Gödel's Program and Large Cardinals

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Outline

1. Gödel's Program
   - The Phenomenon of Incompleteness
   - Gödel's Proposals
2. Large Cardinals
   - General Introduction (Inaccessibility, Mahlo)
   - Compactness
   - Measurability and Elementary Embeddings

The Phenomenon of Incompleteness

Theorem (Gödel's First Incompleteness Theorem)

A sufficiently strong axiomatization of mathematics is incomplete unless it is inconsistent.

- Con(ZFC) is independent from ZFC by the Gödel's Second Incompleteness Theorem.
- Cantor's Continuum Hypothesis (CH): The size of $\mathbb{R}$ is ω_1? Cohen & Gödel: The size of the set of real numbers is not decided by ZFC.

Gödel's Proposals

- Should we satisfy with the results of independence?
- Realism: "...a proof of the undecidability... would by no means solve the problem." A mathematical statement "must be either true or false." "...its undecidability from the axioms being assumed today can only mean that these axioms do not contain a complete description of that reality." [Gödel 1964, 263-264]
- Gödel's Program: Discover new axioms to determine those statements, e.g. to disprove CH.

Intrinsic and Extrinsic Justification

Why don't we just make $\neg$CH an axiom?
Truth is not "playing dice".

- A candidate of new axiom can not be justified mathematically since it must be independent from the current axioms.
- Two kinds of philosophical justification:
  - Intrinsic
    "the axiomatic system of set theory... be supplemented without arbitrariness by new axioms which only unfold the content of the concept of set..." [Gödel 1964, 264]
  - Extrinsic
    "inductively by studying its.. fruitfulness in consequences, in particular,... consequences demonstrable without the new axiom, whose proofs with the help of new axiom... are considerably simpler and easier to discover..." [Gödel 1964, 265]

The Axioms of Large Cardinals

- The axioms of large cardinals (or higher infinity) are those statements which asserts the existence of some large cardinals. They are the stronger versions of the Axiom of Infinity. e.g.:

Definition (Inaccessible Cardinal)

An uncountable cardinal is (strongly) inaccessible if and only if it is regular and strong limit.

Intuition: An inaccessible cardinal can not be reached by the set theory operations from below.

- The axioms of large cardinals are somewhat standard extension of ZFC.

Gödel on Incompleteness

... the true reason for the incompleteness inherent in all formal systems of mathematics is that the formation of ever higher types can be continued into the transfinite... while in any formal system at most denumerably many of them are available. For it can be shown that the undecidable propositions constructed here become decidable whenever appropriate higher types are added (for example, the type ω to the system P [Peano Arithmetic]). An analogous situation prevails for the axiom system of set theory. [Gödel 1931]

The Gödel Hierarchy

Definition (The order of consistency strength)

$T_1 < T_2$ if and only if "T_1 is consistent" is a theorem of T_2.

- The Gödel Hierarchy [Simpson 2009]:
  Gödel Hierarchy
- Note that "P is consistent" is provable in Z_2 and ZFC because "ω exists" is represented in these theory.

Large Cardinals as benchmarks in Gödel Hierarchy

Theorem

ZFC + "there is an inaccessible cardinal" $\vdash$ *Con(ZFC).*

Proof.

If κ is an inaccessible cardinal, then $V_\kappa \models$ ZFC. $\square$

Remark:

- $V_\omega \models \mathrm{ZFC} - \mathrm{Inf}$, thus $\mathrm{ZFC} \vdash \mathrm{Con}(\mathrm{ZFC} - \mathrm{Inf})$.
- The order of large cardinals:
  - $\leq$, implication
  - $<$, relative consistency implication

Mahlo Cardinal

Definition (Mahlo Cardinal)

An inaccessible cardinal κ is Mahlo if and only if the set of all regular cardinals less than κ is stationary.

Theorem (Mahlo>Inaccessible)

If κ is a Mahlo cardinal then

- *the set of all inaccessible cardinals below κ is stationary, thus has size κ*
- $V_\kappa \models ZFC +$ *"there is a inaccessible cardinal"*

Generalization of Compactness I

Definition (Infinitary Predicate Language)

$\mathcal{L}_{\kappa\lambda}$ is the language consists of

- max$\{\kappa, \lambda\}$ variables;
- (finitary) predicate, function, and constant symbols(no restrict on the cardinality of non-logical symbols);
- negative $\neg\varphi$ and infinitary connectives $\bigvee_{\xi<\alpha} \varphi_\xi$ for $\alpha < \kappa$;
- infinitary quantifiers $\exists_{\xi<\alpha} v_\xi$ for $\alpha < \lambda$;

- $\mathcal{L}_{\omega\omega}$ is the usual language of first order logic.
- The semantics of $\mathcal{L}_{\kappa\lambda}$ is naturally defined.

Generalization of Compactness II

Definition (κ-satisfiable)

A set of formula Σ is κ-satisfiable if every subset of Σ with cardinality less than κ is satisfiable.

Definition (Weakly Compact Cardinal)

A uncountable cardinal κ is weakly compact if and only if any collection of formulas of $\mathcal{L}_{\kappa\kappa}$ with cardinality at most κ, if κ-satisfiable, is satisfiable.

Definition (Strongly Compact Cardinal)

A uncountable cardinal κ is strongly compact if and only if any collection of formulas of $\mathcal{L}_{\kappa\kappa}$, if κ-satisfiable, is satisfiable.

Remarks on Weakly/Strongly Compact Cardinal

- ω is "weakly compact".
- Clearly strongly compact $\geq$ weakly compact.

Theorem (Weakly Compact $\geq$ Inaccessible)

If κ is weakly compact, then κ is inaccessible.

Proof.

-

$\square$

Remarks on Weakly/Strongly Compact Cardinal

- ω is "weakly compact".
- Clearly strongly compact $\geq$ weakly compact.

Theorem (Weakly Compact $\geq$ Inaccessible)

If κ is weakly compact, then κ is inaccessible.

Proof.

- Regular: Assume $X \subseteq \kappa$ is unbounded and $|X| < \kappa$.

$$\{c \neq c_\alpha \mid \alpha < \kappa\} \cup \{\bigvee_{\beta \in X} \bigvee_{\alpha < \beta} c = c_\alpha\}$$

is κ-satisfiable, not satisfiable.

□

Remarks on Weakly/Strongly Compact Cardinal

- ω is "weakly compact".
- Clearly strongly compact $\geq$ weakly compact.

Theorem (Weakly Compact $\geq$ Inaccessible)

If κ is weakly compact, then κ is inaccessible.

Proof.

- Strong limit: If $2^\lambda \geq \kappa$ for some $\lambda < \kappa$, then

$$\{\bigwedge_{\alpha<\lambda}((c_\alpha = d_\alpha^0 \vee c_\alpha = d_\alpha^1) \wedge d_\alpha^0 =\neq d_\alpha^1)\} \cup \{\bigvee_{\alpha<\lambda}(c_\alpha \neq d_\alpha^{f(\alpha)a}) \mid f \in 2^\lambda\}$$

is 2^λ-satisfiable and hence κ-satisfiable, not satisfiable.

Equivalent Definitions of Weakly Compact Cardinal I

Theorem

κ is weakly compact if and only if κ is inaccessible and has the tree property: every tree of height κ whose levels have cardinality $< \kappa$ has a branch of cardinality κ.

Theorem

κ is weakly compact if and only if $\kappa \to (\kappa)^2$.

Ramsey Theorem and Ramsey Cardinal

Definition (Arrow Notation)

$\kappa \to (\lambda)^n_m$ means:
For every coloration of points in $[\kappa]^n$ with m colors, there exists a $S \subseteq \kappa$ with cardinality λ such that points in $[S]^n$ have the same color.

Theorem (Ramsey Theorem)

$$\omega \to (\omega)^2.$$

Definition (Ramsey Cardinal)

A cardinal κ is a Ramsey cardinal, if $\kappa \to (\kappa)^{<\omega}$.

Equivalent Definitions of Weakly Compact Cardinal II

Theorem

κ is weakly compact if and only if for any $R \subseteq V_\kappa$ there is a transitive set $M \neq V_\kappa$ and an $S \subseteq M$ such that $(V_\kappa, \in, R) \prec (M, \in, S)$.

Definition (Indescribability)

A cardinal κ is Π^n_m-indescribable if whenever $U \subseteq V_\kappa$ and σ is a Π^n_m sentence such that $(V_\kappa, \in, U) \vDash \sigma$, then there is a $\alpha < \kappa$ such that $(V_\alpha, \in, U \cap V_\alpha) \vDash \sigma$.

Theorem

A cardinal κ is weakly compact if and only if κ is Π^1_1-indescribable.

Weakly Compact Cardinal>Mahlo Cardinal

Theorem (Weakly Compact $>$ Mahlo)

A weakly compact cardinal κ is Mahlo. Moreover, the set of Mahlo cardinal below κ is stationary.

Proof.

In the model $(V_\kappa, \in, C)$, κ being regular is expressible by a Π^1_1 sentence. So is "κ is Mahlo". $\square$

Another Definition of Strongly Compact Cardinal

Theorem

κ is strongly compact if and only if for any set S, every κ-complete filter on S can be extended to a κ-complete ultrafilter.

Proof.

[Kanamori 2003, P.37] $\square$

The Measure Problem I

How to measure the length of an arbitrary subset of $\mathbb{R}$?

Definition (Lebesgue Measure)

A function $\mu : P(\mathbb{R}) \mapsto \mathbb{R}$ is a Lebesgue measure if it satisfies:

- Extends length: For every interval I, $\mu(I) = \text{length}(I)$.
- Monotone: If $A \subseteq B$, then $0 \leq \mu(A) \leq \mu(B) \leq \infty$.
- Translation invariant: $\mu(A + x_0) = \mu(A)$.
- Countably additive: If $\{A_i \mid i < \omega\}$ is pairwise disjoint, then $\mu(\bigcup_{i=0}^{\infty} A_i) = \sum_{i=0}^{\infty} \mu(A_i)$.

Vitali (1905) showed there is no sure measure (AC).

The Measure Problem II

Is there a measure over $\mathbb{R}$ extending Lebesgue measure without translation invariant?

Definition (Measure over Sets)

A function $\mu : P(S) \mapsto \mathbb{R}$ is a measure over S, if

- $\mu(S) > 0$
- Monotone
- nontrivial: $\mu(\{a\}) = 0$ for all $a \in S$.
- Countably additive

Definition (Real-valued Measurable)

A cardinal κ is real-valued measurable if and only if there is a κ-additive measure over κ.

The Measurable Cardinal

Theorem

If there exists a κ-additive measure μ over κ, then

- *either μ is atomless, $\kappa \leq 2^{\aleph_0}$, and there is a measure over $\mathbb{R}$ extending Lebesgue measure;*
- *or there is a two-valued κ-additive measure over μ.*

A κ-additive two-valued measure corresponds to a κ-complete nonprincipal ultrafilter over S.

Definition (Measurable Cardinal)

An uncountable cardinal κ is measurable if there exists a κ-complete nonprincipal ultrafilter on κ.

Strength of Measurable Cardinal

Theorem

- *Every strongly compact cardinal is measurable.*
- *Every measurable cardinal is inaccessible, Mahlo, weakly compact.*

Proof.

-

$\square$

Strength of Measurable Cardinal

Theorem

- *Every strongly compact cardinal is measurable.*
- *Every measurable cardinal is inaccessible, Mahlo, weakly compact.*

Proof.

- *Strongly compact $\geq$ measurable:*
  - *A strongly compact cardinal is regular.*
  - *Then $F = \{X \subseteq \kappa \mid |\kappa \setminus X| < \kappa\}$ is κ-complete.*

□

Strength of Measurable Cardinal

Theorem

- *Every strongly compact cardinal is measurable.*
- *Every measurable cardinal is inaccessible, Mahlo, weakly compact.*

Proof.

- *Measurable $\geq$ inaccessible:*
  - *Regular:* $\emptyset = \bigcap_{\alpha<\mu}(\kappa \setminus \xi_\alpha) \in U$
  - *Strong limit:*

$$h(\alpha) = \begin{cases} 0 & \text{if } \{f \in S \mid f(\alpha) = 0\} \in U, \\ 1 & \text{otherwise.} \end{cases}$$

$$\{h\} = \bigcap_{\alpha<\lambda} \{f \in S \mid f(\alpha) = h(\alpha)\} \in U.$$

Strength of Measurable Cardinal

Theorem

- *Every strongly compact cardinal is measurable.*
- *Every measurable cardinal is inaccessible, Mahlo, weakly compact.*

Proof.

- *Measurable $\geq$ Mahlo:*
  [Jech 2002, P.135]

□

Strength of Measurable Cardinal

Theorem

- *Every strongly compact cardinal is measurable.*
- *Every measurable cardinal is inaccessible, Mahlo, weakly compact.*

Proof.

- *Measurable $\geq$ weakly compact:*
  let T be the tree of height κ and with every level of size $< \kappa$. Then $|T| = \kappa$. Let

  $$B = \{x \in T \mid \{y | y > x\} \in U\}.$$

  Then B is a branch of size κ.

$\square$

Ultrapower and Elementary Embedding I

Definition (Ultrapower)

Let U be an ultrafilter on κ, for $f, g : \kappa \mapsto V$, define

$$f =^* g \text{ iff } \{\alpha < \kappa \mid f(\alpha) = g(\alpha)\} \in U$$
$$f \in^* g \text{ iff } \{\alpha < \kappa \mid f(\alpha) \in g(\alpha)\} \in U$$

$\mathrm{Ult}_U(V)$ = the class of all $[f]$, where $f : \kappa \mapsto V$.

Lemma

If U is an ω_1-complete ultrafilter, then $\in^*$ is well-founded on $\mathrm{Ult}_U(V)$.

Ultrapower and Elementary Embedding II

- By the fundamental theorem of ultraproducts, the mapping $a \mapsto [c_a]$ where c_a is the constant function onto $\{a\}$ is an elementary embedding of V into $\mathrm{Ult}_U(V)$.
- By the Mostowski collapsing, if U is ω_1-complete, there exists a isomorphism π of $\mathrm{Ult}_U(V)$ onto some a transitive class Ult.
- $j(a) = \pi([c_a])$ is the canonical elementary embedding from V into Ult.

Properties of the Canonical Elementary Embedding

- $j(\alpha)$ is an ordinal if α is.
- $j(\alpha) < j(\beta)$ whenever $\alpha < \beta$, hence $\alpha \leq j(\alpha)$.
- If U is λ-complete, then $j(\alpha) = \alpha$ for all $\alpha < \lambda$.
- If κ is measurable and U is a nonprincipal κ-complete ultrafilter on κ, then $\kappa \leq \pi[d] < j(\alpha)$, where d is the diagonal function on κ. Hence j is a nontrivial elementary embedding.

Properties of the Canonical Elementary Embedding

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More on Measurable Cardinal I

Theorem

If there is a measurable cardinal then $V \neq L$.

Proof.

- Ult is an inner model. If $V = L$, then $V = \mathrm{Ult} = L$.
- Since j is an elementary embedding, if κ is the least measurable cardinal, so is $j(\kappa)$ (in Ult). But $\kappa < j(\kappa)$.

$\square$

More on Measurable Cardinal II

Theorem

If j is a nontrivial elementary embedding of V, then there exists a measurable cardinal.

Proof.

Let κ be the first moved ordinal. Define the ultrafilter D on κ as

$$X \in D \text{ if and only if } \kappa \in j(X)$$

$\square$

More on Measurable Cardinal III

Theorem (Measurable $>$ Weakly Compact)

If κ is a measurable cardinal and D is a normal measure on κ then the set of all weakly compact cardinal below κ is in D.

Proof.

- $P^M(\kappa) = P(\kappa)$.
- Thus κ is also weakly compact in Ult_D.
- $[d]_D = \kappa$.
- By the fundamental theorem of ultraproduct, $\{\alpha \mid \alpha \text{ is weakly compact}\} \in D$.

$\square$

Summary

- We have shown so far: strongly compact $\geq$ measurable $>$ weakly compact $>$ Mahlo $>$ inaccessible $> \omega \ldots$.
- But this is only the tip of the iceberg, and these are relatively "small" large cardinal. Chart of Cardinals
- The large cardinals seem to be linearly order by (consistency) strength.
- How to explain the phenomenon?

- Outlook
  - Very large cardinals.
  - Ω-logic.
  - Other approaches in extending the set theory.

Gödel on the Axioms of Infinity

In set theory, e.g. the successive extensions can most conveniently be represented by stronger and stronger axioms of infinity. It is certainly impossible to give a combinatorial and decidable characterization of what an axiom of infinity is but there might exist, e.g. a certain (decidable) formal structure and which in addition is true. Such a concept of demonstrability might have the required closure property, i.e. the following could be true: Any proof for a set-theoretic theorem in the next higher system above set theory... is replaceable by a proof from such an axiom of infinity. It is not impossible that for such a concept of demonstrability some completeness theorem would hold which would say that every proposition expressible in set theory is decidable from the present axioms plus some true assertion about the largeness of the universe of all sets. [Gödel 1946]

For Further Reading

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