Algebraic semantics and its application to Intuitionistic Logic

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need not be given any meaning;

semantics : give the meaning of a formal language;

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question: What is semantics ?(mathematically, formally)

Example 1

Propositional Logic

Symbol $\{\neg\ ,\ \rightarrow\ ,\ (\ ,\ )\} \cup (\mathbf{P} =)\{p_0\ ,\ p_1\ ,\ \ldots\}$

Formula $\phi ::= p \mid \neg\phi \mid (\phi \rightarrow \phi)$, where $p \in \mathbf{P}$

classical sementics for Propositional Logic

- a set $\{0, 1\}$
- $\rightarrow^*$ and $\neg^*$ are the ordinary truth value funtions of $\rightarrow$ and $\neg$
- given an assignment (that is , a mapping from $\mathbf{P}$ to $\{0, 1\}$) $\tilde{\sigma}$, we extend its domain to *Form* in the following way:

$$\sigma(p) = \tilde{\sigma}(p)$$

$$\sigma(\alpha \rightarrow \beta) = \sigma(\alpha) \rightarrow^* \sigma(\beta)$$

$$\sigma(\neg\alpha) = \neg^*\sigma(\alpha)$$

- α is *true* under $(\{0, 1\}, \rightarrow^*, \neg^*)$ and $\tilde{\sigma}$ *iff* $\sigma(\alpha) = 1$.

Example 2

Propositional Logic

Symbol $\{\neg\ ,\ \rightarrow\ ,\ (\ ,\ )\} \cup (\mathbf{P} =)\{p_0\ ,\ p_1\ ,\ ...\}$

Formula $\phi ::= p \mid \neg\phi \mid (\phi \rightarrow \phi)$, where $p \in \mathbf{P}$

BA sementics for Propositional Logic

- a Boolean algebra $\mathfrak{A} = (A, +, -, 0)$
- given an assignment (i.e. a mapping from **P** to A) $\tilde{\sigma}$, we extend its domain to *Form* in the following way:

$$\sigma(p) = \tilde{\sigma}(p)$$

$$\sigma(\alpha \rightarrow \beta) = (-\sigma(\alpha)) + \sigma(\beta)$$

$$\sigma(\neg\alpha) = -\sigma(\alpha)$$

- α is *true* under $\mathfrak{A}$ and $\tilde{\sigma}$ *iff* $\sigma(\alpha) = 1$.

Example 3

Modal Logic

Symbol $\{\neg\ ,\ \vee\ ,\ \Diamond\ ,\ (\ ,\ )\} \cup (\mathbf{P} =)\{p_0\ ,\ p_1\ ,\ ...\}$

Formula $\phi ::= p \mid \neg\phi \mid (\phi \vee \phi) \mid \Diamond\phi$, where $p \in \mathbf{P}$

Kripke sementics for Propositional Logic

- a set W , a binary relation R on W
- given an assignment (i.e. a mapping from **P** to $\mathcal{P}(W)$) $\tilde{\sigma}$, let: $\mathcal{M} = (W, R, \tilde{\sigma})$ and for all $w \in W$,

$$\mathcal{M}, w \vDash p \quad \textit{iff} \quad w \in \tilde{\sigma}(p)$$

$$\mathcal{M}, w \vDash \alpha \vee \beta \quad \textit{iff} \quad \mathcal{M}, w \vDash \alpha \text{ or } \mathcal{M}, w \vDash \beta$$

$$\mathcal{M}, w \vDash \Diamond\alpha \quad \textit{iff} \quad \text{there is an } u \text{ s.t. } wRu \text{ and } \mathcal{M}, w \vDash \alpha$$

- α is *true* under $\mathfrak{A}$ and $\tilde{\sigma}$ *iff* $\sigma(\alpha) = 1$.

Example 4

Intuitionistic Logic

Symbol $\{\neg\ ,\ \wedge\ ,\ \vee\ ,\ \rightarrow\ ,\ (\ ,\ )\} \cup (\mathbf{P} =)\{p_0\ ,\ p_1\ ,\ \ldots\}$

Formula $\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi)$, where $p \in \mathbf{P}$

relational sementics for Intuitionistic Logic

- a set W , a binary relation R on W
- given an assignment (i.e. a mapping from $\mathbf{P}$ to $\mathcal{P}(W)$) $\tilde{\sigma}$, which satisfying : for any $p \in \mathbf{P}$ and for any $w, u \in W$,

$$\text{if } w \in \tilde{\sigma}(p) \text{ and } wRu \text{ , then } u \in \tilde{\sigma}(p)$$

let $\mathcal{M} = (W, R, \tilde{\sigma})$ and for all $w \in W$,

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$$\mathcal{M}, w \vDash \neg\alpha \quad \textit{iff} \quad \text{for all } wRu \text{ , } \mathcal{M}, u \nvDash \alpha$$

- α is *true* under $\mathfrak{A}$ and $\tilde{\sigma}$ *iff* $\sigma(\alpha) = 1$.

Relational semantics vs Algebraic semantics

relational structure $:< A, R_1, ..., R_n >$, where A is a nonempty set and R_i is a relation on A.

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algebraic structure $:< A, f_1, ..., f_n, a_1, ..., a_m >$, where A is a nonempty set , f_i is a relation on A , $a_j \in A$.

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relational semantics : use relational structures to give semantics (Example 3,4)

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relational semantics : use relational structures to give semantics
(Example 3,4)

algebraic semantics : use algebraic structures to give semantics
(Example 1,2)

Example 3 again
Kripke sementics for Propositional Logic

- (W , R) . Given an assignment $\tilde{\sigma}$: $\mathbf{P} \to \mathcal{P}(W)$,

$$\mathcal{M}, w \vDash p \quad \textit{iff} \quad w \in \tilde{\sigma}(p)$$

$$\mathcal{M}, w \vDash \neg\alpha \quad \textit{iff} \quad \mathcal{M}, w \nvDash \alpha$$

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- $\mathfrak{A}, \tilde{\sigma} \vDash \alpha \quad \textit{iff} \quad \sigma(\alpha) = 1.$

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- $\mathfrak{A}, \tilde{\sigma} \vDash \alpha \quad iff \quad \sigma(\alpha) = 1.$

Algebraic version

- $(\mathbf{P}(W), \cup, -, m_R, W)$ .Given an assignment $\tilde{\sigma}$: $\mathbf{P} \to \mathcal{P}(W)$,

$$\sigma(p) = \tilde{\sigma}(p)$$

$$\sigma(\alpha \vee \beta) = \sigma(\alpha) \cup \sigma(\beta)$$

$$\sigma(\neg\alpha) = -\sigma(\alpha)$$

$$\sigma(\Diamond\alpha) = m_R(\sigma(\alpha)) \; (m_R(X) = \{w \in W | \; there \; is \; u \in X, wRu\})$$

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Kripke sementics for Propositional Logic

- (W , R) . Given an assignment $\tilde{\sigma}$: $\mathbf{P} \to \mathcal{P}(W)$,

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$$\mathcal{M}, w \models \neg\alpha \quad \textit{iff} \quad \mathcal{M}, w \not\models \alpha$$

$$\mathcal{M}, w \models \alpha \vee \beta \quad \textit{iff} \quad \mathcal{M}, w \models \alpha \text{ or } \mathcal{M}, w \models \beta$$

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- $\mathfrak{A}, \tilde{\sigma} \models \alpha \quad \textit{iff} \quad \sigma(\alpha) = 1.$

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σ mapping every formula to the set of possible worlds where the formula is true.

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$$\sigma(\alpha \to \beta) = f_R(\sigma(\alpha), \sigma(\beta))$$

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$f_R(X, Y) = \{w \in W \mid h(w) \cap X \subseteq Y\}$, where
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the discussion above shows :

- Relational semantics is essentially Algebraic semantics
- **question:** What is semantics ?(mathematically, formally)
  answer: a mapping from a formal language (the formulas) to a set (the 'interpretation') and a fixed monadic predicate on the set (the 'truth predicate').

Intuitionistic Logic

Symbol $\{\neg\ ,\ \wedge\ ,\ \vee\ ,\ \rightarrow\ ,\ (\ ,\ )\} \cup (\mathbf{P} =)\{p_0\ ,\ p_1\ ,\ ...\}$

Formula $\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi)$, where $p \in \mathbf{P}$

algebra sementics for Intuitionistic Logic

- a algebraic structure $\mathfrak{A} = (A\ ,\ +\ ,\ \cdot\ ,\ f\ ,\ g\ ,\ 1)$,where $+$, $\cdot$, f are binary functions on A , g is a unitary function on A , $1 \in A$.
- given an assignment (i.e. a mapping from $\mathbf{P}$ to A) $\tilde{\sigma}$, we extend its domain to *Form* in the following way:

$$\sigma(p) = \tilde{\sigma}(p)$$

$$\sigma(\alpha \vee \beta) = \sigma(\alpha) + \sigma(\beta)$$

$$\sigma(\alpha \wedge \beta) = \sigma(\alpha) \cdot \sigma(\beta)$$

$$\sigma(\alpha \rightarrow \beta) = f(\sigma(\alpha)\ ,\ \sigma(\beta))$$

$$\sigma(\neg\alpha) = g(\sigma(\alpha))$$

- α is *true* under $\mathfrak{A}$ and $\tilde{\sigma}$ (denoted by $\mathfrak{A}, \tilde{\sigma} \vDash \alpha$) *iff* $\sigma(\alpha) = 1$.
  α is *valid* under $\mathfrak{A}$ (denoted by $\mathfrak{A} \vDash \alpha$) *iff* for any assignment $\tilde{\sigma}, \sigma(\alpha) = 1$.

Intuitionistic Logic

Axiom

- A1 $\beta \to (\alpha \to \beta)$
- A2 $(\alpha \to (\beta \to \gamma)) \to (\alpha \to \beta) \to (\alpha \to \gamma)$
- A3 $\alpha \wedge \beta \to \alpha$
- A4 $\alpha \wedge \beta \to \beta$
- A5 $(\alpha \to \beta) \to (\alpha \to \gamma) \to (\alpha \to \beta \wedge \gamma)$
- A6 $\alpha \to \alpha \vee \beta$
- A7 $\beta \to \alpha \vee \beta$
- A8 $(\alpha \to \gamma) \to (\beta \to \gamma) \to (\alpha \vee \beta \to \gamma)$
- A9 $(\alpha \to \beta) \to (\alpha \to \neg\beta) \to \neg\alpha$
- A10 $\neg\alpha \to (\alpha \to \beta)$

Inference Rule MP: α and $\alpha \to \beta \ \Rightarrow \ \beta$

algebra sementics for Intuitionistic Logic

- a algebraic structure $\mathfrak{A} = (A\ ,\ +\ ,\ \cdot\ ,\ f\ ,\ g\ ,\ 1)$,where $+$, $\cdot$, f are binary functions on A , g is a unitary function on A , $1 \in A$.
- given an assignment (i.e. a mapping from **P** to A) $\tilde{\sigma}$, we extend its domain to *Form* using $+$, $\cdot$, f , g
- $\mathfrak{A}, \tilde{\sigma} \vDash \alpha$ *iff* $\sigma(\alpha) = 1$.
  $\mathfrak{A} \vDash \alpha$ *iff* for any assignment $\tilde{\sigma}, \sigma(\alpha) = 1$.

Theorem (Soundness Theorem(in the algebraic semantics for IL))

Let $\mathfrak{A} = (A\ ,\ +\ ,\ \cdot\ ,\ f\ ,\ g\ ,\ 1)$. If there is an $0 \in A$ s.t.
$\mathfrak{A}' = (A\ ,\ +\ ,\ \cdot\ ,\ 0\ ,\ 1)$ is a bounded distributive lattice ,
$g(x) = f(x\ ,\ 0)$,
and f satisfying:

- *if $x \leq y$ then $f(x\ ,\ y) = 1$*
- *$f(x\ ,\ f(y\ ,\ z)) \leq f(x \cdot y\ ,\ z)$*
- *$f(x\ ,\ y) \cdot x \leq y$*

(where $\leq$ is the partial order of the lattice),
then IL is sound w.r.t $\mathfrak{A}$, i.e. $\vdash \alpha \Rightarrow \mathfrak{A} \vDash \alpha$.

Intuitionistic Logic again

Some results :

	$\vdash \alpha \to \alpha$
deduction th	$\Gamma \cup \{\alpha\} \vdash \beta \;\Rightarrow\; \Gamma \vdash \alpha \to \beta$
syl	$\{\alpha \to \beta \;,\; \beta \to \gamma\} \vdash \alpha \to \gamma$
$\wedge^-$	$\{\alpha \wedge \beta\} \vdash \alpha \;,\; \{\alpha \wedge \beta\} \vdash \beta$
$\wedge^+$	$\{\alpha \;,\; \beta\} \vdash \alpha \wedge \beta$
$\vee^+$	$\{\alpha\} \vdash \alpha \vee \beta \;,\; \{\beta\} \vdash \alpha \vee \beta$
$\vee^-$	$\{\alpha \to \gamma \;,\; \beta \to \gamma\} \vdash \alpha \vee \beta \to \gamma$
	......

Lindenbaum-Tarski algebra for IL

We define a binary relation $\sim$ on *Form*:

$$\alpha \sim \beta \iff (\vdash \alpha \to \beta \text{ and } \vdash \beta \to \alpha)$$

In IL , $\wedge$, $\vee$, $\to$, $\neg$ can be viewed as functions on *Form*
e.g. $\wedge(\alpha\ ,\ \beta) = (\alpha \wedge \beta)$, $\to(\alpha\ ,\ \beta) = (\alpha \to \beta)$ thus ,
(*Form* , $\vee$, $\wedge$, $\to$, $\neg$) is a algebraic structure , called the
formula algebra.
It's easy to check that,

Proposition

$\sim$ *is a congruence on the formula algebra.*

which induces a quotient algebra $(Form/\sim, \vee', \wedge', \to', \neg')$,
where $Form/\sim$ is the set of equivalence classes , $[\alpha] \vee' [\beta] = [\alpha \vee \beta]$,etc.

Lindenbaum-Tarski algebra for IL

We denote the formula $p_1 \to (p_2 \to p_1)$ by α_1 . Algebra $(Form/\sim, \vee', \wedge', \to', \neg' , [\alpha_1])$ is called the *Lindenbaum – Tarski algebra* for IL .
We can show that,

Theorem (soundness w.r.t. LT algebra)

Intuitionistic Logic is sound w.r.t. LT algebra.

Recall the Soundness Theorem . Let $\alpha_0 = \neg\alpha_1$. It suffices to check that,

- $(Form/\sim, \vee', \wedge', \to', [\alpha_0], [\alpha_1])$ is a bounded distributive lattice ,
- $\neg'(x) = \to' (x , [\alpha_0])$
- if $x \leq y$ then $\to' (x , y) = [\alpha_1]$
- $\to' (x , \to' (y , z)) \leq \to' (x \wedge' y , z)$
- $\to' (x , y) \wedge' x \leq y$

□

Lindenbaum-Tarski algebra for IL

but what is more important is that,

Theorem (completeness w.r.t. LT algebra)

Intuitionistic Logic is complete w.r.t. LT algebra.

Given a formula α , if $LT \vDash \alpha$, then by definition ,for all assignment $\tilde{\sigma}$, $\sigma(\alpha) = [\alpha_1]$. Consider this assignment :$\tilde{\sigma}(p) = [p]$, we can show that $\sigma(\alpha) = [\alpha]$ for any formula α , then $[\alpha] = [\alpha_1]$, then $\vdash \alpha$. □

Example 4 again[2]
relational sementics for Intuitionistic Logic

- (W , R) . Given an assignment $\tilde{\sigma}$: $\mathbf{P} \to \mathcal{P}(W)$, satisfying : for any $p \in \mathbf{P}$ $w, u \in W$, if $w \in \tilde{\sigma}(p)$ and wRu , then $u \in \tilde{\sigma}(p)$

$$\ldots$$

$$\mathcal{M}, w \vDash \alpha \to \beta \quad \textit{iff} \quad \text{for all } wRu \text{ , } \mathcal{M}, u \nvDash \alpha \text{ or } \mathcal{M}, u \vDash \beta$$

$$\mathcal{M}, w \vDash \neg\alpha \quad \textit{iff} \quad \text{for all } wRu \text{ , } \mathcal{M}, u \nvDash \alpha$$

Algebraic version (the same idea)

- $(\mathbf{P}(W), \cup, \cap, f_R, g_R, W)$.Given assignment $\tilde{\sigma}$: $\mathbf{P} \to \mathcal{P}(W)$, satisfying : for any $p \in \mathbf{P}, w, u \in W$,
  if $w \in \tilde{\sigma}(p)$ and wRu ,then $u \in \tilde{\sigma}(p)$

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$$\sigma(\alpha \to \beta) = f_R(\sigma(\alpha), \sigma(\beta))$$

$$\sigma(\neg\alpha) = g_R(\sigma(\alpha))$$

$f_R(X, Y) = \{w \in W \mid h(w) \cap X \subseteq Y\}$, where
$h(w) = \{u \in W \mid wRu\}$
$g_R = f_R(X, \emptyset)$

Example 4 again[2]
Algebraic version of the relational sementics for Intuitionistic Logic

- $(\mathbf{P}(W), \cup, \cap, f_R, \emptyset, W)$.Given assignment $\tilde{\sigma}$: $\mathbf{P} \to \mathcal{P}(W)$, satisfying : for any $p \in \mathbf{P}, w, u \in W$, if $w \in \tilde{\sigma}(p)$ and wRu ,then $u \in \tilde{\sigma}(p)$

$$\sigma(\alpha \vee \beta) = \sigma(\alpha) \cup \sigma(\beta)$$

$$\sigma(\alpha \wedge \beta) = \sigma(\alpha) \cap \sigma(\beta)$$

$$\sigma(\alpha \to \beta) = f_R(\sigma(\alpha), \sigma(\beta))$$

$f_R(X, Y) = \{w \in W | h(w) \cap X \subseteq Y\}$,where $h(w) = \{u \in W \mid wRu\}$

$$\sigma(\neg\alpha) = g_R(\sigma(\alpha))$$

$g_R = f_R(X, \emptyset)$.

- To eliminate the restriction on assignment , we replace the algebraic structure $(\mathbf{P}(W), \cup, \cap, f_R, \emptyset, W)$ with its subalgebra .

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Algebraic version of the relational sementics for Intuitionistic Logic

To eliminate the restriction on assignment , we replace the algebraic structure $(\mathbf{P}(W), \cup, \cap, f_R, \emptyset, W)$ with its subalgebra .

Let $(\mathbf{P}(W), \cup, \cap, f_R, \emptyset, W)$ be a algebraic structure where

$f_R(X, Y) = \{w \in W | h(w) \cap X \subseteq Y\}$, $h(w) = \{u \in W \mid wRu\}$.

$\Omega = \{X \in \mathbf{P}(W)|$ for any $w, u \in W$,

if $w \in \tilde{\sigma}(p)$ and wRu ,then $u \in \tilde{\sigma}(p)\}$

Proposition

If R is transitive , then $(\mathbf{P}(W), \cup, \cap, f_R, \emptyset, W)$ has a subalgebra struture above Ω .

Given a transitive frame $F = (W, R)$, we call $(\Omega, \cup, \cap, f_R, \emptyset, W)$ *the generated algebra of F* ,denoted by F^+.

Representation Theorem

Theorem

Let $\mathfrak{A} = (A\ ,\ +\ ,\ \cdot\ ,\ f\ ,\ 0\ ,\ 1)$. *If there is an* $0 \in A$ *s.t.* $\mathfrak{A}' = (A\ ,\ +\ ,\ \cdot\ ,\ 0\ ,\ 1)$ *is a bounded distributive lattice , and f satisfying:*

- *if* $x \leq y$ *then* $f(x\ ,\ y) = 1$
- *if* $z \cdot x \leq y$ *then* $z \leq f(x, y)$
- $f(x\ ,\ y) \cdot x \leq y$

(where $\leq$ *is the partial order of the lattice),*
then there is a set U *,* $\mathfrak{A}$ *can be embedded into* $(U, \leq)^+$

Soundness and Completeness via algebraic semantics

Theorem

1.Given (W , R) ,if R is reflexive and transitve , then IL is sound w.r.t (W,R)
2.IL is complete w.r.t the class of partial order frames.