UNIVERSITÄT BAYREUTH

Dominik Klein[1]
Frederik Van De Putte[2]

Pooling Modalities and Pointwise Intersection: a Survey of Recent Results.

[1]Uni Bayreuth & Uni Bamberg
[2]MSCA Fellow Uni Bayreuth and FWO Postdoc at Ghent University

PKU Logic Seminar May 7th 2019

Neighborhood Modal Logic...

The Idea...

Neighborhood Modal Logic...

UNIVERSITÄT BAYREUTH

The Idea…

- Intersection closed?

Neighborhood Modal Logic...

UNIVERSITÄT BAYREUTH

The Idea...

- Intersection closed?
- Believing trivialities? ($\Box_i \top$?) (or at least something)?

Neighborhood Modal Logic...

UNIVERSITÄT BAYREUTH

The Idea...

- Intersection closed?
- Believing trivialities? ($\Box_i \top$?) (or at least something)?
- Closure under weakening?

Some Examples

	$\wedge$-closed	non-empty	$W \in \mathcal{N}(w)$	$\vee$-closed
Logic of Evidence	-	✓	✓	-
Strategic Powers in Games	-	✓	✓	✓
Deontic norms	-	-	-	-
(Weak) belief (cf. Lottery Paradox)	-	✓	✓	✓

Intersection Modalities

Distributed Knowledge of group G is the knowledge agents could have if they combined their information

$\Rightarrow$ In General not definable.

Intersection Modalities

Distributed Knowledge of group G is the knowledge agents could have if they combined their information

⇒ In General not definable.

- Logic of evidence (van Benthem&Pacuit 2011)
- Forcing Powers and Coalitions (Pauly 2002, Broersen *et al* 2007, van Benthem *et al* 2019)
- Deontic Logic: Norms from possibly different sources (Goble 2005, 2013, Klein & Marra 2019)
- Weak epistemic logics: Knowledge/Belief not closed under intersection, weakening... (Stalnaker 2006, Klein *et al* 2017)

Intersection Modalities

Distributed Knowledge of group G is the knowledge agents could have if they combined their information

$\Rightarrow$ In General not definable.

The General Goal…

…Axomatize Intersection Modalities in Neighborhood Modal Logic

The General Goal...

...Axomatize Intersection Modalities in Neighborhood Modal Logic

Plan

1. Neighborhood Modal Logic
2. Intersection modalities for groups
3. Soundness and completeness
4. Intersection Modalities in stronger logics
5. Axiomatization
6. Special Cases
7. Open ends and Future Work

The Setting: Minimal neighborhood logic

Fix a countable set of agents I, propositional variables $\mathfrak{P}$ and define the Language $\mathfrak{L}$

$$\varphi := p|\varphi \wedge \varphi|\neg\varphi|\Box_i\varphi \text{ for } p \in \mathfrak{P}, i \in I$$

The Setting: Minimal neighborhood logic

Fix a countable set of agents I, propositional variables $\mathfrak{P}$ and define the Language $\mathfrak{L}$

$$\varphi := p|\varphi \wedge \varphi|\neg\varphi|\Box_i\varphi \text{ for } p \in \mathfrak{P}, i \in I$$

Syntactically

The Minimal Neighborhood Logic is defined by

- Propositional tautologies
- $$\frac{\vdash \varphi \leftrightarrow \psi}{\vdash \Box_i\varphi \leftrightarrow \Box_i\psi} \text{ (REG)}$$

The Setting: Minimal neighborhood logic

Fix a countable set of agents I, propositional variables $\mathfrak{P}$ and define the Language $\mathfrak{L}$

$$\varphi := p|\varphi \wedge \varphi|\neg\varphi|\Box_i\varphi \text{ for } p \in \mathfrak{P}, i \in I$$

Syntactically

The Minimal Neighborhood Logic is defined by

- Propositional tautologies
- $\dfrac{\vdash \varphi \leftrightarrow \psi}{\vdash \Box_i\varphi \leftrightarrow \Box_i\psi}$ (REG)

Semantically

Basic Neighborhood models

- $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$
- W set of worlds
- $V : \mathfrak{P} \to \wp(W)$
- $\mathcal{N}_i : W \to \wp(\wp(W))$
- $\mathfrak{M}, w \models \Box_i\varphi \Leftrightarrow \|\varphi\|^{\mathfrak{M}} \in \mathcal{N}_i(w)$

- Minimal Neighborhood Logic is sound and complete w.r.t. the class of basic neighborhood models

The Question formally

Semantically: Intersection neighbourhoods:
For any *finite* set $G = \{i_1, \ldots, i_n\}$ of agents define

$$\mathcal{N}_G(w) = \{\bigcap_{1 \leq j \leq n} X_j | X_j \in \mathcal{N}_{i_j}(w)\}$$

The Question formally

Semantically: Intersection neighbourhoods:
For any *finite* set $G = \{i_1, \ldots, i_n\}$ of agents define

$$\mathcal{N}_G(w) = \{\bigcap_{1 \leq j \leq n} X_j | X_j \in \mathcal{N}_{i_j}(w)\}$$

Syntactically
Extend the language $\mathfrak{L}$ to $\mathfrak{L}_G$, by adding modalities $\Box_G$ for every finite set G and define

$$M, w \vDash \Box_G \varphi \Leftrightarrow \|\varphi\|^{\mathfrak{M}} \in \mathcal{N}_G(w)$$

Question: What is the logic of such intersection modalities, over the class of all frames?

Some Validites and Invalidities

- Propositional tautologies

Some Validites and Invalidities

- Propositional tautologies
- $\Box_G \varphi \wedge \Box_H \psi \rightarrow \Box_{G \cup H}(\varphi \wedge \psi)$ (A1)

Some Validites and Invalidities

- Propositional tautologies
- $\Box_G \varphi \wedge \Box_H \psi \rightarrow \Box_{G \cup H}(\varphi \wedge \psi)$ For G, H disjoint (A1)

Some Validites and Invalidities

- Propositional tautologies
- $\Box_G \varphi \wedge \Box_H \psi \rightarrow \Box_{G \cup H}(\varphi \wedge \psi)$ For G, H disjoint (A1)
- $\Box_{G \cup H} \varphi \not\rightarrow \Box_G \varphi$
- $\Box_G \varphi \not\rightarrow \Box_{G \cup H} \varphi$

Some Validites and Invalidities

- Propositional tautologies
- $\Box_G \varphi \wedge \Box_H \psi \rightarrow \Box_{G \cup H}(\varphi \wedge \psi)$ For G, H disjoint (A1)
- $\Box_{G \cup H} \varphi \not\rightarrow \Box_G \varphi$
- $\Box_G \varphi \not\rightarrow \Box_{G \cup H} \varphi$
- $\Box_{G \cup H} \top \rightarrow \Box_G \top$ (A2)

Some Validites and Invalidities

- Propositional tautologies
- $\Box_G \varphi \wedge \Box_H \psi \to \Box_{G\cup H}(\varphi \wedge \psi)$ For G, H disjoint (A1)
- $\Box_{G\cup H} \varphi \not\to \Box_G \varphi$
- $\Box_G \varphi \not\to \Box_{G\cup H} \varphi$
- $\Box_{G\cup H} \top \to \Box_G \top$ (A2)
- $(\Box_G \varphi \wedge \Box_{G\cup H\cup J} \varphi) \to \Box_{G\cup H} \varphi$ (A3)
- $(\Box_G \varphi \wedge \Box_H(\varphi \vee \psi)) \to \Box_{G\cup H} \varphi$ (A4)

Some Validites and Invalidities

- Propositional tautologies
- $\Box_G\varphi \wedge \Box_H\psi \rightarrow \Box_{G\cup H}(\varphi \wedge \psi)$ For G, H disjoint (A1)
- $\Box_{G\cup H}\varphi \not\rightarrow \Box_G\varphi$
- $\Box_G\varphi \not\rightarrow \Box_{G\cup H}\varphi$
- $\Box_{G\cup H}\top \rightarrow \Box_G\top$ (A2)
- $(\Box_G\varphi \wedge \Box_{G\cup H\cup J}\varphi) \rightarrow \Box_{G\cup H}\varphi$ (A3)
- $(\Box_G\varphi \wedge \Box_H(\varphi \vee \psi)) \rightarrow \Box_{G\cup H}\varphi$ (A4)

The Main Theorem

Minimal neighborhood logic together with the axioms (A1)-(A4) are sound and complete with respect to the class of minimal intersection frames.

The Proof Idea (In Pictures)

Canonical Model construction:

1. Start with set of MCS Λ_i

The Proof Idea (In Pictures)

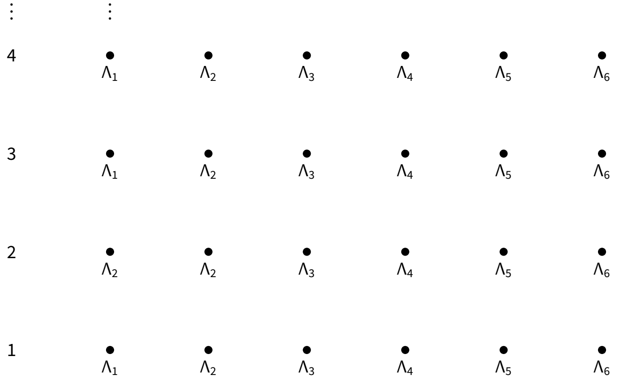
Canonical Model construction:

1. Start with set of MCS Λ_i
2. Get infinitely many copies of each

The Proof Idea (In Pictures)

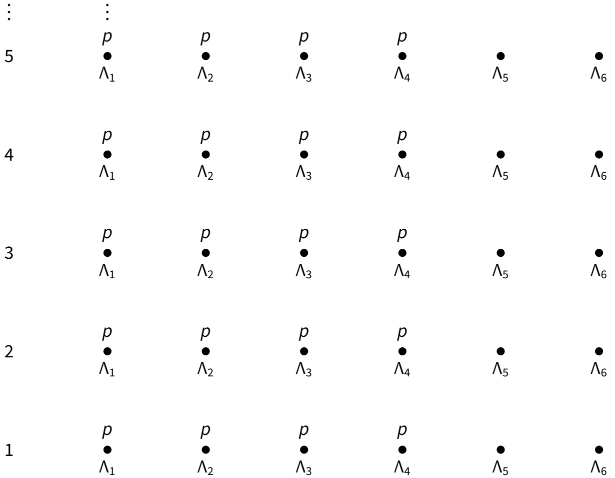
Canonical Model construction:

1. Start with set of MCS Λ_i
2. Get infinitely many copies of each
3. Cut puzzle pieces



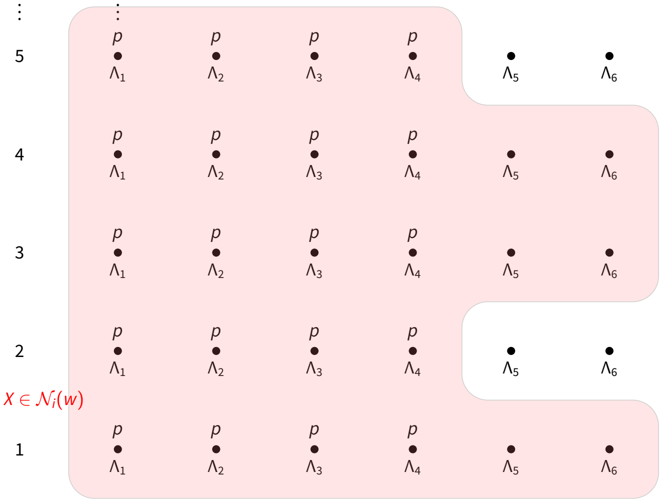
The Proof Idea

Goal: $\Box_{\{i,j,k\}}p, \quad \neg\Box_{\{i,j\}}p, \neg\Box_{\{i,k\}}p, \neg\Box_{\{j,k\}}p$



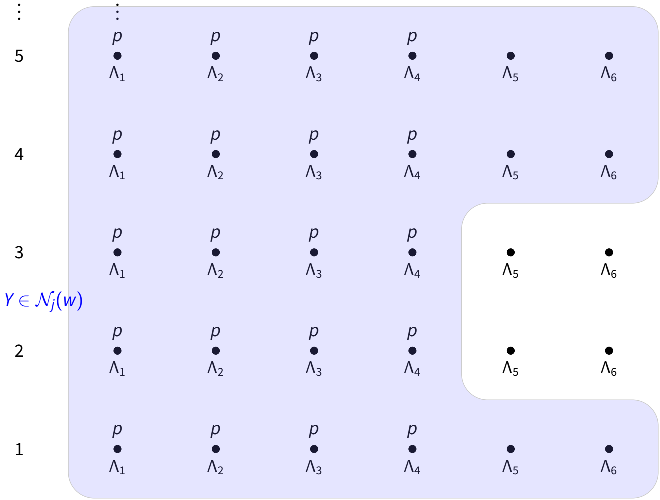
The Proof Idea

Goal: $\Box_{\{i,j,k\}}p$, $\neg\Box_{\{i,j\}}p, \neg\Box_{\{i,k\}}p, \neg\Box_{\{j,k\}}p$



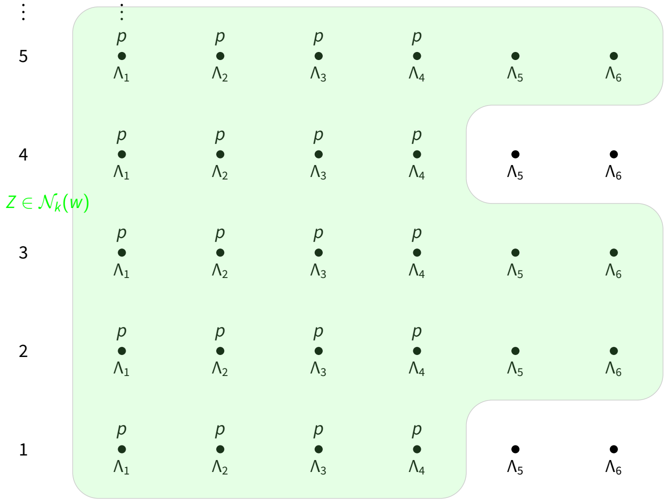
The Proof Idea

Goal: $\Box_{\{i,j,k\}}p, \quad \neg\Box_{\{i,j\}}p, \neg\Box_{\{i,k\}}p, \neg\Box_{\{j,k\}}p$



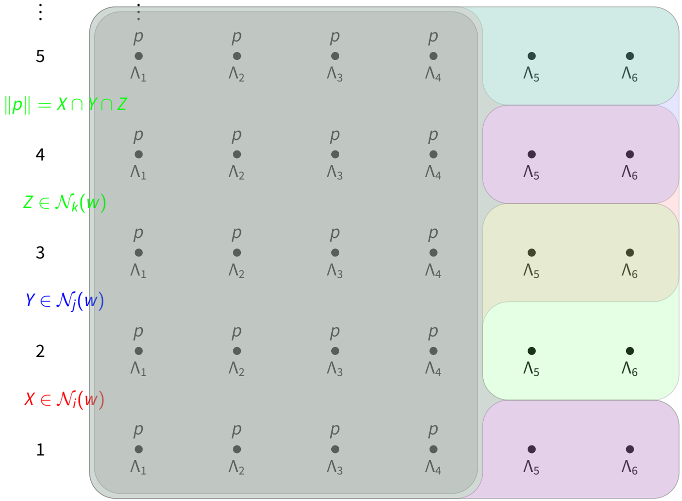
The Proof Idea

Goal: $\Box_{\{i,j,k\}}p, \quad \neg\Box_{\{i,j\}}p, \neg\Box_{\{i,k\}}p, \neg\Box_{\{j,k\}}p$



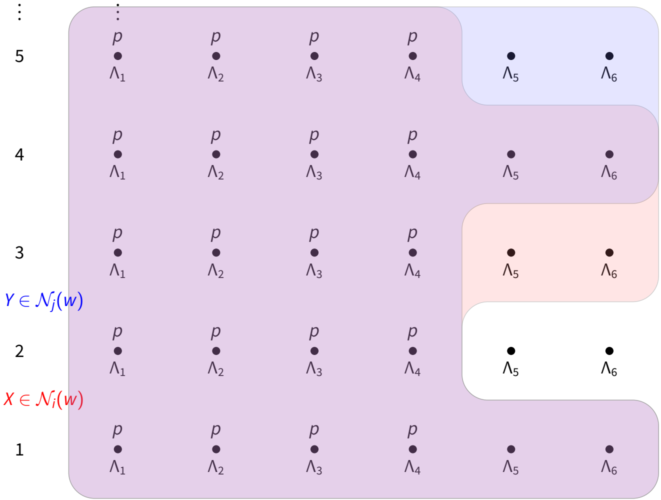
The Proof Idea

Goal: $\Box_{\{i,j,k\}}p, \quad \neg\Box_{\{i,j\}}p, \neg\Box_{\{i,k\}}p, \neg\Box_{\{j,k\}}p$



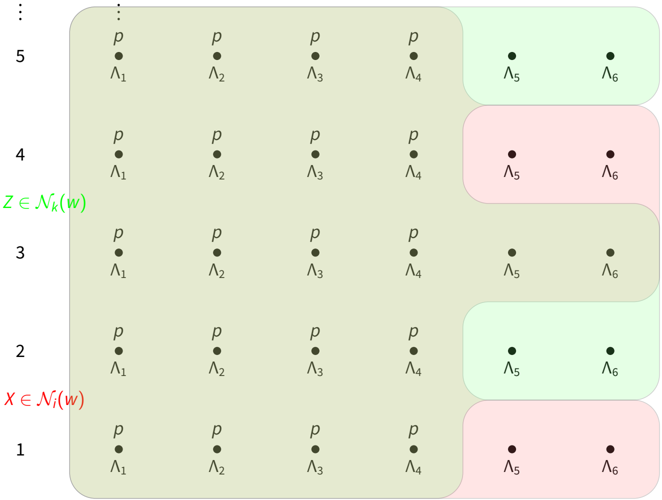
The Proof Idea

Goal: $\Box_{\{i,j,k\}}p$, $\neg\Box_{\{i,j\}}p$, $\neg\Box_{\{i,k\}}p$, $\neg\Box_{\{j,k\}}p$



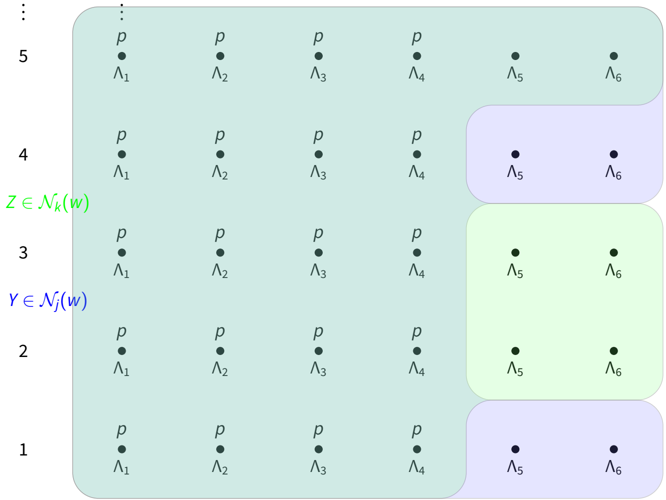
The Proof Idea

Goal: $\Box_{\{i,j,k\}} p, \quad \neg\Box_{\{i,j\}} p, \neg\Box_{\{i,k\}} p, \neg\Box_{\{j,k\}} p$

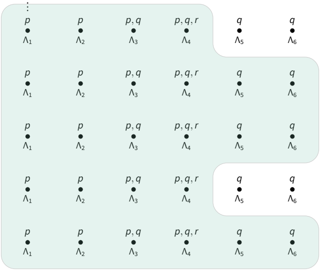


The Proof Idea

Goal: $\Box_{\{i,j,k\}}p, \quad \neg\Box_{\{i,j\}}p, \neg\Box_{\{i,k\}}p, \neg\Box_{\{j,k\}}p$



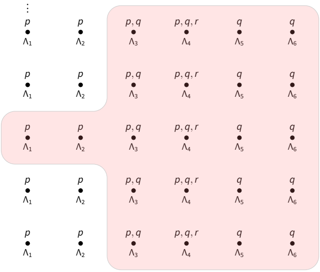
The Proof Idea



- Distribute puzzle pieces to agents:
  Agent i gets one for each $\Box_G \varphi$ with $i \in G, \varphi \in \mathcal{L}$
- Show that combining wrong pieces can't combine create additional beliefs

$\Rightarrow$ Some Combinatorics (Chinese Remainder Theorem)

The Proof Idea

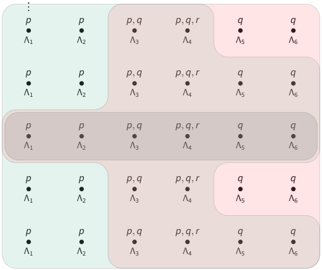


- Distribute puzzle pieces to agents:
  Agent i gets one for each $\Box_G\varphi$ with $i \in G, \varphi \in \mathcal{L}$
- Show that combining wrong pieces can't combine create additional beliefs

$\Rightarrow$ Some Combinatorics (Chinese Remainder Theorem)

The Proof Idea

UNIVERSITÄT BAYREUTH



- Distribute puzzle pieces to agents:
  Agent *i* gets one for each $\Box_G \varphi$ with $i \in G$, $\varphi \in \mathcal{L}$
- Show that combining wrong pieces can't combine create additional beliefs

$\Rightarrow$ Some Combinatorics (Chinese Remainder Theorem)

Plan

1. Neighborhood Modal Logic ✓
2. Intersection modalities for groups ✓
3. Soundness and completeness ✓
4. Intersection Modalities in stronger logics
5. Axiomatization
6. Special Cases
7. Open ends and Future Work
8. Generalizations and open ends

Richer Languages (1)

UNIVERSITÄT BAYREUTH

$\mathcal{N}_i(w)$ need not be closed under intersections ($\neq$ Kripke-semantics!)

Richer Languages (1)

UNIVERSITÄT BAYREUTH

$\mathcal{N}_i(w)$ need not be closed under intersections ($\neq$ Kripke-semantics!)

So evidence, norms, beliefs, powers, etc. of a single agent may not be closed under aggregation

Richer Languages (1)

$\mathcal{N}_i(w)$ need not be closed under intersections ($\neq$ Kripke-semantics!)

So evidence, norms, beliefs, powers, etc. of a single agent may not be closed under aggregation

How can we reason about their (hypothetical) intra-agent aggregation?

Richer Languages (1)

UNIVERSITÄT BAYREUTH

$\mathcal{N}_i(w)$ need not be closed under intersections ($\neq$ Kripke-semantics!)

So evidence, norms, beliefs, powers, etc. of a single agent may not be closed under aggregation

How can we reason about their (hypothetical) intra-agent aggregation?

E.g. $\Box_{i,i,i}\varphi$ could stand for: "combining three pieces of evidence of agent i, one arrives at φ"

Richer Languages (2)

We replace groups G with multisets M

$M : I \to \mathbb{N} \cup \{*\}$ (or $M \subseteq I \times \mathbb{N}$)
with

(i) $M(i) > 0$ for at most finitely many $i \in I$

(ii) $M(i) > 0$ for at least one $i \in I$

$M(i) = *$ means:
one can use an arbitrary number of neighbourhoods from $\mathcal{N}_i(w)$

$\mathbb{M}_*$; $\mathbb{M}_f$

$I(M) =_{\mathrm{df}} \{i \in I \mid M(i) \neq 0\}$.

Richer Languages (3)

$\mathfrak{L}_*$	$\varphi := p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box_M \varphi$	$p \in \mathfrak{P}$ and $M \in \mathbb{M}_*$
$\mathfrak{L}_*^{[\forall]}$	$\varphi := p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box_M \varphi \mid [\forall]\varphi$	$p \in \mathfrak{P}$ and $M \in \mathbb{M}_*$
$\mathfrak{L}_f$	$\varphi := p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box_M \varphi$	$p \in \mathfrak{P}$ and $M \in \mathbb{M}_f$
$\mathfrak{L}_f^{[\forall]}$	$\varphi := p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box_M \varphi \mid [\forall]\varphi$	$p \in \mathfrak{P}$ and $M \in \mathbb{M}_f$

Richer Languages (3)

UNIVERSITÄT BAYREUTH

Definition

Let D be a set, let $\mathcal{X}, \mathcal{Y} \subseteq \wp(\wp(D))$, and let $k \in \mathbb{N}$.

1. $\mathcal{X} \Cap \mathcal{Y} = \{X \cap Y \mid X \in \mathcal{X}, Y \in \mathcal{Y}\}$ is the pointwise intersection of $\mathcal{X}$ and $\mathcal{Y}$.
2. $\Cap^k \mathcal{X} = \{X_1 \cap \ldots \cap X_k \mid X_1, \ldots, X_k \in \mathcal{Y}\}$ is the pointwise k-intersection of $\mathcal{X}$ with itself.
3. $\Cap^* \mathcal{X} = \{\bigcap \mathcal{Y} \mid \mathcal{Y} \subseteq \mathcal{X}\}$ is the pointwise arbitrary intersection of $\mathcal{X}$ with itself.

Richer Languages (4')

Some facts (for $k, l \in \mathbb{N}$):

1. If $k < l$, then $\mathcal{X} = \Cap^1\mathcal{X} \subseteq \Cap^k\mathcal{X} \subseteq \Cap^l\mathcal{X} \subseteq \Cap^*\mathcal{X}$.
2. $(\Cap^k\mathcal{X}) \Cap (\Cap^l\mathcal{X}) = \Cap^{k+l}\mathcal{X}$
3. $(\Cap^k\mathcal{X}) \Cap (\Cap^*\mathcal{X}) = \Cap^*\mathcal{X}$
4. $\mathcal{X} \Cap \mathcal{Y} = \mathcal{Y} \Cap \mathcal{X}$
5. $(\mathcal{X} \Cap \mathcal{Y}) \Cap \mathcal{Z} = \mathcal{X} \Cap (\mathcal{Y} \Cap \mathcal{Z})$

Richer Languages (4')

Some facts (for $k, l \in \mathbb{N}$):

1. If $k < l$, then $\mathcal{X} = \Cap^1 \mathcal{X} \subseteq \Cap^k \mathcal{X} \subseteq \Cap^l \mathcal{X} \subseteq \Cap^* \mathcal{X}$.
2. $(\Cap^k \mathcal{X}) \Cap (\Cap^l \mathcal{X}) = \Cap^{k+l} \mathcal{X}$
3. $(\Cap^k \mathcal{X}) \Cap (\Cap^* \mathcal{X}) = \Cap^* \mathcal{X}$
4. $\mathcal{X} \Cap \mathcal{Y} = \mathcal{Y} \Cap \mathcal{X}$
5. $(\mathcal{X} \Cap \mathcal{Y}) \Cap \mathcal{Z} = \mathcal{X} \Cap (\mathcal{Y} \Cap \mathcal{Z})$

Remark. $\mathcal{X} \Cap \mathcal{Y}$ need not contain all members of $\mathcal{X}$ or $\mathcal{Y}$!

Richer Languages (4)

UNIVERSITÄT BAYREUTH

Definition

Let $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ be a neighbourhood model and let $M \in \mathbb{M}_*$, with $I(M) = \{i_1, \ldots, i_n\}$. The neighbourhood function $\mathcal{N}_M$ is defined as follows: for every $w \in W$,

$$\mathcal{N}_M(w) = (\Cap^{M(i_1)} \mathcal{N}_{i_1}(w)) \Cap \ldots \Cap (\Cap^{M(i_n)} \mathcal{N}_{i_n}(w))$$

Richer Languages (4)

Definition

Let $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ be a neighbourhood model and let $M \in \mathbb{M}_*$, with $I(M) = \{i_1, \ldots, i_n\}$. The neighbourhood function $\mathcal{N}_M$ is defined as follows: for every $w \in W$,

$$\mathcal{N}_M(w) = (\Cap^{M(i_1)} \mathcal{N}_{i_1}(w)) \Cap \ldots \Cap (\Cap^{M(i_n)} \mathcal{N}_{i_n}(w))$$

Definition

Where $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ and $w \in W$:

4. $\mathfrak{M}, w \models \Box_M \varphi$ iff $\|\varphi\|^{\mathfrak{M}} \in \mathcal{N}_M(w)$
5. $\mathfrak{M}, w \models [\forall]\varphi$ iff for all $w' \in W$, $\mathfrak{M}, w' \models \varphi$

Richer Languages (5)

New applications:

1. Evidence aggregation for single agents, belief-based evidence
2. resource-bounded abilities:

   $\Box_{1¥,1h}\varphi \wedge \Box_{3¥,2h}\psi \vdash \Box_{4¥,3h}(\varphi \wedge \psi)$

3. implicit norms of an agent, in view of its explicitly endorsed norms

Richer Languages (5)

New applications:

1. Evidence aggregation for single agents, belief-based evidence
2. resource-bounded abilities:

   $\Box_{1¥,1h}\varphi \wedge \Box_{3¥,2h}\psi \vdash \Box_{4¥,3h}(\varphi \wedge \psi)$

3. implicit norms of an agent, in view of its explicitly endorsed norms

Expressive power:

- level of models: just like intersection modalities over Kripke-models, pooling modalities are strictly more expressive
- for NBH semantics: already in the single-agent case!
- frame level: i.a. closure under arbitrary intersections

Axiomatization (1)

UNIVERSITÄT BAYREUTH

label	axiom/rule	$\mathbf{BL}_*^{[\forall]}$	$\mathbf{BL}_f^{[\forall]}$	$\mathbf{BL}_*$	$\mathbf{BL}_f$
(B1)	$(\Box_M\varphi \wedge \Box_N\psi) \rightarrow \Box_{M\sqcup N}(\varphi \wedge \psi)$	✓	✓	✓	✓
(B2)	$\Box_{M\sqcup N}\top \rightarrow \Box_M\top$	✓	✓	✓	✓
(B3)	$(\Box_M\varphi \wedge \Box_{M\sqcup N\sqcup N'}\varphi) \rightarrow \Box_{M\sqcup N}\varphi$	✓	✓	✓	✓
(B4)	$\Box_M\varphi \rightarrow \Box_{M^*}\varphi$	✓		✓	
(RE)	if $\vdash \varphi \leftrightarrow \psi$, then $\Box_M\varphi \rightarrow \Box_M\psi$	D	D	✓	✓
(RGE)	$[\forall](\varphi \leftrightarrow \psi) \rightarrow (\Box_M\varphi \rightarrow \Box_M\psi)$	✓	✓		
(S5)	all **S5**-axioms and rules for $[\forall]$	✓	✓		

Axiomatization (2)

frame condition	**BL** + axiom/rule
$\mathcal{N}_i(w) =\uparrow \mathcal{N}_i(w)$	$\Box_M\varphi \to \Box_M(\varphi \vee \psi)$
$\mathcal{N}_i(w) = \Cap^f \mathcal{N}_i(w)$	$\Box_M\varphi \to \Box_{M_f^-}\varphi$
$\mathcal{N}_i(w) = \Cap^* \mathcal{N}_i(w)$	$\Box_M\varphi \to \Box_{M_f}\varphi$
$\emptyset \notin \mathcal{N}_i(w)$	$\neg\Box_i\bot$
$\emptyset \notin \mathcal{N}_M(w)$	$\neg\Box_M\bot$
$\top \in \mathcal{N}_M(w)$ / $\top \in \mathcal{N}_i(w)$	$\Box_M\top$ / $\Box_i\top$
$w \in \mathcal{N}_i(w)$	$\Box_M\varphi \to \varphi$
$\mathcal{N}_i(w) = \mathcal{N}_i(w')$	$\Box_M\varphi \to [\forall]\Box_M\varphi$
$\vdots$	$\vdots$

Axiomatization (2)

UNIVERSITÄT BAYREUTH

frame condition	**BL** + axiom/rule
$\mathcal{N}_i(w) = \uparrow \mathcal{N}_i(w)$	$\Box_M\varphi \rightarrow \Box_M(\varphi \vee \psi)$
$\mathcal{N}_i(w) = \Cap^f \mathcal{N}_i(w)$	$\Box_M\varphi \rightarrow \Box_{M_f^-}\varphi$
$\mathcal{N}_i(w) = \Cap^* \mathcal{N}_i(w)$	$\Box_M\varphi \rightarrow \Box_{M_f}\varphi$
$\emptyset \notin \mathcal{N}_i(w)$	$\neg\Box_i\bot$
$\emptyset \notin \mathcal{N}_M(w)$	$\neg\Box_M\bot$
$\top \in \mathcal{N}_M(w)$ / $\top \in \mathcal{N}_i(w)$	$\Box_M\top$ / $\Box_i\top$
$w \in \mathcal{N}_i(w)$	$\Box_M\varphi \rightarrow \varphi$
$\mathcal{N}_i(w) = \mathcal{N}_i(w')$	$\Box_M\varphi \rightarrow [\forall]\Box_M\varphi$
$\vdots$	$\vdots$

Axiomatization (2')

Remark. For many variants, the single-index axiom does not entail the multiset-counterpart (and hence completeness requires the multiset-counterparts).

Axiomatization (3)

UNIVERSITÄT BAYREUTH

Theorem

All the mentioned logis are sound and strongly complete and enjoy the finite model property.

Proof. By playing with puzzle pieces!

Axiomatization (3)

Theorem

All the mentioned logis are sound and strongly complete and enjoy the finite model property.

Proof. By playing with puzzle pieces!

Corollary

All the mentioned logics are decidable.

Axiomatization (4)

cf. the group case: create copies, make puzzle pieces for each agent $i \in G$ and each proposition φ

Axiomatization (4)

UNIVERSITÄT BAYREUTH

cf. the group case: create copies, make puzzle pieces for each agent $i \in G$ and each proposition φ

Let $\delta(M) = \{(i,k) \mid i \in I(M), k \leq M(i)\}$; the members of $\delta(M)$ will play the role of “agents”; players & roles

Axiomatization (4)

cf. the group case: create copies, make puzzle pieces for each agent $i \in G$ and each proposition φ

Let $\delta(M) = \{(i,k) \mid i \in I(M), k \leq M(i)\}$; the members of $\delta(M)$ will play the role of "agents"; players & roles

Labels for worlds: $f : \mathbb{M}_* \times \mathfrak{L} \to I \times \mathbb{N}$ where, for all (M, φ), $f(M, \varphi) \in \delta(M)$

Axiomatization (4)

UNIVERSITÄT BAYREUTH

cf. the group case: create copies, make puzzle pieces for each agent $i \in G$ and each proposition φ

Let $\delta(M) = \{(i, k) \mid i \in I(M), k \leq M(i)\}$; the members of $\delta(M)$ will play the role of "agents"; players & roles

Labels for worlds: $f : \mathbb{M}_* \times \mathfrak{L} \to I \times \mathbb{N}$ where, for all (M, φ), $f(M, \varphi) \in \delta(M)$

Game-theoretic interpretation: if $f(M, \varphi) = (i, k)$ then, (only) in role k, (only) player i can cut away $\neg\varphi$-worlds

Axiomatization (4)

cf. the group case: create copies, make puzzle pieces for each agent $i \in G$ and each proposition φ

Let $\delta(M) = \{(i, k) \mid i \in I(M), k \leq M(i)\}$; the members of $\delta(M)$ will play the role of "agents"; players & roles

Labels for worlds: $f : \mathbb{M}_* \times \mathfrak{L} \to I \times \mathbb{N}$ where, for all (M, φ), $f(M, \varphi) \in \delta(M)$

Game-theoretic interpretation: if $f(M, \varphi) = (i, k)$ then, (only) in role k, (only) player i can cut away $\neg\varphi$-worlds

Only if some $\delta(N)$ contains all pairs $(i, k) \in \delta(M)$, it can cut away all the $\neg\varphi$-worlds, thus obtaining $\|\varphi\|$

Axiomatization (5)

UNIVERSITÄT BAYREUTH

For every $i \in I$,

$$\mathcal{N}_i(\Lambda, f) = \{X_{i,k}^{M,\varphi} \mid \Box_M \varphi \in \Lambda, (i,k) \in \delta^{\Theta}(M)\}$$

where, for all $(M, \varphi) \in \mathcal{D}_{\Theta}$ and $(i,k) \in \delta^{\Theta}(M)$,

$$X_{i,k}^{M,\varphi} = \{(\Lambda, f) \in W \mid \varphi \in \Lambda \text{ or } f(M, \varphi) \neq (i,k)\}$$

Axiomatization (5)

For every $i \in I$,

$$\mathcal{N}_i(\Lambda, f) = \{X_{i,k}^{M,\varphi} \mid \Box_M \varphi \in \Lambda, (i,k) \in \delta^{\Theta}(M)\}$$

where, for all $(M, \varphi) \in \mathcal{D}_{\Theta}$ and $(i,k) \in \delta^{\Theta}(M)$,

$$X_{i,k}^{M,\varphi} = \{(\Lambda, f) \in W \mid \varphi \in \Lambda \text{ or } f(M, \varphi) \neq (i,k)\}$$

Finite model property: reduce the domain and range of the f's, i.v.o. the formula to be falsified...

Special Cases (0)

Recall:

1. Kripke-semantics is a special case of NBH semantics: augmented models (cf. infra)
2. intersection modalities in Kripke-semantics:

$$\mathfrak{M}, w \models \Box_G \varphi \text{ iff } \bigcap_{i \in G} R_i(w) \subseteq \|\varphi\|^{\mathfrak{M}}$$

(study of distributed belief, distributed knowledge, etc.)

Special Cases (0)

Recall:

1. Kripke-semantics is a special case of NBH semantics: augmented models (cf. infra)
2. intersection modalities in Kripke-semantics:

$$\mathfrak{M}, w \models \Box_G \varphi \text{ iff } \bigcap_{i \in G} R_i(w) \subseteq \|\varphi\|^{\mathfrak{M}}$$

(study of distributed belief, distributed knowledge, etc.)

How do intersection modalities and intersections of relations relate to pooling modalities and pointwise intersection?

Special Cases (1)

Definition

$\mathcal{X} \subseteq \wp(D)$ is augmented iff

(i) $\mathcal{X} = \mathcal{X}^{\uparrow}$

(ii) $D \in \mathcal{X}$

(iii) $\mathcal{X} = \Cap^{*}\mathcal{X}$

$\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ is augmented iff for all $w \in W$ and all $i \in I$, $\mathcal{N}_i(w)$ is augmented.

Lemma

For augmented NBH models, $\Box_M \varphi$ and $\Box_{I(M)} \varphi$ are equivalent.

Special Cases (1)

UNIVERSITÄT BAYREUTH

Definition

$\mathcal{X} \subseteq \wp(D)$ is augmented iff

(i) $\mathcal{X} = \mathcal{X}^{\uparrow}$

(ii) $D \in \mathcal{X}$

(iii) $\mathcal{X} = \Cap^{*}\mathcal{X}$

$\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ is augmented iff for all $w \in W$ and all $i \in I$, $\mathcal{N}_i(w)$ is augmented.

Lemma

For augmented NBH models, $\Box_M \varphi$ and $\Box_{I(M)} \varphi$ are equivalent.

Standard transformation:

- $\mathcal{R}_{\mathcal{N}}(w) = \bigcap \mathcal{N}(w)$
- $\mathcal{N}_{\mathcal{R}}(w) = \{X \subseteq W \mid \mathcal{R}(w) \subseteq X\}$.

Special Cases (2)

UNIVERSITÄT BAYREUTH

Theorem

Every augmented NBH model can be transformed into an $\mathfrak{L}_G^{[\forall]}$-equivalent relational model (using the standard transformation), and vice versa.

Special Cases (2)

UNIVERSITÄT BAYREUTH

Theorem
Every augmented NBH model can be transformed into an $\mathfrak{L}_G^{[\forall]}$-equivalent relational model (using the standard transformation), and vice versa.

Corollary
For all $\varphi \in \mathfrak{L}_G^{[\forall]}$, φ is valid on the class of all relational models iff φ is valid on the class of all augmented neighbourhood models.

Special Cases (3)

Where $G \in \mathbb{G}$, let $G^* = \{(i,*) \mid i \in G\} \cup \{(i,0) \mid i \notin G\}$.

Define $tr : \mathfrak{L}_G^{[\forall]} \to \mathfrak{L}_*^{[\forall]}$ as follows:

$$tr(\varphi) = \varphi \text{ for all } \varphi \in \mathfrak{P}$$

$$tr(\neg\varphi) = \neg tr(\varphi)$$

$$tr(\varphi \vee \psi) = tr(\varphi) \vee tr(\psi)$$

$$tr([\forall]\varphi) = [\forall] tr(\varphi)$$

$$tr(\Box_G \varphi) = \bigvee_{\emptyset \subset H \subseteq G} \Box_{H^*} tr(\varphi) \vee [\forall] tr(\varphi)$$

Theorem

For all $\varphi \in \mathfrak{L}_G^{[\forall]}$, φ is valid on the class of all relational models iff $tr(\varphi)$ is valid on the class of all monotonic models.

Special Cases (4)

Monotonic models $\Leftrightarrow$ inexact semantic clause

Definition

Let $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ be a model and $w \in W$. Then

$$\mathfrak{M}, w \vDash_i \Box_M \varphi \text{ iff there is an } X \in \mathcal{N}_M(w) \text{ such that } X \subseteq \|\varphi\|^{\mathfrak{M}}.$$

Lemma

Let $\mathbb{X} \subseteq \wp(\wp(\wp(D)))$ for some non-empty domain D. Then $\Cap\{\mathcal{X}^{\uparrow} \mid \mathcal{X} \in \mathbb{X}\} = (\Cap \mathbb{X})^{\uparrow}$.

Corollary

For all models $\mathfrak{M}$: $\mathfrak{M}, w \vDash_i \varphi$ iff $\mathfrak{M}^{\uparrow}, w \vDash \varphi$.

Corollary

φ is valid using the inexact semantic clause iff φ is valid on the class of monotonic models.

Open Issues and Future Work

UNIVERSITÄT BAYREUTH

- pointwise boolean operations, e.g. $\Box_{(\alpha \sqcup \beta) \sqcap (\overline{\gamma})^*} \varphi$; relation to truthmaker semantics
- non-empty intersections for the inexact semantic clause?
- other frame conditions: positive & negative introspection, . . .
- beliefs based on (co-compatible) distributed evidence
- shared vs. distributed information over NBH models
- alternative proof via quasi-models?
- conditional modal logics?

$\vdots$

UNIVERSITÄT
BAYREUTH

Thank you!

References

Frederik Van De Putte and Dominik Klein. *Pointwise Intersection in Neighbourhood Modal Logic*. Advances in Modal Logic (AiML 12), College Publications pp. 591-610.
→ dominikklein.dk

Eric Pacuit. *Neighborhood Semantics for Modal Logic*. Springer 2017.

David Lewis. *Intensional logics without iterative axioms*. Journal of Philosophical Logic 3, 457 – 466..