Introduction to PDL & Some related work–EPDL

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Background of PDL(propositional dynamic logic)

- What is a programme?
  Calculate the greatest common divisor of two integers.

```
while(y ≠ 0)
{
    z = x mod y;
    x = y;
    y = z;
}
```

$$\xrightarrow{\textit{input}} \boxed{\text{programme}} \xrightarrow{\textit{output}}$$

$$\text{input} \xrightarrow{\textit{programme}} \text{output}$$

- How to verify its correctness?
  - Specification of correctness
    If the two inputs are not both zero, after the programming, the output should be their *gcd*.
  - Formal verification
    $\phi \to [\alpha]\psi$

Language of PDL

The language is defined by mutual induction:

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid [\pi]\phi$$
$$\pi ::= a \mid ?\phi \mid \pi;\pi \mid \pi + \pi \mid \pi^*$$

where $p \in \Phi_0$, $a \in \Pi_0$.

Example ($\phi \rightarrow [\alpha]\psi$)

```
while(y ≠ 0)
{
    z = x mod y;
    x = y;
    y = z;
}
```

$$\begin{aligned} \neg p &:= y \neq 0 \\ a &:= z = x \bmod y \\ b &:= x = y \\ c &:= y = z \\ \alpha &:= (?\neg p; a; b; c)^* \end{aligned}$$

Semantics of PDL

- Model

  The model $\mathfrak{M}$ of PDL is a Kripke model:

  $$\langle S, \{\xrightarrow{a} \mid a \in \Pi_0\}, V\rangle$$

  where S is the set of states, $\xrightarrow{a} \subseteq S \times S$ and $V : \Phi \to \mathscr{P}(S)$.

- Satisfiability

$$
\begin{array}{rcl}
\mathfrak{M}, s \Vdash \top & \iff & \text{always} \\
\mathfrak{M}, s \Vdash p & \iff & s \in V(p) \\
\mathfrak{M}, s \Vdash \neg\phi & \iff & \mathfrak{M}, s \nVdash \phi \\
\mathfrak{M}, s \Vdash \phi \wedge \psi & \iff & \mathfrak{M}, s \Vdash \phi \text{ and } \mathfrak{M}, s \Vdash \phi \\
\mathfrak{M}, s \Vdash [\pi]\phi & \iff & \text{for all } t,\ (s,t) \in [\![\pi]\!] \text{ implies } \mathfrak{M}, t \Vdash \phi \\
(s,t) \in [\![a]\!] & \iff & s \xrightarrow{a} t \\
(s,t) \in [\![?\phi]\!] & \iff & s = t \text{ and } \mathfrak{M}, s \Vdash \phi \\
(s,t) \in [\![\pi_1;\pi_2]\!] & \iff & (s,t) \in [\![\pi_1]\!] \circ [\![\pi_2]\!] \\
(s,t) \in [\![\pi_1 + \pi_2]\!] & \iff & (s,t) \in [\![\pi_1]\!] \cup [\![\pi_2]\!] \\
(s,t) \in [\![\pi^*]\!] & \iff & (s,t) \in [\![\pi]\!]^\star
\end{array}
$$

Model theoretical properties

Proposition (Structure invariance)

For any two PDL models $\mathfrak{M}, s$ and $\mathfrak{M}', s'$, if $\mathfrak{M}, s \leftrightarroweq \mathfrak{M}', s'$,

- *for any π, there is t such that $s \xrightarrow{\pi} t$ iff there is t' such that $s' \xrightarrow{\pi} t'$. And $\mathfrak{M}, t \leftrightarroweq \mathfrak{M}', t'$.*
- *for any ϕ, $\mathfrak{M}, s \Vdash \phi$ iff $\mathfrak{M}', s' \Vdash \phi$.*

Proposition (Finite model property)

If ϕ is satisfiable, then it is satisfiable on a finite model.

Finite model property by filtration

Proposition

- *For all* $[\alpha]\phi \in \Sigma$,
  - *if* $(s,t) \in [\![\pi]\!]$, *then* $(|s|,|t|) \in [\![\pi]\!]$.
  - *if* $(|s|,|t|) \in [\![\pi]\!]$ *and* $\mathfrak{M}, s \Vdash [\alpha]\phi$, *then* $\mathfrak{M}, t \Vdash \phi$.
- *For all* $\phi \in \Sigma$, $\mathfrak{M}, s \Vdash \phi$ *iff* $\mathfrak{M}_{/\Sigma}, |s| \Vdash \phi$.

Definition (Fischer-Ladner closure)

$$
\begin{aligned}
FL(p) &= \{p\} \\
FL(\neg\phi) &= \{\neg\phi\} \cup FL(\phi) \\
FL(\phi \wedge \psi) &= \{\phi \wedge \psi\} \cup FL(\phi) \cup FL(\psi) \\
FL([\pi]\phi) &= FL^{\Box}([\pi]\phi) \cup FL(\phi) \\
FL^{\Box}([a]\phi) &= \{[a]\phi\} \\
FL^{\Box}([?\psi]\phi) &= \{[?\psi]\phi\} \cup FL(\psi) \\
FL^{\Box}([\pi_1 + \pi_2]\phi) &= \{[\pi_1 + \pi_2]\phi\} \cup FL^{\Box}([\pi_1]\phi) \cup FL^{\Box}([\pi_2]\phi) \\
FL^{\Box}([\pi_1 ; \pi_2]\phi) &= \{[\pi_1 ; \pi_2]\phi\} \cup FL^{\Box}([\pi_1][\pi_2]\phi) \cup FL^{\Box}([\pi_2]\phi) \\
FL^{\Box}([\pi^*]\phi) &= \{[\pi^*]\phi\} \cup FL^{\Box}([\pi][\pi^*]\phi)
\end{aligned}
$$

Axiomatization

Axioms

TAUT all the axioms of propositional logic

DIST $$[\pi](\phi \to \psi) \to ([\pi]\phi \to [\pi]\psi)$$

$$[\pi_1 + \pi_2]\phi \leftrightarrow [\pi_1]\phi \wedge [\pi_2]\phi$$

$$[\pi_1 ; \pi_2]\phi \leftrightarrow [\pi_1][\pi_2]\phi$$

$$[?\psi]\phi \leftrightarrow \psi \to \phi$$

$$\phi \wedge [\pi][\pi^*]\phi \leftrightarrow [\pi^*]\phi$$

IND $$\phi \wedge [\pi^*](\phi \to [\pi]\phi) \to [\pi^*]\phi$$

Rules

MP $$\frac{\phi, \phi \to \psi}{\psi}$$

GEN $$\frac{\phi}{[\pi]\phi}$$

Soundness and Completeness

Proposition (Soundness)

If $\vdash \phi$, *then* $\Vdash \phi$.

Proposition (weaker completeness)

If $\Vdash \phi$, *then* $\vdash \phi$.

Proof.

1) $\phi \in s \iff \mathfrak{M}^c, s \Vdash \phi \iff \mathfrak{M}^c_{/\Sigma}, |s| \Vdash \phi$

2) $\mathfrak{M}^c_{/\Sigma} \Vdash \phi \wedge [\pi^*](\phi \to [\pi]\phi) \to [\pi^*]\phi$

3) Construct a PDL model $(\mathfrak{M}^c_{/\Sigma})'$ based on $\mathfrak{M}^c_{/\Sigma}$, then we have that

- for all $[\pi]\phi \in \Sigma$, $|s| \xrightarrow{\pi} |t|$ iff $|s| \xrightarrow{\pi} |t|$.
- for all $\phi \in \Sigma$, $\mathfrak{M}^c_{/\Sigma}, |s| \Vdash \phi$ iff $(\mathfrak{M}^c_{/\Sigma})', |s| \Vdash \phi$.

$\square$

Motivation: Lost with a map at hand

The secret agent sneaking in an enemy building is guided by his headquarters. Suddenly, the communication with the HQ is lost due to some emergency. Now the agent must reach a safe place as soon as possible.

$$\begin{tikzcd}
s_7 & s_6 \arrow[l, "l"'] & s_8{:}\text{Safe} & s_9{:}\text{Safe} & \\
s_1 \arrow[r, "r"] & s_2 \arrow[r, "r"] \arrow[u, "u"] & \underline{s_3} \arrow[r, "r"] \arrow[u, "u"] & s_4{:}\text{Safe} \arrow[r, "r"] \arrow[u, "u"] & s_5
\end{tikzcd}$$

Language and Semantics

- The EAL language with action and knowledge as modalities:

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid [a]\phi \mid K\phi$$

  where $p \in \mathbf{P}$, $a \in \mathbf{A}$.

- Model: an *uncertainty map (UM)*

$$\mathcal{M} = \langle S, \{\xrightarrow{a} \mid a \in \mathbf{A}\}, V, U\rangle$$

  where $U \neq \emptyset$, $U \subseteq S$ such that $\forall s, t \in U$, $e(s) = e(t)$.

  $\mathcal{M}, s$ is a *pointed* UM model, if $s \in U$.

- The satisfiable relation on pointed UM model $\mathscr{M}, s$ is defined as:

$$\begin{aligned}
\mathscr{M}, s \vDash \top &\iff \text{always} \\
\mathscr{M}, s \vDash p &\iff s \in V(p) \\
\mathscr{M}, s \vDash \neg\phi &\iff \mathscr{M}, s \nvDash \phi \\
\mathscr{M}, s \vDash \phi \wedge \psi &\iff \mathscr{M}, s \vDash \phi \text{ and } \mathscr{M}, s \vDash \phi \\
\mathscr{M}, s \vDash K\phi &\iff \forall u \in U : \mathscr{M}, u \vDash \phi \\
\mathscr{M}, s \vDash [a]\phi &\iff \forall t \in S : s \xrightarrow{a} t \text{ implies } \mathscr{M}|^a_t, t \vDash \phi
\end{aligned}$$

- $\mathscr{M}|^a_t = \langle S, \{R_a \mid a \in \mathbf{A}\}, V, U|^a_t \rangle$
- $U|^a_t = U|^a \cap E(t)$
- $U|^a = \{r' \mid \exists r \in U \text{ such that } r \xrightarrow{a} r'\}$
- $E(t) = \{t' \mid e(t') = e(t)\}$

Main results

- Axiomatization and Completeness
- Structure invariance
- Normal form
- Finite model property
- Comparisons

Language of EPDL(epistemic propositional dynamic logic)

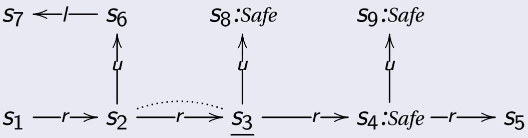
The language of EPDL is defined by mutual induction:

$$\phi ::= \top \mid p \mid \neg\phi \mid \phi \wedge \phi \mid K\phi \mid [\pi]\phi$$
$$\pi ::= a \mid ?\phi \mid \pi;\pi \mid \pi + \pi \mid \pi^*$$

where $p \in \Phi_0$, $a \in \Pi_0$.

Remark

EPDL makes its application more natural and convenient.



$\phi := \langle\alpha\rangle safe \wedge K\, safe$

Model

An *uncertainty map (UM)*

$$\mathcal{M} = \langle S, \{\xrightarrow{a} \mid a \in \Pi_0\}, V, U\rangle$$

such that

- $U \neq \emptyset$ and $U \subseteq S$
- for all $s, t \in U$, $o(s) = o(t)$

$\mathcal{M}, s$ is a *pointed* UM model, if $s \in U$.

Remark

The definition makes the uncertainty set more controllable.

$$\begin{tikzcd}
s_7 & s_6 \arrow[l, "l"'] & s_8{:}\mathit{Safe} & s_9{:}\mathit{Safe} & \\
s_1 \arrow[r, "r"] & s_2 \arrow[u, "u"] \arrow[r, "r"] & \underline{s_3} \arrow[u, "u"] \arrow[r, "r"] & s_4{:}\mathit{Safe} \arrow[u, "u"] \arrow[r, "r"] & s_5
\end{tikzcd}$$

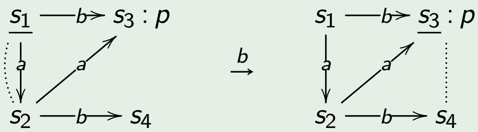
Satifiablity

The satisfiable relation on pointed $\mathtt{UM}$ model $\mathcal{M}, s$ is defined by mutual induction:

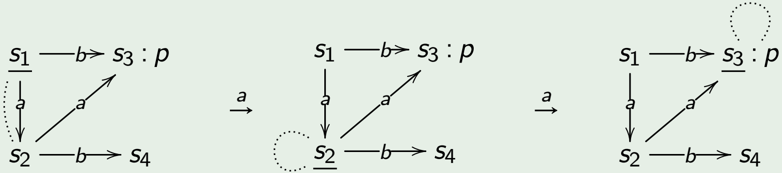
$$
\begin{aligned}
\mathcal{M}, s \models \top &\iff \text{always} \\
\mathcal{M}, s \models p &\iff s \in V(p) \\
\mathcal{M}, s \models \neg\phi &\iff \mathcal{M}, s \not\models \phi \\
\mathcal{M}, s \models \phi \wedge \psi &\iff \mathcal{M}, s \models \phi \text{ and } \mathcal{M}, s \models \phi \\
\mathcal{M}, s \models K\phi &\iff \text{for all } s',\ s' \in U \text{ implies } \mathcal{M}, s' \models \phi \\
\mathcal{M}, s \models [\pi]\phi &\iff \text{for all } \mathcal{M}', s' : (\mathcal{M}, s)[\![\pi]\!](\mathcal{M}', s') \\
&\qquad \text{implies } \mathcal{M}', s' \models \phi \\
(\mathcal{M}, s)[\![a]\!](\mathcal{M}', s') &\iff \mathcal{M}' = \mathcal{M}|^{a}_{s'} \text{ and } s \xrightarrow{a} s' \\
(\mathcal{M}, s)[\![?\psi]\!](\mathcal{M}', s') &\iff (\mathcal{M}', s') = (\mathcal{M}, s) \text{ and } \mathcal{M}, s \models \psi \\
(\mathcal{M}, s)[\![\pi_1 ; \pi_2]\!](\mathcal{M}', s') &\iff (\mathcal{M}, s)[\![\pi_1]\!] \circ [\![\pi_2]\!](\mathcal{M}', s') \\
(\mathcal{M}, s)[\![\pi_1 + \pi_2]\!](\mathcal{M}', s') &\iff (\mathcal{M}, s)[\![\pi_1]\!] \cup [\![\pi_2]\!](\mathcal{M}', s') \\
(\mathcal{M}, s)[\![\pi^*]\!](\mathcal{M}', s') &\iff (\mathcal{M}, s)[\![\pi]\!]^\star(\mathcal{M}', s')
\end{aligned}
$$

Examples

Example ($\mathscr{M}, s_1 \vDash K\neg p \wedge \langle b \rangle \neg Kp$)



Example ($\mathscr{M}, s_1 \vDash K\neg p \wedge \langle a; a \rangle Kp$)



Remark

- *Truth value of* $\mathtt{EAL}$ *formulas are not defined on all the states in a model.*
- *We say a formula* ϕ *is valid* ($\models \phi$)*, if for any pointed UM model* $\mathcal{M}, s$: $\mathcal{M}, s \models \phi$.

Structure invariance

Given an UM model $\mathscr{M} = \langle S, \{R_a\}_{a \in \Pi_0}, V, U \rangle$, let $\mathscr{M}^{\mathrm{ML}} = \langle S, \{R_a\}_{a \in \Pi_0}, V \rangle$.

Definition

For any $\mathscr{M} = \langle S, \{R_a\}_{a \in \Pi_0}, V, U \rangle$, $\mathscr{M}' = \langle S', \{R'_a\}_{a \in \Pi_0}, V', U' \rangle$, we say that $\mathscr{M}$ is U-bisimilar to $\mathscr{M}'$ (notation: $\mathscr{M} \rightleftarrows \mathscr{N}$) iff:

- for any $u \in U$, there is a $u' \in U'$, such that $\mathscr{M}^{\mathrm{ML}}, u \leftrightarroweq \mathscr{N}^{\mathrm{ML}}, u'$,
- for any $u' \in U'$, there is a $u \in U$, such that $\mathscr{M}^{\mathrm{ML}}, u \leftrightarroweq \mathscr{N}^{\mathrm{ML}}, u'$.

We say two pointed UM models are *U-bisimilar* ($\mathscr{M}, u \rightleftarrows \mathscr{N}, u'$) iff $\mathscr{M}^{\mathrm{ML}}, u \leftrightarroweq \mathscr{N}^{\mathrm{ML}}, u'$ and $\mathscr{M} \rightleftarrows \mathscr{N}$.

Proposition

If $\mathscr{M}, s \rightleftarrows \mathscr{N}, u$,

- *for any* π, *there is a pointed* $\mathtt{UM}$ *model* $\mathscr{M}', s'$, *such that* $(\mathscr{M}, s)[\![\pi]\!](\mathscr{M}', s')$, *iff there is* $\mathscr{N}', u'$, *such that* $(\mathscr{N}, u)[\![\pi]\!](\mathscr{N}', u')$. *And* $\mathscr{M}', s' \rightleftarrows \mathscr{N}', u'$.
- *for any* ϕ, $\mathscr{M}, s \vDash \phi$ *iff* $\mathscr{N}, u \vDash \phi$.

Proof.

By mutual induction. $\square$

Compare with ETS

- An ETS (*Epistemic Temporal Structure*) model is a PDL model with an equivalent relation. Formally, an ETS model $\mathfrak{M}$ is a tuple

$$\mathfrak{M} = \langle S, \{R_a \mid a \in \Pi_0\}, \sim, V \rangle$$

  where $\sim$ is an equivalent relation on S.
- The satisfiable relation of an EPDL formula ϕ on an ETS model $\mathfrak{M}$ is the same as PDL besides that:

$$\mathfrak{M}, s \Vdash K\phi \iff \text{ for all } t,\ s \sim t \text{ implies } \mathfrak{M}, t \Vdash \phi$$

Proposition

For any UM models $\mathscr{M} = \langle S, \{R_a \mid a \in \Pi_0\}, V, U \rangle$, $\mathscr{M}$ can be unravelled as an ETS model $\mathscr{M}^{ETS}$.

Definition

Given a UM model $\mathcal{M} = \langle S, \{R_a \mid a \in \Pi_0\}, V, U\rangle$, we define $\mathcal{M}^{\mathrm{ETS}}$ as $\langle S^\bullet, \{R_a^\bullet \mid a \in \Pi_0\}, \sim^\bullet, V^\bullet\rangle$ where:

- $S^\bullet = \{\rho \mid \rho$ is a path in $\mathcal{M}$ starting with some $s \in U\}$
- $(\rho, \rho') \in R_a^\bullet$ iff $\rho' = \rho a t$ for some $t \in S$ and $a \in \Pi_0$.
- For any two paths $\rho = s_0 a_1 \cdots a_n s_n$, $\rho' = t_0 a_1 \cdots a_n t_n$ in $S^\bullet$: $\rho \sim^\bullet \rho'$ iff $n = 0$ or $o(s_i) = o(t_i)$ for each $1 \le i \le n$.
- $V^\bullet(s_0 a_1 \cdots a_n s_n) = V(s_n)$

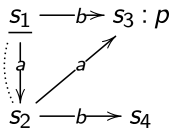
Proposition

Let $\mathcal{M} = \langle S, \{R_a \mid a \in \Phi\}, U, V\rangle$ and $s \in U$, then

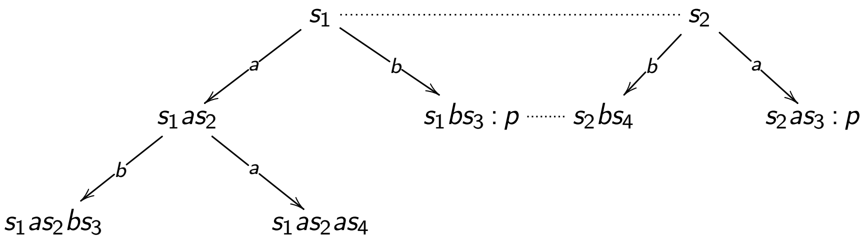
- *For any π, there is $\mathcal{M}', s'$ such that $\mathcal{M}, s[\![\pi]\!]\mathcal{M}', s'$ iff there is $\rho' \in S^\bullet$ such that $s \xrightarrow{\pi} \rho'$ in $\mathcal{M}^{ETS}$. And $\mathcal{M}^{ETS}, \rho' \leftrightarroweq (\mathcal{M}')^{ETS}, s'$.*
- *For any EPDL formula ϕ: $\mathcal{M}, s \vDash \phi$ iff $\mathcal{M}^{ETS}, s \Vdash \phi$.*

An example

An UM model: $\mathscr{M}, s_1 \vDash K\neg p \wedge \langle b \rangle \neg Kp$



Its unravelled ETS model: $\mathscr{M}^{\mathrm{ETS}}, s_1 \Vdash K\neg p \wedge \langle b \rangle \neg Kp$



Proposition

$\mathcal{M}^{ETS}$ is the ETS model, which is constructed by the UM model $\mathcal{M}$, then

- *If $\rho_1 \sim \rho_2$, then $o(\rho_1) = o(\rho_2)$.*
- *If $\rho_1 \xrightarrow{a} \rho_2$, for any $a \in \Pi_0$, and $\rho_2 \sim \rho_4$, then there is ρ_3 such that $\rho_1 \sim \rho_3$ and $\rho_3 \xrightarrow{a} \rho_4$.*
- *If $\rho_1 \sim \rho_3$ and $\rho_3 \xrightarrow{a} \rho_4$, for any $a \in \Pi_0$, then for any ρ_2, $\rho_1 \xrightarrow{a} \rho_2$ and $o(\rho_2) = o(\rho_4)$ implies $\rho_2 \sim \rho_4$.*

Let $\mathbb{C}$ be the ETS models which has the three properties, then we can get that

Proposition

For any EPDL formula ϕ, if $\mathbb{C} \Vdash \phi$, then $\vDash \phi$.

How about the other direction?

Proposition

For any pointed ETS *model* $(\mathfrak{M}, s)$*, where* $\mathfrak{M} \in \mathbb{C}$*, there is a pointed* UM $\mathfrak{M}_s^{UM}, s$.

Definition

Given a pointed ETS model $\mathfrak{M} = \langle S, \{R_a \mid a \in \Pi_0\}, \sim, V\rangle$, $s \in S$, we define the UM model $\mathfrak{M}_s^{\mathrm{UM}}$ as $\langle S^{\blacksquare}, \{R_a^{\blacksquare} \mid a \in \Pi_0\}, V^{\blacksquare}, U^{\blacksquare}\rangle$ where:

- $U^{\blacksquare} = \{s' \mid s' \sim s\}$
- $S^{\blacksquare} = \bigcup\{S_{s'}$ is the domain of the pointed generated model of $\mathfrak{M}^{\mathrm{ML}}$ from $s' \mid s' \in U^{\blacksquare}\}$
- $R_a^{\blacksquare} = R_a \cap S^{\blacksquare} \times S^{\blacksquare}$
- $V^{\blacksquare}(s) = V(s)$

Example

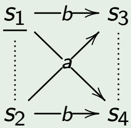


Figure: $\mathfrak{M}, s_1$

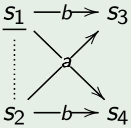


Figure: $\mathfrak{M}^{\mathtt{UM}}_{s_1}, s_1$

Proposition

Let $\mathfrak{M} = \langle S, \{R_a \mid a \in \Pi_0\}, \sim, V\rangle$ *and* $s_1 \in S$.

- *For any* π*, there is* $s_2 \in S$ *such that* $s_1 \xrightarrow{\pi} s_2$ *in* $\mathfrak{M}$ *iff there is* $((\mathfrak{M}^{UM}_{s_1})', s_1')$ *such that* $(\mathfrak{M}^{UM}_{s_1}, s_1)[\![\pi]\!]((\mathfrak{M}^{UM}_{s_1})', s_1')$. *And* $\mathfrak{M}^{UM}_{s_2}, s_2 \rightleftarrows (\mathfrak{M}^{UM}_{s_1})', s_1'$.
- *For any* EPDL *formula* ϕ: $\mathfrak{M}, s_1 \Vdash \phi$ *iff* $\mathfrak{M}^{UM}_{s_1}, s_1 \vDash \phi$.

Proposition

For any EPDL *formula* ϕ, *if* $\vDash \phi$, *then* $\mathbb{C} \Vdash \phi$.

Theorem

For any EPDL *formula* ϕ, $\vDash \phi \iff \mathbb{C} \Vdash \phi$.

Axiomatization

Axioms:

TAUT	all the axioms of propositional logic
DISTK	$K(\phi \to \psi) \to (K\phi \to K\psi)$
DISTπ	$[\pi](\phi \to \psi) \to ([\pi]\phi \to [\pi]\psi)$
T	$K\phi \to \phi$
4	$K\phi \to KK\phi$
5	$\neg K\phi \to K\neg K\phi$
	$[\pi_1 + \pi_2]\phi \leftrightarrow [\pi_1]\phi \wedge [\pi_2]\phi$
	$[\pi_1;\pi_2]\phi \leftrightarrow [\pi_1][\pi_2]\phi$
	$[?\psi]\phi \leftrightarrow \psi \to \phi$
	$\phi \wedge [\pi][\pi^*]\phi \leftrightarrow [\pi^*]\phi$
IND	$\phi \wedge [\pi^*](\phi \to [\pi]\phi) \to [\pi^*]\phi$
OBS(p)	$Kp \vee K\neg p$
PR(a)	$K[a]\phi \to [a]K\phi$
NM(a)	$\langle a\rangle(\psi_{\mathbf{Q}} \wedge K\phi) \to K[a](\psi_{\mathbf{Q}} \to \phi)$

Rules:

Soundness and Completeness

Proposition

If $\vdash \phi$, *then* $\mathbb{C} \Vdash \phi$.

Proposition (Ongoing)

If $\mathbb{C} \Vdash \phi$, *then* $\vdash \phi$.

Thanks!