Algorithmic correspondence and canonicity for possibility semantics

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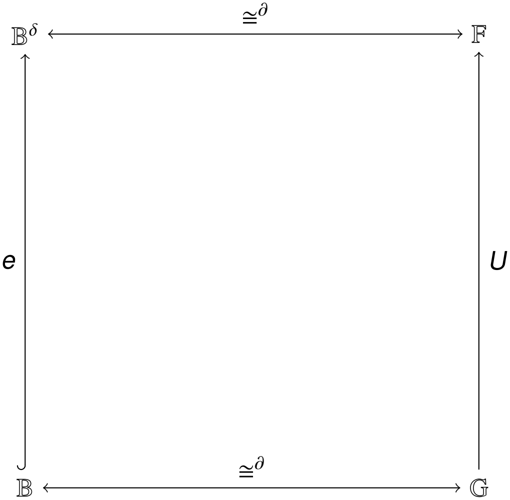
Logic Seminar, 2017.11.7, Room 417

The big picture of duality theory in modal logic



The big picture of duality theory in modal logic

$\mathbb{F}$:Kripke frame

Relational semantics for interpreting modal logic

Kripke frame and model

- Kripke frame $\mathbb{F} = (W, R)$, where W is the set of states and R is a binary relation on W
- Kripke model $\mathbb{M} = (F, V)$, where $V : \mathrm{Prop} \to \mathcal{P}(W)$ is the valuation on $\mathbb{F}$

Relational semantics for interpreting modal logic

Satisfaction relation

- $\mathbb{F}, V, w \Vdash p$ iff $w \in V(p)$;
- $\mathbb{F}, V, w \Vdash \bot$: never;
- $\mathbb{F}, V, w \Vdash \top$: always;
- $\mathbb{F}, V, w \Vdash \neg\varphi$ iff $(\mathbb{F}, V, w \nVdash \varphi)$;
- $\mathbb{F}, V, w \Vdash \varphi \wedge \psi$ iff $(\mathbb{F}, V, w \Vdash \varphi$ and $\mathbb{F}, V, w \Vdash \psi)$;
- $\mathbb{F}, V, w \Vdash \varphi \vee \psi$ iff $(\mathbb{F}, V, w \Vdash \varphi$ or $\mathbb{F}, V, w \Vdash \psi)$;
- $\mathbb{F}, V, w \Vdash \varphi \rightarrow \psi$ iff $(\mathbb{F}, V, w \Vdash \varphi \Rightarrow \mathbb{F}, V, w \Vdash \psi)$;
- $\mathbb{F}, V, w \Vdash \Diamond\varphi$ iff $\exists v(Rwv$ and $\mathbb{F}, V, v \Vdash \varphi)$;
- $\mathbb{F}, V, w \Vdash \Box\varphi$ iff $\forall v(Rwv \Rightarrow \mathbb{F}, V, v \Vdash \varphi)$.

From relational semantics to algebraic semantics

We can extend V to the set of all modal formulas such that $V(\varphi) = \{w \in W \mid \mathbb{F}, V, w \Vdash \varphi\}$.

Proposition

- $V(\bot) = \varnothing$;
- $V(\top) = W$;
- $V(\neg\varphi) = W - V(\varphi)$;
- $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$;
- $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$;
- $V(\varphi \rightarrow \psi) = (W - V(\varphi)) \cup V(\psi)$;
- $V(\Diamond\varphi) = m(V(\varphi))$, where $m(X) = \{w \in W \mid \text{there is a } v \in W \text{ such that } Rwv \text{ and } v \in X\}$;
- $V(\Box\varphi) = W - m(W - V(\varphi))$.

The big picture of duality theory in modal logic

$\mathbb{F}$

$\mathbb{B}$:Boolean algebra with operator (BAO)

Algebraic semantics for interpreting modal logic

Boolean algebra with operator

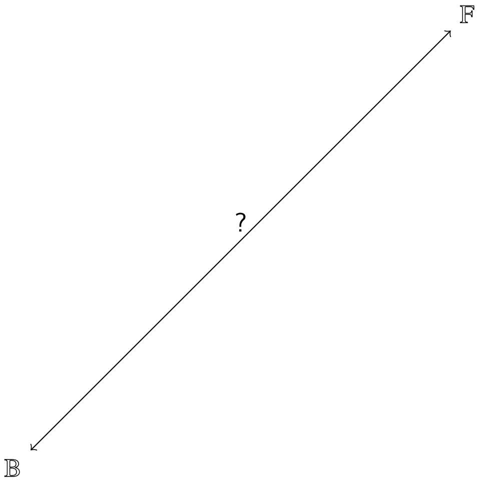
- Boolean algebra with operator (BAO) $\mathbb{B} = (B, \bot^{\mathbb{B}}, \top^{\mathbb{B}}, \wedge^{\mathbb{B}}, \vee^{\mathbb{B}}, -^{\mathbb{B}}, \Diamond^{\mathbb{B}})$, where $(B, \bot^{\mathbb{B}}, \top^{\mathbb{B}}, \wedge^{\mathbb{B}}, \vee^{\mathbb{B}}, -^{\mathbb{B}})$ is a Boolean algebra, $\Diamond^{\mathbb{B}}$ satisfies the following two conditions:
  - (normality) $\Diamond^{\mathbb{B}}\bot^{\mathbb{B}} = \bot^{\mathbb{B}}$;
  - (additivity) $\Diamond^{\mathbb{B}}(a \vee^{\mathbb{B}} b) = \Diamond^{\mathbb{B}}a \vee^{\mathbb{B}} \Diamond^{\mathbb{B}}b$.
- Assignment $\theta : \mathrm{Prop} \to B$, which can be extended to all modal formulas as follows:

Algebraic semantics for interpreting modal logic

Interpretation

- $\theta(\bot) = \bot^{\mathbb{B}}$;
- $\theta(\top) = \top^{\mathbb{B}}$;
- $\theta(\neg\varphi) = -^{\mathbb{B}}\theta(\varphi)$;
- $\theta(\varphi \wedge \psi) = \theta(\varphi) \wedge^{\mathbb{B}} \theta(\psi)$;
- $\theta(\varphi \vee \psi) = \theta(\varphi) \vee^{\mathbb{B}} \theta(\psi)$;
- $\theta(\varphi \rightarrow \psi) = (-^{\mathbb{B}}\theta(\varphi)) \vee^{\mathbb{B}} \theta(\psi)$;
- $\theta(\Diamond\varphi) = \Diamond^{\mathbb{B}}\theta(\varphi)$;
- $\theta(\Box\varphi) = -^{\mathbb{B}}\Diamond^{\mathbb{B}} -^{\mathbb{B}} \theta(\varphi)$.

The big picture of duality theory in modal logic



The big picture of duality theory in modal logic

$\mathbb{F}^+$:complex algebra

$\mathbb{F}$

$\mathbb{B}$

The complex algebra of Kripke frame

Complex algebra

Given a Kripke frame $\mathbb{F} = (W, R)$, its complex algebra is given by $\mathbb{F}^+ = (\mathcal{P}(W), \varnothing, W, \cap, \cup, (\cdot)^c, \Diamond^{\mathbb{F}^+})$, where

- $(X)^c = W - X$;
- $\Diamond^{\mathbb{F}^+} X = m(X)$.

Proposition

Given a Kripke frame $\mathbb{F}$, its complex frame $\mathbb{F}^+$ is a BAO.

The equivalence between Kripke frame and its complex algebra

For any Kripke frame $\mathbb{F} = (W, R)$ and its complex algebra $\mathbb{F}^+$, any valuation $V : \mathrm{Prop} \to \mathcal{P}(W)$ on $\mathbb{F}$, any assignment $\theta : \mathrm{Prop} \to \mathcal{P}(W)$ on $\mathbb{F}^+$, any $w \in W$, any formula φ,

- $\{w \in W \mid \mathbb{F}, V, w \Vdash \varphi\} = \theta_V(\varphi)$ where $\theta_V(p) = V(p)$ for all $p \in \mathrm{Prop}$;
- $\{w \in W \mid \mathbb{F}, V_\theta, w \Vdash \varphi\} = \theta(\varphi)$ where $V_\theta(p) = \theta(p)$ for all $p \in \mathrm{Prop}$;
- $\mathbb{F} \Vdash \varphi$ iff $\mathbb{F}^+ \vDash \varphi$.

Complex algebras are not arbitrary BAOs

An auxiliary definition

- In a Boolean algebra $\mathbb{B}$, we say that an element $a \in B$ is an atom if $a \neq \bot^{\mathbb{B}}$ and there is no element $b \in B$ such that $\bot^{\mathbb{B}} < b < a$.
- Example: singleton in the power set algebra
- Boolean algebra with no atoms: the Lindenbaum-Tarski algebra of classical propositional logic with countably many propositional variables, i.e. the Boolean algebra of propositional formulas (in the language of countably many propositional variables) modulo provability equivalence

Complex algebras are not arbitrary BAOs

Properties of BAOs

We say that a BAO $\mathbb{B}$ is

- complete, if arbitrary meet and arbitrary join exist in $\mathbb{B}$;
- atomic, if for any element $a \neq \bot$ there is an atom $b \in B$ such that $\bot < b < a$;
- completely additive, if $\Diamond^{\mathbb{B}}(\bigvee X) = \bigvee\{\Diamond^{\mathbb{B}} a \mid a \in X\}$.

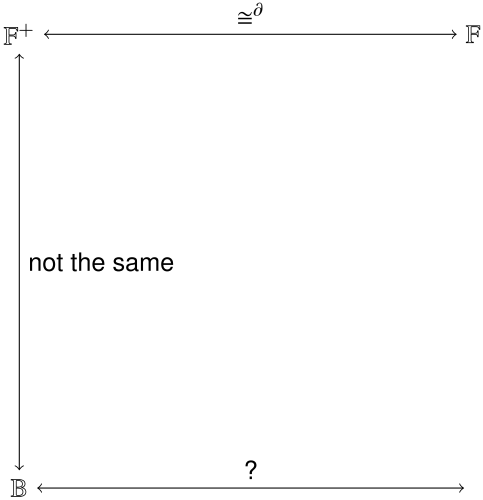
Theorem

For any Kripke frame $\mathbb{F}$, its complex algebra $\mathbb{F}^+$ is complete, atomic and completely additive (CABACO), in addition, every element of $\mathbb{F}^+$ can be represented as a (possibly infinite) join of atoms.

One-to-one correspondence between Kripke frames and CABACOs

- Indeed, given a complete, atomic and completely additive BAO, we can construct an equivalent Kripke frame out of it. (Hint: a complete atomic Boolean algebra is isomorphic to a powerset algebra)
- Furthermore, we have a one-to-one correspondence between Kripke frames and complete, atomic and completely additive BAOs, by the constructions we give.

The big picture of duality theory in modal logic



The big picture of duality theory in modal logic

$$\mathbb{F}^+ \xleftrightarrow{\cong^\partial} \mathbb{F}$$

$$\mathbb{B} \longleftrightarrow \mathbb{G}$$:descriptive general frame

General frame

General frame and general frame-based model

- A general frame $\mathbb{G} = (W, R, A)$, where $A \subseteq \mathcal{P}(W)$ is called the set of admissible values such that
  - if $X, Y \in A$, then $X \cap Y \in A$ (close under intersection);
  - if $X \in A$, then $W - X \in A$ (close under complementation);
  - if $X \in A$, then $m(X) \in A$ (close under modality).
- Admissible valuation $V : \mathrm{Prop} \to A$
- General frame-based model: $\mathbb{M} = (\mathbb{G}, V)$ where V is an admissible valuation

The underlying BAO of a general frame

Given a general frame $\mathbb{G} = (W, R, A)$, its underlying BAO is given by $\mathbb{G}^* = (A, \varnothing, W, \cap, \cup, (\cdot)^c, \Diamond^{\mathbb{F}^+})$.

How to construct a general frame from a BAO?

General ultrafilter frame

Ultrafilter

Given a Boolean algebra $\mathbb{B}$, an ultrafilter u on $\mathbb{B}$ is a proper subset of B such that

- $\top^{\mathbb{B}} \in u$;
- if $a \in u$ and $a \leq b$, then $b \in u$ (upward closeness);
- if $a, b \in u$, then $a \wedge b \in u$ (close under meet);
- for any $a \in B$, either $a \in u$ or $-^{\mathbb{B}} a \in u$.

General ultrafilter frame

Definition

Given a BAO $\mathbb{B}$, its general ultrafilter frame $\mathbb{B}_+ = (W, R, A)$ is defined as follows:

- W is the collection of all ultrafilters on $\mathbb{B}$;
- Ruv iff (for all $a \in \mathbb{B}$, if $a \in v$ then $\Diamond^{\mathbb{B}} a \in u$);
- $A = \{\hat{a} \mid a \in \mathbb{B}\}$, where $\hat{a} = \{u \in W \mid a \in u\}$.

What is special of this general frame?

Descriptive general frame

Some properties of general frame

Given a general frame $\mathbb{G} = (W, R, A)$, it is

- differentiated, if for all $u, v \in W$, $u = v$ iff $(\forall a \in A)(u \in A \Leftrightarrow v \in A)$;
- tight, if for all $u, v \in W$, Ruv iff $(\forall a \in A)(a \in v \Rightarrow m(a) \in u)$;
- compact, if $\bigcap A_0 \neq \varnothing$ for every subset A_0 of A which has the finite intersection property;
- descriptive, if it is differentiated, tight and descriptive.

Theorem

Given a BAO $\mathbb{B}$, its general ultrafilter frame $\mathbb{B}_+$ is descriptive.

One-to-one correspondence between descriptive general frames and BAOs

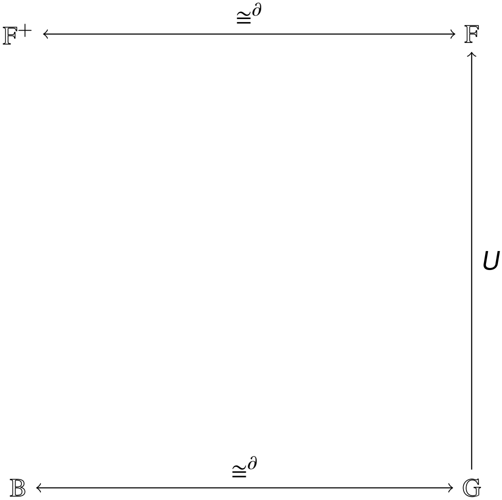
By the constructions we give, there is a one-to-one correspondence between descriptive general frames and BAOs.

The big picture of duality theory in modal logic

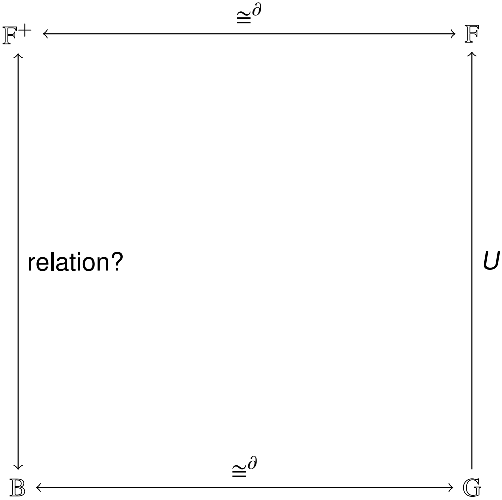
$$\mathbb{F}^{+} \overset{\cong^{\partial}}{\longleftrightarrow} \mathbb{F}$$

$$\mathbb{B} \overset{\cong^{\partial}}{\longleftrightarrow} \mathbb{B}_{+}$$

The big picture of duality theory in modal logic



The big picture of duality theory in modal logic



Canonical extension of Boolean algebra

Definition

Given a Boolean algebra $\mathbb{B}$, its canonical extension is a Boolean algebra $\mathbb{A}$ such that $\mathbb{B}$ is a sub-Boolean algebra of $\mathbb{A}$, and satisfies the following two properties:

- (denseness) every element of $\mathbb{B}^\delta$ is both a meet of joins and a join of meets of elements from $\mathbb{B}$;
- (compactness) for all $S, T \subseteq \mathbb{B}$ with $\bigwedge S \leq \bigvee T$ in $\mathbb{B}^\delta$, there exist some finite subsets $F \subseteq S$ and $G \subseteq T$ such that $\bigwedge F \leq \bigvee G$.

Theorem

Given a Boolean algebra, its canonical extension is unique up to isomorphism.

Canonical extension of map

An element $x \in \mathbb{B}^\delta$ is closed (resp. open) if it is the meet (resp. join) of some subset of $\mathbb{B}$. We let $\mathsf{K}(\mathbb{B}^\delta)$ and $\mathsf{O}(\mathbb{B}^\delta)$ respectively denote the sets of closed and open elements of $\mathbb{B}^\delta$. It is easy to see that elements in $\mathbb{B}$ are exactly the ones which are both closed and open (i.e. clopen).

Canonical extension of map

For any order-preserving map $f : \mathbb{A} \to \mathbb{B}$ and all $u \in \mathbb{A}^\delta$, we define

$$f^\sigma(u) = \bigvee\{\bigwedge\{f(a) : x \leq a \in \mathbb{A}\} : u \geq x \in \mathsf{K}(\mathbb{A}^\delta)\}$$

$$f^\pi(u) = \bigwedge\{\bigvee\{f(a) : y \geq a \in \mathbb{A}\} : u \leq y \in \mathsf{O}(\mathbb{A}^\delta)\}.$$

Since in a BAO, $\Diamond^\mathbb{B}$ is join-preserving, it is smooth, i.e. $(\Diamond^\mathbb{B})^\sigma = (\Diamond^\mathbb{B})^\pi$.

Canonical extension of BAO

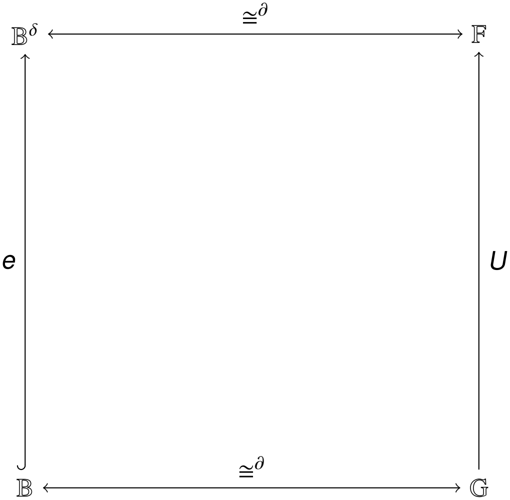
Definition

Given a BAO $\mathbb{B}$, its canonical extension $\mathbb{B}^\delta$ is defined as its canonical extension of the Boolean part together with $(\Diamond^{\mathbb{B}})^\sigma$.

Theorem

Given any BAO $\mathbb{B}$, $\mathbb{B}^\delta = (U(\mathbb{B}_+))^+$.

The big picture of duality theory in modal logic



Correspondence theory

- one of the three pillars of modal logic
- we say that a modal formula φ corresponds to a first-order formula α if they are valid on the same class of Kripke frames
- Example: $\Box p \to \Box\Box p$ corresponds to $\forall xyz(Rxy \wedge Ryz \to Rxz)$

Unified correspondence theory

- unified correspondence: moving from relational to algebraic
- Component:
  - A modular definition of Sahlqvist and inductive formulas/inequalities, which can be adapted easily to different semantic environment
  - An algorithm ALBA to compute first-order correspondents, which can also be adapted easily
- benifit: modularity and easy to generalize
  - distributive lattice
  - general lattice
  - regular modal logic
  - modal mu calculus
  - many-valued modal logic
  - hybrid logic
  - possibility semantics
- extra benefit: applications in proof theory

Unified correspondence: from relational to algebraic

$$
\begin{array}{ccc}
\mathbb{F}^{+} \models \varphi(\vec{p}) & \Leftrightarrow & \mathbb{F} \models \varphi(\vec{p}) \\
\Updownarrow & & \\
\mathbb{F}^{+} \models \mathrm{Pure}(\varphi(\vec{p})) & \Leftrightarrow & \mathbb{F} \models \mathrm{FO}(\mathrm{Pure}(\varphi(\vec{p})))
\end{array}
$$

The expanded language

$$\varphi ::= p \mid \mathbf{i} \mid \mathbf{m} \mid \bot \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \blacklozenge\varphi \mid \blacksquare\varphi$$

i is nominal, **m** is co-nominal, interpreted as follows:
$V(\mathbf{i}) = \{w\}$ for some $w \in W$, $V(\mathbf{m}) = W - \{v\}$ for some $v \in W$.
$\blacklozenge$ and $\blacksquare$ are interpreted as the tense "backward looking" modalities, i.e.:

- $\mathbb{F}, V, w \Vdash \blacklozenge\varphi$ iff $\exists v(Rvw \text{ and } \mathbb{F}, V, v \Vdash \varphi)$.
- $\mathbb{F}, V, w \Vdash \blacksquare\varphi$ iff $\forall v(Rvw \Rightarrow \mathbb{F}, V, v \Vdash \varphi)$.

Nominals and co-nominals: algebraically

What is the interpretation of nominals, co-nominals and the black connectives in Boolean algebras?

- In a Boolean algebra $\mathbb{B}$, we say that an element $a \in B$ is an co-atom if $a \neq \top^{\mathbb{B}}$ and there is no element $b \in B$ such that $a < b < \top^{\mathbb{B}}$.
- In $\mathbb{F}^{+}$, $V(\mathbf{i})$ is an atom, and $V(\mathbf{m})$ is a co-atom.
- Indeed, nominals and co-nominals cannot be interpreted in arbitrary Boolean algebras, since arbitrary Boolean algebra might not have atoms or co-atoms.
- In our setting, we only allow nominals and co-nominals to be interpreted in CABACO.

Black connectives: algebraically

We expand $\mathbb{F}^+$ by defining $\blacklozenge^{\mathbb{F}^+}$ and $\blacksquare^{\mathbb{F}^+}$ as follows:

- $\blacklozenge^{\mathbb{F}^+}(X) = \{w \in W \mid \exists v(Rvw \text{ and } v \in X)\}$;
- $\blacksquare^{\mathbb{F}^+}(X) = \{w \in W \mid \forall v(Rvw \Rightarrow v \in X)\}$.

Observation

- $\blacklozenge^{\mathbb{F}^+}(X) \subseteq Y$ iff $X \subseteq \Box^{\mathbb{F}^+}(Y)$;
- $\Diamond^{\mathbb{F}^+}(X) \subseteq Y$ iff $X \subseteq \blacksquare^{\mathbb{F}^+}(Y)$.

Adjoints

Definition

Given any two complete lattices $\mathbb{C}$ and $\mathbb{C}'$, the monotone maps $f : \mathbb{C} \to \mathbb{C}'$ and $g : \mathbb{C}' \to \mathbb{C}$ form an adjoint pair (notation: $f \dashv g$), if for every $x \in \mathbb{C}, y \in \mathbb{C}'$,

$$f(x) \leq_{\mathbb{C}'} y \;\text{ iff }\; x \leq_{\mathbb{C}} g(y).$$

Whenever $f \dashv g$, f is called the left adjoint of g and g the right adjoint of f. We also say f is a left adjoint and g is a right adjoint.

Proposition

- $\blacklozenge^{\mathbb{F}} \dashv \Box^{\mathbb{F}^+}$;
- $\Diamond^{\mathbb{F}^+} \dashv \blacksquare^{\mathbb{F}^+}$.

Adjoints

Proposition

For any complete lattice map $f : \mathbb{C} \to \mathbb{C}'$,

- f is completely join-preserving iff it has a right adjoint;
- f is completely meet-preserving iff it has a left adjoint.

We cannot guarantee that $\blacklozenge^{\mathbb{B}}$ and $\blacksquare^{\mathbb{B}}$ to be well-defined in arbitrary BAOs, because $\Diamond^{\mathbb{B}}$ (resp. $\Box^{\mathbb{B}}$) is not necessarily completely join- (resp. meet-) preserving, but only finite join- (resp. finite meet-) preserving.

Summary

In the BAO setting,

- Nominals and co-nominals are respectively interpreted as atoms and co-atoms in the CABACO;
- Black connectives are interpreted as adjoints of standard modal operators (in the CABACO);
- All the interpretations happen in CABACO instead of BAO.

The algorithm, an informal example

The Church-Rosser formula

$$\forall p(\Diamond\Box p \leq \Box\Diamond p)$$
$$\forall p \forall \mathbf{i} \forall \mathbf{m}(\mathbf{i} \leq \Diamond\Box p \ \& \ \Box\Diamond p \leq \mathbf{m} \ \Rightarrow \ \mathbf{i} \leq \mathbf{m})$$
$$\forall p \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}(\mathbf{i} \leq \Diamond\mathbf{j} \ \& \ \mathbf{j} \leq \Box p \ \& \ \Box\Diamond p \leq \mathbf{m} \ \Rightarrow \ \mathbf{i} \leq \mathbf{m})$$
$$\forall p \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}(\mathbf{i} \leq \Diamond\mathbf{j} \ \& \ \blacklozenge\mathbf{j} \leq p \ \& \ \Box\Diamond p \leq \mathbf{m} \ \Rightarrow \ \mathbf{i} \leq \mathbf{m})$$
$$\forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}(\mathbf{i} \leq \Diamond\mathbf{j} \ \& \ \ \& \ \Box\Diamond\blacklozenge\mathbf{j} \leq \mathbf{m} \ \Rightarrow \ \mathbf{i} \leq \mathbf{m})$$
$$\forall \mathbf{j}(\Diamond\mathbf{j} \leq \Box\Diamond\blacklozenge\mathbf{j})$$
$$\forall \mathbf{j}(\blacklozenge\Diamond\mathbf{j} \leq \Diamond\blacklozenge\mathbf{j})$$
$$\forall x(R[R^{-1}[\{x\}]] \subseteq R^{-1}[R[\{x\}]])$$
$$\forall x \forall y \forall z(Rxy \wedge Rxz \Rightarrow \exists w(Ryw \wedge Rzw))$$

Some auxiliary definitions

Signed generation tree

Given a modal formula φ, the positive (resp. negative) generation tree is defined by first labelling the root of the generation tree of φ with $+$ (resp. $-$), and then labelling the children nodes as follows:

- Assign the same sign to the children nodes of any node labelled with $\Box, \Diamond, \vee, \wedge$;
- Assign the opposite sign to the child node of any node labelled with $\neg$;
- Assign the opposite sign to the first child node and the same sign to the second child node of any node labelled with $\rightarrow$.

Nodes in signed generation trees are positive (resp. negative) if they are signed $+$ (resp. $-$).

Signed generation trees will be used in the inequalities $\varphi \leq \psi$, where positive generation tree $+\varphi$ and negative generation tree $-\psi$ will be considered.

The algorithm, formally

The algorithm receives a modal formula $\varphi \to \psi$ as input. The algorithm first transform the input formula into an inequality $\varphi \leq \psi$.

Stage 1: Preprocessing and initialization. In the signed generation tree of $+\varphi$ and $-\psi$,

- Apply the distribution rules:
  - Push down $+\Diamond, +\wedge, -\neg$ and $- \to$, by distributing them over nodes labelled with $+\vee$, and
  - Push down $-\Box, -\vee, +\neg$ and $- \to$, by distributing them over nodes labelled with $-\wedge$.
- Apply the monotone and antitone variable-elimination rules:

$$\frac{\alpha(p) \leq \beta(p)}{\alpha(\bot) \leq \beta(\bot)} \qquad \frac{\beta(p) \leq \alpha(p)}{\beta(\top) \leq \alpha(\top)}$$

  where $\beta(p)$ is positive in p and $\alpha(p)$ is negative in p.
- Apply the splitting rules:

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \quad \alpha \leq \gamma} \qquad \frac{\alpha \vee \beta \leq \gamma}{\alpha \leq \gamma \quad \beta \leq \gamma}$$

The algorithm, formally

Let Preprocess$(\varphi \leq \psi) := \{\varphi_i \leq \psi_i \mid 1 \leq i \leq n\}$ be the set of inequalities obtained by applying the above rules exhaustively. Then the following rule (which is called the first approximation rule) is applied to each $\varphi_i \leq \psi_i$ in Preprocess$(\varphi \leq \psi)$:

$$\frac{\varphi \leq \psi}{\mathbf{i}_0 \leq \varphi \quad \psi \leq \mathbf{m}_0}$$

where $\mathbf{i}_0$ is a nominal and $\mathbf{m}_0$ is a co-nominal. After the first approximation rule, for each inequality $\varphi_i \leq \psi_i \in$ Preprocess$(\varphi \leq \psi)$, the algorithm produces a system of inequalities $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$.

The algorithm, formally

Stage 2: reduction and elimination stage. The present stage aims at eliminating all propositional variables from each system obtained in the previous stage. The variables are eliminated by the so called Ackermann rules, and the other rules in this stage are applied in order to reach the shape to apply the Ackermann rule.
In this stage, for each $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$, we apply the following rules together with the splitting rules in the previous stage:

The algorithm, formally

Residuation rules:

$$\frac{\alpha \wedge \beta \leq \gamma}{\alpha \leq \beta \to \gamma} \qquad \frac{\alpha \leq \beta \vee \gamma}{\alpha \wedge \neg\beta \leq \gamma} \qquad \frac{\Diamond\alpha \leq \beta}{\alpha \leq \blacksquare\beta} \qquad \frac{\alpha \leq \Box\beta}{\blacklozenge\alpha \leq \beta}$$

$$\frac{\alpha \wedge \beta \leq \gamma}{\beta \leq \alpha \to \gamma} \qquad \frac{\alpha \leq \beta \vee \gamma}{\alpha \wedge \neg\gamma \leq \beta} \qquad \frac{\neg\alpha \leq \beta}{\alpha \leq \neg\beta} \qquad \frac{\alpha \leq \neg\beta}{\neg\alpha \leq \beta}$$

$$\frac{\alpha \leq \beta \to \gamma}{\alpha \wedge \beta \leq \gamma} \qquad \frac{\alpha \leq \beta \to \gamma}{\beta \leq \alpha \to \gamma}$$

The algorithm, formally

Approximation rules:

$$\frac{\mathbf{i} \leq \Diamond\alpha}{\mathbf{j} \leq \alpha \quad \mathbf{i} \leq \Diamond\mathbf{j}} \qquad \frac{\Box\alpha \leq \mathbf{m}}{\alpha \leq \mathbf{n} \quad \Box\mathbf{n} \leq \mathbf{m}}$$

The nominals and co-nominals introduced by the approximation rules must not occur in the system before applying the rule.

The algorithm, formally

The Ackermann rules: These two rules are the core of ALBA, since their application eliminates proposition variables. In fact, all the preceding steps are aimed at reaching a shape in which the rules can be applied. Notice that an important feature of these rules is that they are executed on the whole set of inequalities, and not on a single inequality.

The algorithm, formally

The right-handed Ackermann rule:

The system $\left\{\begin{array}{l}\alpha_1 \leq p \\ \vdots \\ \alpha_n \leq p \\ \beta_1 \leq \gamma_1 \\ \vdots \\ \beta_m \leq \gamma_m\end{array}\right.$ is replaced by

$$\left\{\begin{array}{l}\beta_1((\alpha_1 \vee \ldots \vee \alpha_n)/p) \leq \gamma_1((\alpha_1 \vee \ldots \vee \alpha_n)/p) \\ \vdots \\ \beta_m((\alpha_1 \vee \ldots \vee \alpha_n)/p) \leq \gamma_m((\alpha_1 \vee \ldots \vee \alpha_n)/p)\end{array}\right.$$

where:

- p does not occur in $\alpha_1, \ldots, \alpha_n$;
- Each β_i is positive, and each γ_i negative in p, for $1 \leq i \leq m$.

The algorithm, formally

The left-handed Ackermann rule:

The system $\left\{\begin{array}{l} p \leq \alpha_1 \\ \vdots \\ p \leq \alpha_n \\ \beta_1 \leq \gamma_1 \\ \vdots \\ \beta_m \leq \gamma_m \end{array}\right.$ is replaced by

$$\left\{\begin{array}{l} \beta_1((\alpha_1 \wedge \ldots \wedge \alpha_n)/p) \leq \gamma_1((\alpha_1 \wedge \ldots \wedge \alpha_n)/p) \\ \vdots \\ \beta_m((\alpha_1 \wedge \ldots \wedge \alpha_n)/p) \leq \gamma_m((\alpha_1 \wedge \ldots \wedge \alpha_n)/p) \end{array}\right.$$ where:

- p does not occur in $\alpha_1, \ldots, \alpha_n$;
- Each β_i is negative, and each γ_i positive in p, for $1 \leq i \leq m$.

The algorithm, formally

Stage 3: output stage. If in the previous stage, for some $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$, the algorithm gets stuck, i.e. some proposition variables cannot be eliminated by the application of the reduction rules, then the algorithm halts and output "failure". Otherwise, each initial tuple $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$ of inequalities after the first approximation has been reduced to a set of pure inequalities $\mathrm{Reduce}(\varphi_i \leq \psi_i)$, and then the output is a set of quasi-inequalities
$\{\&\mathrm{Reduce}(\varphi_i \leq \psi_i) \Rightarrow \mathbf{i}_0 \leq \mathbf{m}_0 : \varphi_i \leq \psi_i \in \mathrm{Preprocess}(\varphi \leq \psi)\}$.
If one is interested in the first-order correspondent, the resulting quasi-inequalities can be translated into first-order logic.

Unified correspondence: from relational to algebraic

$$
\begin{array}{ccc}
\mathbb{F}^{+} \models \varphi(\vec{p}) & \Leftrightarrow & \mathbb{F} \models \varphi(\vec{p}) \\
\Updownarrow & & \\
\mathbb{F}^{+} \models \mathrm{Pure}(\varphi(\vec{p})) & \Leftrightarrow & \mathbb{F} \models \mathrm{FO}(\mathrm{Pure}(\varphi(\vec{p})))
\end{array}
$$

Sahlqvist correspondence

- Question: for which class of modal formulas can we get first-order correspondents?
- Chagrova: It is not decidable whether a modal formula is frame-equivalent to a first-order formula. What we can do is to approximate.
- Sahlqvist and inductive formulas are famous examples.

Some auxiliary notions

For an n-tuple $(p_1, \ldots, p_n)$ of propositional variables, the order-type ε of $(p_1, \ldots, p_n)$ is an element in $\{1, \partial\}^n$. We say that p_i has order-type 1 if $\varepsilon_i = 1$, and denote $\varepsilon(p_i) = 1$ or $\varepsilon(i) = 1$, otherwise p_i has order-type ∂ if $\varepsilon_i = \partial$, and denote $\varepsilon(p_i) = \partial$ or $\varepsilon(i) = \partial$.
For any given formula $\varphi(p_1, \ldots p_n)$, any order-type ε over n, and any $1 \leq i \leq n$, an ε-critical node in a signed generation tree of φ is a leaf node $+p_i$ when $\varepsilon_i = 1$ or $-p_i$ when $\varepsilon_i = \partial$. An ε-critical branch in a signed generation tree is a branch from an ε-critical node.

Classification of nodes in signed generation tree

Nodes in signed generation trees are called Δ-adjoints, syntactically left residual (SLR), syntactically right adjoint (SRA), and syntactically right residual (SRR), according to the table below.

Skeleton	PIA
Δ-adjoints	SRA
$+$ $\vee$ $\wedge$	$+$ $\wedge$ $\Box$ $\neg$
$-$ $\wedge$ $\vee$	$-$ $\vee$ $\Diamond$ $\neg$
SLR	SRR
$+$ $\wedge$ $\Diamond$ $\neg$	$+$ $\vee$ $\rightarrow$
$-$ $\vee$ $\Box$ $\neg$ $\rightarrow$	$-$ $\wedge$

Table: Skeleton and PIA nodes.

Some auxiliary notions

A branch in a signed generation tree is called a good branch if it is the concatenation of two paths P_1 and P_2, one of which might be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) of PIA-nodes only, and P_2 consists (apart from variable nodes) of Skeleton-nodes only. A good branch will be called excellent if P_1 consists of SRA nodes only.

Sahlqvist and inductive inequality

Inductive

For any order-type ε and any irreflexive and transitive binary relation $<_\Omega$ on $p_1, \ldots p_n$, the signed generation tree $*\varphi$ $(* \in \{-, +\})$ of a formula $\varphi(p_1, \ldots p_n)$ is (Ω, ε)-inductive if

- for all $1 \leq i \leq n$, every ε-critical branch with leaf p_i is good;
- every SRR-node in the critical branch is either $\bigstar(\gamma, \beta)$ or $\bigstar(\beta, \gamma)$, where the critical branch is in β, and
  - $\varepsilon^\partial(\gamma) \prec *\varphi$, and
  - $p_k <_\Omega p_i$ for every p_k that occurs in γ.

Sahlqvist

For any order-type ε, the signed generation tree $*\varphi$ of a formula $\varphi(p_1, \ldots p_n)$ is ε-Sahlqvist if for all $1 \leq i \leq n$, every ε-critical branch with leaf p_i is excellent.

Sahlqvist and inductive inequality

We will refer to $<_\Omega$ as the dependence order on the variables. An inequality $\varphi \leq \psi$ is (Ω, ε)-inductive (resp. ε-Sahlqvist) if the signed generation trees $+\varphi$ and $-\psi$ are (Ω, ε)-inductive (resp. ε-Sahlqvist). An inequality $\varphi \leq \psi$ is inductive (resp. Sahlqvist) if it is (Ω, ε)-inductive (ε-Sahlqvist) for some (Ω, ε).

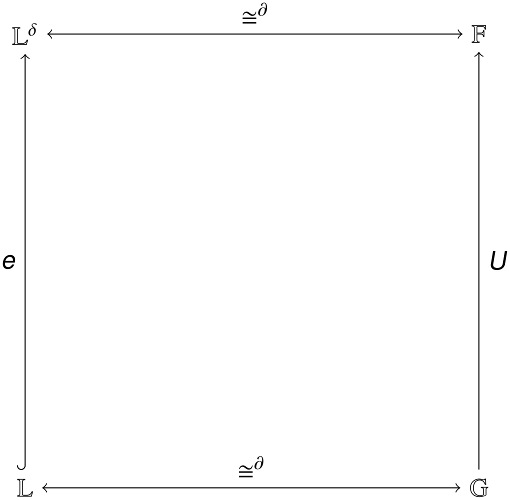
Theorem

The algorithm ALBA succeeds on all Sahlqvist and inductive formulas.

Generalizing to other semantic setting

- Analyze semantic environment
- Change ALBA rules
- Change the classification table

The duality picture for distributive modal logic



Distributive modal logic: formulas and semantics

$$\varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \lhd\varphi \mid \rhd\varphi$$

Distributive modal algebra: $\mathbb{L} = (L, \bot^{\mathbb{L}}, \top^{\mathbb{L}}, \wedge^{\mathbb{L}}, \vee^{\mathbb{L}}, \Diamond^{\mathbb{L}}, \Box^{\mathbb{L}}, \lhd^{\mathbb{L}}, \rhd^{\mathbb{L}})$ where

- $(L, \bot^{\mathbb{L}}, \top^{\mathbb{L}}, \wedge^{\mathbb{L}}, \vee^{\mathbb{L}})$ is a distributive lattice;
- $\Diamond^{\mathbb{L}}\bot^{\mathbb{L}} = \bot^{\mathbb{L}}$, $\Diamond^{\mathbb{L}}(a \vee^{\mathbb{L}} b) = \Diamond^{\mathbb{L}}a \vee^{\mathbb{L}} \Diamond^{\mathbb{L}}b$;
- $\Box^{\mathbb{L}}\top^{\mathbb{L}} = \top^{\mathbb{L}}$, $\Box^{\mathbb{L}}(a \wedge^{\mathbb{L}} b) = \Box^{\mathbb{L}}a \wedge^{\mathbb{L}} \Box^{\mathbb{L}}b$;
- $\lhd^{\mathbb{L}}\top^{\mathbb{L}} = \bot^{\mathbb{L}}$, $\lhd^{\mathbb{L}}(a \wedge^{\mathbb{L}} b) = \lhd^{\mathbb{L}}a \vee^{\mathbb{L}} \lhd^{\mathbb{L}}b$;
- $\rhd^{\mathbb{L}}\bot^{\mathbb{L}} = \top^{\mathbb{L}}$, $\rhd^{\mathbb{L}}(a \vee^{\mathbb{L}} b) = \rhd^{\mathbb{L}}a \wedge^{\mathbb{L}} \rhd^{\mathbb{L}}b$.

Some auxiliary notions

Definition

In a lattice $\mathbb{L}$,

- an element a is complete join-irreducible if whenever $a = \bigvee X$, we have $a = x$ for some $x \in X$. Complete meet-irreducible is defined similarly.
- an element a is complete join-prime if whenever $a \leq \bigvee X$ then there exists a $b \in X$ such that $a \leq b$. Complete meet-prime is defined similarly.

Proposition

In distributive lattices, an element is complete join- (resp. meet-) irreducible iff it is complete join- (resp. meet-) prime.

Some auxiliary notions

Definition

We say that a distributive modal algebra $\mathbb{L}$ is perfect if it is complete, completely distributive and every element is join-generated by some completely join-irreducible elements, and is meet-generated by some completely meet-irreducible elements, and all modal operations preserve arbitrary joins or meets according to their finite preservation.

Proposition

The canonical extension of a distributive modal algebra is perfect.

The expanded language and interpretations

$$\varphi ::= p \mid \mathbf{i} \mid \mathbf{m} \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \varphi - \varphi \mid$$

$$\Diamond\varphi \mid \Box\varphi \mid {\lhd}\varphi \mid {\rhd}\varphi \mid \blacklozenge\varphi \mid \blacksquare\varphi \mid \blacktriangleleft \varphi \mid \blacktriangleright \varphi$$

- In perfect distributive modal algebras, we do not necessarily have enough atoms/coatoms, but we have enough complete join-(resp. meet-) irreducibles such that every element can be represented both as a join of complete join-irreducibles and a meet of complete meet-irreducibles
- We interpret nominals (resp. co-nominals) as complete join-(resp. meet-) irreducibles
- We interpret the black connectives as the adjoints of the white connectives, since the white connectives are completely join- or meet-preserving

The design of ALBA rules

In Stage 1,

- The distribution rules depend on the finite join- or meet-preservation of connectives, ✓
- The monotone and antitone rules depend on the semantic monotonicity/antitonicity of formulas, ✓
- The splitting rules depend on the properties of meet and join in lattices, ✓
- For the first approximation rule, by the join- or meet-presentation, we have the following equivalence: $\varphi \leq \psi$
  iff $\{\mathbf{i} \mid \mathbf{i} \leq \varphi\} \subseteq \{\mathbf{i} \mid \mathbf{i} \leq \psi\}$
  iff $\forall \mathbf{i}(\mathbf{i} \leq \varphi \ \Rightarrow \ \mathbf{i} \leq \psi)$
  iff $\forall \mathbf{i}(\mathbf{i} \leq \varphi \ \Rightarrow \ \{\mathbf{m} \mid \psi \leq \mathbf{m}\} \subseteq \{\mathbf{m} \mid \mathbf{i} \leq \mathbf{m}\})$
  iff $\forall \mathbf{i}(\mathbf{i} \leq \varphi \ \Rightarrow \ \forall \mathbf{m}(\psi \leq \mathbf{m} \ \Rightarrow \ \mathbf{i} \leq \mathbf{m}))$
  iff $\forall \mathbf{i} \forall \mathbf{m}(\mathbf{i} \leq \varphi \ \& \ \psi \leq \mathbf{m} \ \Rightarrow \ \mathbf{i} \leq \mathbf{m})$, ✓

The design of ALBA rules

In Stage 2,

- The splitting rules, ✓
- The Ackermann rules depend on semantic monotonicity, ✓
- The residuation rules depend on the fact that black connectives are interpreted as the adjoints of white connectives, which are given below:

$$\frac{\alpha \wedge \beta \leq \gamma}{\alpha \leq \beta \to \gamma} \qquad \frac{\alpha \leq \beta \vee \gamma}{\alpha - \beta \leq \gamma} \qquad \frac{\Diamond\alpha \leq \beta}{\alpha \leq \blacksquare\beta} \qquad \frac{\alpha \leq \Box\beta}{\blacklozenge\alpha \leq \beta} \qquad \frac{\lhd\alpha \leq \beta}{\blacktriangleleft\beta \leq \alpha} \qquad \frac{\alpha \leq \rhd\beta}{\beta \leq \blacktriangleright\alpha}$$

The design of ALBA rules

In Stage 2,

- For the approximation rules, we have the following analysis:
  $\mathbf{i} \leq \Diamond\alpha$
  iff $\mathbf{i} \leq \Diamond \bigvee\{\mathbf{j} \mid \mathbf{j} \leq \alpha\}$
  iff $\mathbf{i} \leq \bigvee\{\Diamond\mathbf{j} \mid \mathbf{j} \leq \alpha\}$
  iff there exists $\mathbf{j} \leq \alpha$ such that $\mathbf{i} \leq \Diamond\mathbf{j}$

The first step requires join-presentation, the second step requires that $\Diamond$ is completely join-preserving, the third step requires that nominals are interpreted as complete join-primes. In distributive lattices, complete join-primes are the same as complete join-irreducibles, ✓
Therefore, we have the following approximation rules:

$$\frac{\mathbf{i} \leq \Diamond\alpha}{\mathbf{j} \leq \alpha \quad \mathbf{i} \leq \Diamond\mathbf{j}} \qquad \frac{\Box\alpha \leq \mathbf{m}}{\alpha \leq \mathbf{n} \quad \Box\mathbf{n} \leq \mathbf{m}} \qquad \frac{\mathbf{i} \leq \lhd\alpha}{\alpha \leq \mathbf{m} \quad \mathbf{i} \leq \lhd\mathbf{m}} \qquad \frac{\rhd\alpha \leq \mathbf{m}}{\mathbf{i} \leq \alpha \quad \rhd\mathbf{i} \leq \mathbf{m}}$$

Classification of nodes in signed generation tree

Nodes in signed generation trees are called Δ-adjoints, syntactically left residual (SLR), syntactically right adjoint (SRA), and syntactically right residual (SRR), according to the table below.

Table: Skeleton nodes and PIA nodes for distributive modal logic.

Skeleton	PIA
Δ-adjoints $+$ $\vee$ $\wedge$ $-$ $\wedge$ $\vee$	SRA $+$ $\wedge$ $\Box$ $\triangleright$ $-$ $\vee$ $\Diamond$ $\triangleleft$
SLR $+$ $\wedge$ $\Diamond$ $\triangleleft$ $-$ $\vee$ $\Box$ $\triangleright$	SRR $+$ $\vee$ $-$ $\wedge$

Possibility semantics

Possibility semantics

- a variant of standard Kripke semantics for modal logic
- motivation: partial possibilities vs total worlds
- constructive study of classical (modal) logic:
  - intuitionistic-style semantics: refinement relation
  - constructive completeness proofs
  - relation to constructive canonical extension

Possibility semantics

- possibility frame: $\mathbb{F} = (W, R, \sqsubseteq, \mathrm{RO}(W, \sqsubseteq))$
- possibility model: $\mathbb{M} = (\mathbb{F}, V)$ where $V : \mathrm{Prop} \to \mathrm{RO}(W, \sqsubseteq)$
- refinement relation $\sqsubseteq$: partial order on W
- accessibility relation R: binary relation on W
- $\mathrm{RO}(W, \sqsubseteq)$: set of admissible valuations
- intuition behind $\mathrm{RO}(W, \sqsubseteq)$: subsets equal to their “double negation”

Possibility semantics

Satisfaction relation

- $\mathbb{F}, V, w \Vdash p$ iff $w \in V(p)$;
- $\mathbb{F}, V, w \Vdash \varphi \wedge \psi$ iff $\mathbb{F}, V, w \Vdash \varphi$ and $\mathbb{F}, V, w \Vdash \psi$;
- $\mathbb{F}, V, w \Vdash \varphi \vee \psi$ iff $(\forall v \sqsubseteq w)(\exists u \sqsubseteq v)(\mathbb{F}, V, u \Vdash \varphi$ or $\mathbb{F}, V, u \Vdash \psi)$;
- $\mathbb{F}, V, w \Vdash \varphi \rightarrow \psi$ iff $(\forall v \sqsubseteq w)(\mathbb{F}, V, v \Vdash \varphi \Rightarrow \mathbb{F}, V, v \Vdash \psi)$;
- $\mathbb{F}, V, w \Vdash \neg\varphi$ iff $(\forall v \sqsubseteq w)(\mathbb{F}, V, v \nVdash \varphi)$;
- $\mathbb{F}, V, w \Vdash \Box\varphi$ iff $\forall v(Rwv \Rightarrow \mathbb{F}, V, v \Vdash \varphi)$.

Algebraic correspondence: from frames to algebras

$$\mathbb{B} \Vdash \forall\vec{p}(\varphi(\vec{p})) \quad \Leftrightarrow \quad \mathbb{F} \Vdash \varphi(\vec{p})$$

$$\Updownarrow$$

$$\mathbb{B} \Vdash \forall\vec{\mathbf{i}}\mathrm{Pure}(\varphi(\vec{p})) \quad \Leftrightarrow \quad \mathbb{F} \Vdash \mathrm{FO}(\mathrm{Pure}(\varphi(\vec{p})))$$

- In the dual BAO of Kripke frames, nominals are interpreted as atoms.
- How about possibility semantics?

Dual algebras

Given $\mathbb{F} = (W, \sqsubseteq, R, \mathrm{RO}(W, \sqsubseteq))$, the regular open dual BAO $\mathbb{B}_{\mathrm{RO}}$

- $\mathbb{B}_{\mathrm{RO}}$ is a complete and completely additive BAO, but not necessarily atomic.
- lack of atomicity: what is the consequence in correspondence theory?

Nominals and their interpretations

Algebraic setting	**Interpretation for nominals**	**Dually corresponding to**
perfect Boolean algebras	atoms	singletons
perfect distributive lattices	complete join-primes	$w\uparrow$
perfect general lattices	complete join-irreducibles	Galois closure of singletons
constructive canonical extensions	closed elements	N.A.
complex algebras of possibility frames		regular open closures of singletons