters. Any social choice function that satisfies independence of irrelevant alternatives and Pareto condition is a dictatorship.

- Endriss, U. (2011). *Logic and social choice theory*. In A. Gupta & J. van Benthem (Eds.), Logic and Philosophy Today .
- Taylor, A. D. (2005). *Social Choice and the Mathematics of Manipulation*. Cambridge : Cambridge University Press.

The idea of this work

Cinà, G., & Endriss, U. (2016).
Proving Classical Theorems of Social Choice Theory in Modal Logic.
Autonomous Agents and Multi-Agent Systems

- As the scope of social choice theory is being widened, sociologists are designing ever more specialized social choice mechanisms for novel types of tasks, better tools to analyze the formal properties of these mechanisms are needed.
- There is now a growing literature on the formal verification of social choice mechanisms by means of logical modeling and the use of techniques from automated reasoning.
- An obvious yardstick against which to measure different approaches to the formalization of social choice frameworks is Arrow's Theorem.

Social choice functions

- Let $N = \{1, ..., n\}$ be a finite set of agents (or individuals) and let X be a finite set of alternatives (or candidates).
- To vote, each agent $i \in N$ expresses her preferences by supplying a linear order $\geqslant_i$ over X, i. e., a binary relation that is reflective, antisymmetric, complete, and transitive.
- Let $L(X)$ denote the set of all such linear orders. We shall also refer to $\geqslant_i$ as the ballot provided by agent i.
- A *profile* is an n-tuple $(\geqslant_1, ... \geqslant_n) \in L(X)^n$ of such ballots, one for each agent.

Social choice functions

Definition 1

- A resolute social choice function is a function $F: L(X)^n \rightarrow X$ mapping any given profile of ballots to a single winning alternative.

CL-PC

- The logic is basically a modal logic, which derives inspiration from the Coalition Logic of Propositional Control (CL-PC).
- Troquard, N., van der Hoek, W., & Wooldridge, M. (2011). *Reasoning about social choice functions*. Journal of Philosophical Logic.

CL-PC

- CL-PC is a variant of cooperation logic which is intended to represent and reason about strategic powers of agents and coalitions of agents in game-like multi-agent systems.
- CL-PC can allow us to express the fact that a group of agents can cooperate to bring about a certain state of affairs with the allocation of proposition to each agent.

Language

- This language is built on top of two types of atomic propositions :
  1. for every $i \in N$ and $x, y \in X$, $P^i_{x \geqslant y}$ is an atomic proposition, $Pref[N, X] := \{P^i_{x \geqslant y} | i \in N \text{ and } x, y \in X\}$ is the set of all such propositions.
  2. every alternative $x \in X$ is also an atomic proposition.
- We have a modal operator $\Diamond_C$ for every coalition of agents $C \subseteq N$ (with the intuitive meaning that C can ensure the truth of a given formula, provided the others do not alter their ballots).

Language

Definition 2

- The set of well-formed formulas ϕ in the language of $L[N, X]$ is generated by the following Backus-Naur Form (where $p \in Pref[N, X], x \in X$ and $C \subseteq N$) :

$$\phi ::= p \mid x \mid \top \mid \neg\phi \mid \phi \vee \phi \mid \Diamond_C \phi$$

- For $i \in N$, we write $\Diamond_i$ as a shorthand for $\Diamond_{\{i\}}$, and $\Box_i$ as a shorthand for $\Box_{\{i\}}$.

Semantics

Definition 3

- A model is a tripe $M =< N, X, F >$, consisting of a finite set of agents N with $|N| = n$, a finite set of alternatives X, and a SCF $F : L(X)^n \rightarrow X$

- For fixed sets N and X, we sometimes write M_F for the model $M =< N, X, F >$ based on the SCF F.
  A *state* (or a preference profile) is an element of $L(X)^n$. The terms '*state*' and '*profile*' can be used interchangeably, that is *state* $w = (\geqslant_1, ... \geqslant_n)$.

Semantics

Definition 4

- Let $M =< N, X, F >$ be a model, the satisfaction relation $\vDash$ is defined inductively :
  - $M, w \vDash p^i_{x \geqslant y}$ iff $x \geqslant_i y$
  - $M, w \vDash x$ iff $F(\geqslant_1, \dots \geqslant_n) = x$
  - $M, w \vDash \neg\phi$ iff $M, w \nvDash \phi$
  - $M, w \vDash \phi \vee \psi$ iff $M, w \vDash \phi$ or $M, w \vDash \psi$
  - $M, w \vDash \Diamond_C \phi$ iff $M, w' \vDash \phi$ for some world $w' = (\geqslant'_1, \dots \geqslant'_n) \in L(X)^n$ with $\geqslant_i = \geqslant'_i$ for all agents $i \in N \backslash C$

- Let ϕ be a formula in the language based on N and X.Then ϕ is called *satisfiable*, if there exists a SCF F and a state $w \in L(X)^n$ such that $M_F, w \vDash \phi$. It is called *true* in the model M, denoted $M \vDash \phi$, if $M \vDash \phi$ for every world $w \in L(X)^n$. Finally, it is called *valid*, denoted $\vDash \phi$, if $M \vDash \phi$ for every model M based on N and X.

Semantics

Proposition

- Determining whether a formula in the language of $L[N, X]$ is valid is a decidable problem.

- **Proof** Since N and X are fixed, we have $|L(X)^n| = |X|!^{|N|}$, then the number of SCF based on N and X is $|X|^{|L(X)^n|}$. We can enumerate all models and check for each of them whether our formula is true at every state in the model.

Axiomatisation and completeness

- Constraints of control

(*refl*)	$p^i_{x \geqslant x}$	
(*antisym* − *total*)	$p^i_{x \geqslant y} \leftrightarrow \neg p^i_{y \geqslant x}$	, where $x \neq y$
(*trans*)	$p^i_{x \geqslant y} \wedge p^i_{y \geqslant z} \rightarrow p^i_{x \geqslant z}$	

Axiomatisation and completeness

- Consider a profile $w = (\geqslant_1, \ldots \geqslant_n) \in L(X)^n$. For a given agent $i \in N$, let $x_1, x_2, \ldots, x_m$ be a permutation of the elements of X such that $x_1 \geqslant_i x_2 \geqslant_i \ldots \geqslant_i x_m$.

$$ballot_i(w) := p^i_{x_1 \geqslant x_2} \wedge p^i_{x_2 \geqslant x_3} \wedge \ldots \wedge p^i_{x_{m-1} \geqslant x_m}$$

$$profile(w) := ballot_1(w) \wedge ballot_2(w) \wedge \ldots \wedge ballot_n(w)$$

- Let $N^w_{x \geqslant y} := \{i \in N | x \geqslant_i y\}$ denote the set of agents that prefer x over y in profile $w = (\geqslant_1, \ldots \geqslant_n)$. By a slight abuse of notation, we use the same expression as a construct of our language :

$$N^w_{x \geqslant y} := \bigwedge \left\{ p^i_{x \geqslant y} | x \geqslant_i y \text{ in } w \right\}$$

$$profile(w)(x, y) := N^w_{x \geqslant y} \wedge N^w_{y \geqslant x}$$

Axiomatisation and completeness

- Propositional control

(*Prop*)	ϕ	,where ϕ is a propositional tautology
$(K(i))$	$\Box_i(\phi \to \psi) \to (\Box_i\phi \to \Box_i\psi)$	
$(T(i))$	$\Box_i\phi \to \phi$	
$(B(i))$	$\phi \to \Box_i\Diamond_i\phi$	
(*union*)	$\Box_{C_1}\Box_{C_2}\phi \leftrightarrow \Box_{C_1\cup C_2}\phi$	
(*empty*)	$\Box_\emptyset\phi \leftrightarrow \phi$	
(*exclusiveness*)	$(\Diamond_i p \wedge \Diamond_i\neg p) \to (\Box_j p \vee \Box_j\neg p)$	,where $j \neq i, p \in Pref[N, X]$
(*ballot*)	$\Diamond_i ballot_i(w)$	
(*cooperation*)	$\Diamond_{C_1}\delta_1 \wedge \Diamond_{C_2}\delta_2 \to \Diamond_{C_1\cup C_2}(\delta_1 \wedge \delta_2)$	

,where δ_1, δ_2 do not contain any common atoms, and the atoms like $p^i_{x\geqslant y}$, $p^i_{y\geqslant x}$ should not show up respectively in $\delta_1,\ \delta_2$

Axiomatisation and completeness

- Outcomes

(*resoluteness*)	$\bigvee_{x\in X}(x \wedge \bigwedge_{y\in X\setminus\{x\}} \neg y)$
(*functionality*)	$(profile(w) \wedge \phi) \rightarrow \Box_N(profile(w) \rightarrow \phi)$

- Rules

(*MP*)	from$\phi \rightarrow \psi$ and ϕ infer $\vdash \psi$
(*Nec$_i$*)	from$\vdash \phi$ infer $\vdash \Box_i\phi$

Axiomatisation and completeness

Theorem

- (Soundness) The logic $L[N, X]$ is sound and complete w.r.t. the class of models of SCF's.

- **Proof** First, we show for every SCF F based on N and X, $M_F \vDash \Box_{C_1}\Box_{C_2}\phi \leftrightarrow \Box_{C_1\cup C_2}\phi$. Suppose there is a $w \in L(X)^n$ such that $M_F, w \vDash \Box_{C_1}\Box_{C_2}\phi$ and $M_F, w \nvDash \Box_{C_1\cup C_2}\phi$ or $M_F, w \vDash \Box_{C_1\cup C_2}\phi$ and $M_F, w \nvDash \Box_{C_1}\Box_{C_2}\phi$, consider the former condition, there exists a $w* \in L(X)^n$,for every $i \in N\backslash(C_1 \cup C_2)$, $\geqslant_i^* = \geqslant_i$ and $M_F, w^* \vDash \neg\phi$. And for every $w', w'', \ i \in N\backslash C_1, \ j \in N\backslash C_2, \ \geqslant_i' = \geqslant_i, \ \geqslant_j'' = \geqslant_j'$, we have $M_F, w'' \vDash \phi$, then $M_F, w^* \vDash \phi$, contradiction. The proof of the latter condition is similar. Second, we can prove $M_F \vDash (\Diamond_i p \wedge \Diamond_i \neg p) \to (\Box_j p \vee \Box_j \neg p)$ $(i \neq j)$. Suppose there is a w such that $M_F, w \vDash \Diamond_i p \wedge \Diamond_i \neg p$, and $M_F, w \nvDash \Box_j p \vee \Box_j \neg p$, then p is an atom with the form of $p^i_{x\geqslant y}$, no matter how j changes his ballot, the value of p will not be affected. The rest of the proof is trivial.

Unrestricted Domain Lemma

Lemma 1

- For every possible profile $w \in L(X)^n$, we have that $\vdash \Diamond_N profile(w)$.

- **Proof** Take any profile w,
  $\vdash \Diamond_1 ballot_1(w)$ $(ballot)$
  $\vdash \Diamond_2 ballot_2(w)$ $(ballot)$
  $\vdash \Diamond_{\{1,2\}}(ballot_1(w) \wedge ballot_2(w))$ $(cooperation)$
  ...
  $\vdash \Diamond_N profile(w)$

Pareto efficiency

Definition 5

- A SCF F is Pareto efficient if, for every profile $w \in L(X)^n$ and every pair of distinct alternatives $x, y \in X$ with $N^w_{x \geqslant y} = N$, we obtain $F(w) \neq y$.

- This is formalized as follows :

$$Par := \bigwedge_{x \in X} \bigwedge_{y \in X \setminus \{x\}} \left[\left(\bigwedge_{i \in N} p^i_{x \geqslant y} \right) \to \neg y \right]$$

Lemma 2

- For every SCF F, $M_F \vDash Par$ if and only if F is Pareto efficient.

- **Proof** Straightforward.

Independence of irrelevant alternatives

Definition 6

- A SCF F satisfies IIA if, for every pair of profiles $w, w' \in L(X)^n$ and every pair of distinct alternatives $x, y \in X$ with $N^{w}_{x \geqslant y} = N^{w'}_{x \geqslant y}$, it is the case that $F(w) = x$ implies $F(w') \neq y$.

- This is formalized as follows :

$$IIA := \bigwedge_{w \in L(X)^n} \bigwedge_{x \in X} \bigwedge_{y \in X \setminus \{x\}} [\Diamond_N(\mathit{profile}(w) \wedge x) \rightarrow (\mathit{profile}(w)(x, y) \rightarrow \neg y)]$$

Independence of irrelevant alternatives

Lemma 3

- For every SCF F, $M_F \vDash IIA$ if and only if F satisfies the property of independence of irrelevant alternatives.

- **Proof** $\Leftarrow$ Assume F satisfies IIA, we want to prove every conjunct of the formula IIA. So take any world such that $M_F, w' \vDash \Diamond_N(profile(w) \wedge x)$, we want to show that $M_F, w' \vDash profile(w)(x, y) \rightarrow \neg y$.
  So suppose $M_F, w' \vDash profile(w)(x, y)$, which entails $N^{w}_{x \geqslant y} = N^{w'}_{x \geqslant y}$. By the semantic of $\Diamond_N$, there is a world w'' such that $M_F, w'' \vDash profile(w)$, which entails $N^{w}_{x \geqslant y} = N^{w''}_{x \geqslant y}$. Thus also $N^{w'}_{x \geqslant y} = N^{w''}_{x \geqslant y}$. From $M_F, w'' \vDash x$, we can infer $F(w'') = x$. Now we can apply IIA to w'' and w' and obtain $F(w') \neq y$. Again by the semantics, we have $M_F, w' \vDash \neg y$.
  $\Rightarrow$ Assume $M_F \vDash IIA$. Take any two profiles w, w' and two alternatives x, y with $N^{w}_{x \geqslant y} = N^{w'}_{x \geqslant y}$. Now assume $F(w) = x$, then we have $M_F, w \vDash profile(w) \wedge x$, by the semantics of $\Diamond_N$, also $M_F, w' \vDash \Diamond_N(profile(w) \wedge x)$. Then we get $M_F, w' \vDash \neg y$, which by the semantics entails $F(w') \neq y$.

Dictatorships

Definition 7

- A SCF F is a dictatorship if there exists an agent $i \in N$ (the dictator) such that for every profile $w \in L(X)^n$, we obtain $F(w) = top_i^w$.
  (Denote with top_i^w that alternative $x \in X$ for which $x \geqslant_i y$ for all other alternatives $y \in X$ in profile $w = (\geqslant_1, ..., \geqslant_n)$.)

- This property is encoded by the following formula :

$$Dic := \bigvee_{i \in N} \bigwedge_{x \in X} \bigwedge_{y \in X \setminus \{x\}} (p^i_{x \geqslant y} \to \neg y)$$

Dictatorships

Lemma 4

- For every SCF F, $M_F \vDash Dic$ if and only if F is a dictatorship.

- **Proof** $\Rightarrow$ Suppose $M_F \vDash Dic$, then one of the disjuncts must be valid, say for agent i. Suppose $x = top_i^w$ under profile w, then $M_F, w \vDash \bigwedge_{y \in X \setminus \{x\}} p^i_{x \geqslant y}$, $M_F, w \vDash \bigwedge_{y \in X \setminus \{x\}} \neg y$. By resoluteness, this entails $M_F, w \vDash x$ and thus $F(w) = x$.
  $\Leftarrow$ Suppose F is a dictatorship, and call the agent i. Take any world $w = (\geqslant_1, ..., \geqslant_n)$. We want to show that the disjunct corresponding to i is true at w. Thus for any two distinct alternative x, y, if $x \geqslant_i y$, then $top_i^w \neq y$, due to F being a dictatorship of i, we have $F(w) \neq y,\ M_F, w \vDash \neg y$. If $x \ngeqslant_i y,\ M_F, w \nvDash p^i_{x \geqslant y}$, and the implication holds vacuously.

Arrow's Impossibility Theorem

Theorem (Arrow,1951) Suppose that there are at least three candidates and finitely many voters. Any social choice function that satisfies independence of irrelevant alternatives and Pareto condition is a dictatorship.

Decisive coalitions

- We call a coalition of agents $C \subseteq N$ *decisive* over a pair of alternatives $(x, y) \in X^2$ if the member of C preferring x to y is a sufficient condition for preventing y from winning.

$$C_{dec}(x, y) := \left(\bigwedge_{i \in C} p^i_{x \geqslant y} \right) \to \neg y$$

If C is decisive on every pair, we will simply write C_{dec}. Note that :

$$\vdash \bigwedge_{x \in X} \bigwedge_{y \in X \setminus \{x\}} \left[\left(\bigwedge_{i \in N} p^i_{x \geqslant y} \right) \to \neg y \right] \leftrightarrow N_{dec}$$

$$\vdash \bigvee_{i \in N} \bigwedge_{x \in X} \bigwedge_{y \in X \setminus \{x\}} (p^i_{x \geqslant y} \to \neg y) \leftrightarrow \bigvee_{i \in N} \{i\}_{dec}$$

- We define a *weakly decisive* coalition C for (x, y) as a coalition that can bar y from winning if *exactly* the agents in C prefer x to y.

$$C_{wdec}(x, y) := \left(\bigwedge_{i \in C} p^i_{x \geqslant y} \wedge \bigwedge_{i \notin C} p^i_{y \geqslant x} \right) \to \neg y$$

Field expansion lemma

Lemma 4

- Consider a language parametrized by X such that $|X| \geqslant 3$. Then for any coalition $C \subseteq N$ and any two distinct alternatives $x, y \in X$, we have that

$$\vdash Par \wedge IIA \wedge C_{wdec}(x, y) \rightarrow C_{dec}$$

- **Proof** To prove C_{dec}, we need to prove each of the conjuncts in the following formula :

$$\bigwedge_{x \in X} \bigwedge_{y \in X \setminus \{x\}} \left[\left(\bigwedge_{i \in C} p^i_{x \geqslant y} \right) \rightarrow \neg y \right]$$

Suppose x, y, x' and y' are distinct alternatives. Denote by C' one of the possible subsets of $N \setminus C$ preferring x' over y'. Now consider the following derivation :

Field expansion lemma

- **Proof**

(1) $\bigwedge_{i\in C} P^i_{x'\geqslant y'} \to \bigwedge_{i\in C} P^i_{x'\geqslant y'} \wedge \bigvee_{C'\subseteq N\setminus C}(\bigwedge_{i\in C'} P^i_{x'\geqslant y'} \wedge \bigwedge_{i\notin C\cup C'} p^i_{y'\geqslant x'})$

(2) $\bigwedge_{i\in C} P^i_{x'\geqslant y'} \to \bigvee_{C'\subseteq N\setminus C}(\bigwedge_{i\in C} P^i_{x'\geqslant y'} \wedge \bigwedge_{i\in C'} P^i_{x'\geqslant y'} \wedge \bigwedge_{i\notin C\cup C'} p^i_{y'\geqslant x'})$

(3) for every coalition $C' \subseteq N\backslash C$:

$Par \wedge IIA \wedge C_{wdec}(x,y) \to \bigwedge_{i\in C} P^i_{x'\geqslant y'} \wedge \bigwedge_{i\in C'} P^i_{x'\geqslant y'} \wedge \bigwedge_{i\notin C\cup C'} p^i_{y'\geqslant x'} \to \neg y'$

(4) $Par \wedge IIA \wedge C_{wdec}(x,y) \to$

$\bigvee_{C'\subseteq N\setminus C}(\bigwedge_{i\in C} P^i_{x'\geqslant y'} \wedge \bigwedge_{i\in C'} P^i_{x'\geqslant y'} \wedge \bigwedge_{i\notin C\cup C'} p^i_{y'\geqslant x'}) \to \neg y'$

(5) $Par \wedge IIA \wedge C_{wdec}(x,y) \to \bigwedge_{i\in C} P^i_{x'\geqslant y'} \to \neg y'$

Field expansion lemma

- **Proof** We need to show (all the finitely many instances of) step (3), prove each of them in the following way.
  Consider a specific profile $w = (\geqslant_1, ..., \geqslant_n)$ for which we can rearrange the conjuncts in the formula *profile*(w) as follows :

$$profile(w) = \bigwedge_{i \in C} p^i_{x \geqslant y} \wedge \bigwedge_{i \notin C} p^i_{y \geqslant x} \wedge \bigwedge_{i \in C \cup C'} p^i_{x' \geqslant y'} \wedge \bigwedge_{i \notin C \cup C'} p^i_{y' \geqslant x'} \wedge \bigwedge_{i \in N} (p^i_{x' \geqslant x} \wedge p^i_{y \geqslant y'}) \wedge \alpha$$

y	x'		yx'	y
y'	x		xy'	y'
x'	y			x'
x	y'			x

α is the formula expressing the fact that all the other alternatives (if any) are ranked by all agents below x, y, x', y'.

Field expansion lemma

- **Proof**

y	x'		yx'	y
y'	x		xy'	y'
x'	y			x'
x	y'			x

(a) for any $z \in X \backslash [x, y, x', y']$:
$Par \wedge profile(w) \rightarrow \neg x \wedge \neg y' \wedge \neg z$
(b)$C_{wdec}(x, y) \wedge profile(w) \rightarrow \neg y$
(c)$Par \wedge C_{wdec}(x, y) \rightarrow profile(w) \rightarrow x'$
(d)$\Diamond_N profile(w)$
(e)$Par \wedge C_{wdec}(x, y) \rightarrow \Diamond_N(profile(w) \wedge x')$
(f)$Par \wedge IIA \wedge C_{wdec}(x, y) \rightarrow profile(w)(x', y') \rightarrow \neg y'$

Field expansion lemma

- **Proof** With three alternatives x, y, z, the proof of the case $\bigwedge_{i \in C} p^i_{y \geqslant z} \to \neg z$ is nontrivial. First we need to show $\bigwedge_{i \in C} p^i_{x \geqslant z} \to \neg z$. The key step is (3), still consider a specific profile w which has the following form :

y	x		y	y
z	y		x	z
x	z		z	x

(a)$Par \wedge C_{wdec}(x, y) \to profile(w) \to x$
(d)$\Diamond_N profile(w)$
(e)$Par \wedge C_{wdec}(x, y) \to \Diamond_N(profile(w) \wedge x)$
(f)$Par \wedge IIA \wedge C_{wdec}(x, y) \to profile(w)(x, z) \to \neg z$
Follow the same way, we have $Par \wedge IIA \wedge C_{wdec}(x, y) \to \bigwedge_{i \in C} p^i_{x \geqslant z} \to \neg z$ $(\star)$

Field expansion lemma

- **Proof** Consider another specific profile w' which has the following form :

y z	y	yz
x	x	x
	z	

According to $(\star)$, we have :

(1) $Par \wedge IIA \wedge C_{wdec}(x, y) \rightarrow profile(w') \rightarrow y$

(2) $\Diamond_N profile(w')$

(3) $Par \wedge IIA \wedge C_{wdec}(x, y) \rightarrow \Diamond_N(profile(w') \wedge y)$

(4) $Par \wedge IIA \wedge C_{wdec}(x, y) \rightarrow profile(w')(y, z) \rightarrow \neg z$

(5) for every coalition $C'' \subseteq N \backslash C$ (C'' is one of the possible subsets of $N \backslash C$ preferring y over z) :

$Par \wedge IIA \wedge C_{wdec}(x, y) \rightarrow \bigwedge_{i \in C} P^i_{y \geqslant z} \wedge \bigwedge_{i \in C''} P^i_{y \geqslant z} \wedge \bigwedge_{i \notin C \cup C''} p^i_{z \geqslant y} \rightarrow \neg z$

(6) $\bigwedge_{i \in C} p^i_{y \geqslant z} \rightarrow \bigvee_{C'' \subseteq N \backslash C} (\bigwedge_{i \in C} P^i_{y \geqslant z} \wedge \bigwedge_{i \in C''} P^i_{y \geqslant z} \wedge \bigwedge_{i \notin C \cup C''} p^i_{z \geqslant y})$

(7) $Par \wedge IIA \wedge C_{wdec}(x, y) \rightarrow \bigwedge_{i \in C} P^i_{y \geqslant z} \rightarrow \neg z$

Contraction lemma

Lemma 5

- Consider a language parametrized by X such that $|X| \geqslant 3$. Then for any coalition $C \subseteq N$ and any two coalitions C_1 and C_2 that form a partition of C, we have that

$$\vdash Par \wedge IIA \wedge C_{dec} \rightarrow C_{1dec} \vee C_{2dec}$$

- **Proof** Consider C, C_1 and C_2, $C = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$. Let x, y, z be three distinct alternatives, consider any profile w for which $profile(w)$ has the following form :

$$profile(w) = \bigwedge_{i \in C_1} p^i_{x \geqslant z} \wedge \bigwedge_{i \notin C_1} p^i_{z \geqslant x} \wedge \bigwedge_{i \in C_2} p^i_{y \geqslant x} \wedge \bigwedge_{i \notin C_2} p^i_{x \geqslant y} \wedge \bigwedge_{i \in C_1 \cup C_2} p^i_{y \geqslant z} \wedge \bigwedge_{i \notin C_1 \cup C_2} p^i_{z \geqslant y} \wedge \alpha$$

α encodes the fact that all other alternatives (if any) are ranked by all agents below x, y, z.

Contraction lemma

- **Proof**

z	x	y	z
x	y	z	x
y	z	x	y

(1) $Par \wedge C_{dec} \rightarrow (profile(w) \rightarrow x) \vee (profile(w) \rightarrow y)$
(which is equivalent to $Par \wedge C_{dec} \wedge profile(w) \rightarrow x \vee y$)
(2) $Par \wedge C_{dec} \rightarrow \Diamond_N(profile(w) \wedge x) \vee \Diamond_N(profile(w) \wedge y)$
(3) $Par \wedge IIA \wedge C_{dec} \rightarrow (profile(w)(x,z) \rightarrow \neg z) \vee (profile(w)(y,x) \rightarrow \neg x)$
(4) $Par \wedge IIA \wedge C_{dec} \rightarrow C_{1wdec}(x,z) \vee C_{2wdec}(y,x)$
(5) $Par \wedge IIA \wedge C_{dec} \rightarrow C_{1dec} \vee C_{2dec}$

Arrow's Theorem

Consider a logic L[N,X] with a language parametrized by X such that $|X| \geqslant 3$. Then we have :

$$\vdash Par \wedge IIA \rightarrow Dic$$

- **Proof**
  (1) $Par \wedge IIA$
  (2) $Par \leftrightarrow N_{dec}$
  (3) $Par \wedge IIA \wedge N_{dec}$
  (4) $N_{1dec} \vee N_{2dec}$ $(N = N_1 \cup N_2, N_1 \cap N_2 = \emptyset)$
  ...
  (k) $\{i\}_{dec} \vee \{j\}_{dec}$ $(i, j \in N,\ i \neq j)$
  (k+1) $\bigvee_{i \in N} \{i\}_{dec}$
  (k+2) Dic
  Due to the correctness of the representation of the Arrovian conditions within the logic, the usual, semantic, rendering of Arrow's Theorem for $SCF's$ has been proved.

Ways out of Arrow's impossibility theorem

- Infinitely many voters (Fishburn 1970, Kirman and Sondermann (1972), Armstrong (1980, 1985) and Lauwers and Van Liedekerke (1995))
- Restricting the number of alternatives to 2 (Majority rule/Plurality rule)

...

The proofs of Arrow's Theorem

...
Amartya Sen : "Decisive coalition" (1986)
Graciela Chichilnisky, Yuliy Baryshnikov : "Topology" (1993)
Pingzhong Tang, Fangzhen Lin : "Induction" & "Automated reasoning" .(2008)
...

The formalization of Arrovian framework

- generate logical formalization of the theorem
- verify an existing proof of this theorem in a given logic (natural deduction)
- use the logic to derive proofs for Arrow's theorem and similar results

Related work

- First-order logic :
  Ariel Rubinstein. (1984) *The single profile analogues to multi profile theorems : Mathematical logic's approach*.
  International economic review.
  Grandi, U., & Endriss, U. (2013). *First – order logic formalization of impossibility theorems in preference aggregation*.
  Journal of Philosophical Logic.
  Eric Pacuit & Fan Yang (2016).
  Dependence and Independence in Social Choice : Arrow's Theorem.
  Dependence Logic
- Higher-order logic :
  Nipkow, T. (2009).
  Social choice theory in HOL : Arrow and Gibbard – Satterthwaite.
  Journal of Automated Reasoning
- Modal logic :
  Ågotnes, T., van der Hoek, W., & Wooldridge, M. (2011).
  On the logic of preference and judgment aggregation.
  Journal of Autonomous Agents and Multiagent Systems.

Completeness

Theorem

- (Completeness) The logic $L[N, X]$ is sound and complete w.r.t. the class of models of SCF's.

- **Proof** We need to show that if a formula is consistent, it is provable in the system $L[N, X]$. First, we show the existence of an isomorphism between the models of Definition 3 and particular kripke models. The latter structures are tuples $< W, (R_C)_{C \subseteq N} >$ where W is the set of profiles and $R_C \subseteq W^2$ are relations defined as

$$wR_C w' \ iff \ w \restriction N \backslash C = w' \restriction N \backslash C$$

Second, given a consistent formula ϕ, we build a maximally consistent set Γ_ϕ containing it using the usual Lindenbaum construction. Define $Cluster(\Gamma_\phi)$ to be the set of maximally consistent sets that describe the same SCF.

$$Cluster(\Gamma_\phi) := \{\Gamma | \forall w \in L(X)^n, \forall x \in X : \Diamond_N(profile(w) \wedge x) \in \Gamma iff \Diamond_N(profile(w) \wedge x) \in \Gamma_\phi\}$$

Finally, we consider the submodel of the canonical model generated by $Cluster(\Gamma_\phi)$. Let us call this submodel M_ϕ. It remains to be checked that :
-the Truth lemma holds for M_ϕ
-there is a bijection between profiles and states of M_ϕ
-M_ϕ is one of the aforementioned particular Kripke models corresponding to the models of our logic