Typos ‘ *Principia*
On the Elimination of Incomplete Symbols

Principia Mathematic goes to great lengths to hide its type theory and to make it appear as if its incomplete symbols (definite descriptions, class expressions) are well-behaved. But well-hidden as they are, we cannot understand the proofs in *Principia* unless with bring them into focus. When we do, some rather surprising results emerge—which is the subject of this paper.

§0 Introduction

In a work of three volumes and thousands of pages, it is not at all very interesting to herald that not all the typos in *Principia Mathematica* have been found. This paper is not about typos as such, it is about fruitful typos and *faux pas*— ones that help to reveal very important features of the formal logic and philosophy of mathematics couched in *Principia*. The surprising thing is that apart from what some Poles liquidated during the war and a few assimilated Texans may have once seen,[1] these “fruitful typos” (if we may so call them) have not already been discovered. This reveals that *Principia’s* theory of incomplete symbols has not been properly understood. Whitehead and Russell offer a “no-classes” theory couched in a type-theory. In looking at notations alone, one will find very little syntactic markers for types. *Principia* hides its type theory and makes it appear as if its incomplete symbols (definite descriptions, class expressions) are well-behaved. But well-hidden as they are, we cannot understand the *Principia* or its proofs unless with bring them into focus. When we do, some rather wonderful results emerge.

§1 Schematic Letters versus Predicate Variables in *Principia*

The syntax *of Principia’s* formal language is difficult to discern because its formal language is not set out explicitly. Instead, it employs a technique of suppressing its order\ type indices under conventions of restoration. This is the technique it calls “typical ambiguity.” The conventions of restoration are, however, far from clear. There are entire sections (e.g., sections

[1] For the comment about Poles and Texans being the few who have read *Principia*, see Russell (1959), p. 86.

*63-5 of volume I) devoted to introducing and explaining the conventions. The *Prefatory Statement* of volume II is a nightmare to disentangle.

In spite of the many formidable difficulties, it is reassuring to know that there is ample historical evidence that the syntax intended by Whitehead and Russell is that of simple type theory. A recursive definition of simple type symbols can be set out as follows:

i) *o* is a simple type symbol.
ii) If $t_1,\ldots, t_n$ are simple-type symbols, then $(t_1,\ldots, t_n)$ is a simple-type symbol.
iii) There are no other simple-type symbols.

The order of a simple-type symbol may be defined as follows. The order of *o* is 0; the order of a simple type symbol then $(t_1,\ldots, t_n)$ is *n*+1 where *n* is the highest order of any of the simple-type symbols $t_1,\ldots, t_n$. An object-language variable is "predicative" when its order is the order of its simple type index. Now the historical evidence for this can be found scattered throughout *Principia*, and it will be too much of distraction to recount all the evidence here. [2] But consider the following passage. Whitehead and Russell write (*PM*, vol. 1, p. 165):

> … it is unnecessary to introduce a special notation for non-predicative function of a given order and taking arguments of a given order. … It is possible, therefore, without loss of generality, to use no apparent variables except such as are predicative. We require, however, a means of symbolizing a function whose order is not assigned. We shall use "φx" of "$f(\chi!\hat{z})$" or etc., to express a function (ϕ or *f*) whose order, relatively to its argument, is not given. Such a function cannot be made into an apparent variable, unless we suppose its order previously fixed. As the only purpose of the notation is to avoid the necessity of fixing the order, such a function will not be used as an apparent variable; the only functions which will be so used will be predicative functions, because, as we have just seen, this restriction involves no loss of generality.

Principia adopts no object-language variables whose order in not the order of their simple type. In *Principia*, all and only genuine variables are predicative. Our thesis is that bindable predicate variables φ, ψ, f , g and the like always come with the exclamation (shriek !). When such letters occur *without* the shriek they are schematic for *wffs* (well-formed formulas) and are not object-language predicate variables.

Given that only predicative predicate variables are allowed, ramification is *not* coded into the syntax of the theory. Under the influence of Church, many have imagined that *Principia* adopts a syntax in which encodes ramification—a syntax of *r*-types (ramified types) that adopts

[2] See Landini (1998).

predicate variables whose order is not the order of its simple type. In light of Church's view, which has become orthodoxy, our thesis that all and only variables in *Principia* are have the order of their simple type may at first seem shocking. On our view, ramification is not coded into the syntax; it is a product of the informal substitutional semantics that Whitehead and Russell offered for the theory. The correct way to put the matter of ramification to say that the intended informal substitutional semantics offered by Whitehead and Russell does not validate *Principia's* comprehension axiom(s). The axioms can be amalgamated and rendered as follows:

12.*n* $(\exists f)(\ f!^{(t_1,\ldots,t_n)}(x_1^{t_1},\ldots,\ x_n^{t_n}) \equiv\ \varphi(x_1^{t_1},\ldots,\ x_n^{t_n}))$,

where $f!^{(t_1,\ldots,t_n)}$ is not free in the *wff* $\varphi(x_1^{t_1},\ldots,\ x_n^{t_n})$. This has come to be called Principia's "axiom of reducibility." Note that $\varphi(x_1^{t_1},\ldots,\ x_n^{t_n})$ is schematic for a *wff* and that φ is *not* a predicate variable. It is not historically accurate to follow Church's reconstruction of *Principia* which codes ramification into the syntax with non-predicative predicate variables, comprehension axiom schemata written in virtue of them, and an "Axiom of Reducibility." The serpent of ramification enters the garden because it is only those instances of *12.*n* that embody predicative restrictions on the formula

$\varphi(x_1^{t_1},\ldots,\ x_n^{t_n})$

that are valid in the (nominalistic) semantics set out by Whitehead and Russell in volume I of the first edition of *Principia*. Only instances which are so restricted can be seen, as Ramsey (1925) would put it, as "generalized tautologies." It is in virtue of this, not its syntax, that *Principia* is a ramified type theory. That is, 12.*n* ought to have placed restrictions, in accordance with Whitehead and Russell's intended semantics, on the *wffs* $\varphi(x_1^{t_1},\ldots,\ x_n^{t_n})$ that are allowed.

It is important to understand that typical ambiguity applies only to *genuine* variables of the object-language of *Principia*, not to schematic letters. Only the genuine variables (individual variables and predicate variables) of *Principia* are typically ambiguous (with conventions and contextual clues governing the rules for their proper restoration). Schematic letters are not typically ambiguous—they have no types at all! Thus, it is of utmost importance distinguish genuine object-language variables from schematic letters.

It follows from our thesis concerning the letters variables φ, ψ, *f* , *g,* that when *Principia* uses expressions such as

$f(\iota x \varphi x)$
$\psi(\iota x \varphi x)$

we are to understand that they are used schematically for some formula in which $\iota x\varphi x$ occurs in a subject position. We are not in a position to know its scope until we know the formula in question. This is made clear from the fact that *Principia* says explicitly that without adopting a convention, $f(\iota x\varphi x)$ can represent a number of different sorts of embeddings of the expression $\iota x\varphi x$. We find the following (*PM vol. 1* p. 69):

$$\psi\,(\iota x\varphi x) \supset p \;\; =df \;\; [\iota x\varphi x][\,\psi\,(\iota x\varphi x)] \supset p$$
$$p \supset \psi\,(\iota x\varphi x) \; =df \;\; p \supset [\iota x\varphi x][\psi\,(\iota x\varphi x)]$$
$$\psi\,(\iota x\varphi x) \supset \chi\,(\iota x\varphi x) \; =df \;\; [\iota x\varphi x][\psi\,(\iota x\varphi x)] \supset [\iota x\varphi x][\chi\,(\iota x\varphi x)].$$

Principia tells us, therefore, that it must set out a convention governing scope. *Principia* has:

$$*14.01 \quad [\iota x\varphi x][f\,(\iota x\varphi x)] \;\; =df \;\; (\exists x)(\varphi z \equiv_z z = x \;.\&.\; fx)$$
$$*14.02 \quad \mathrm{E}!(\iota x\varphi x)] \;\; =df \;\; (\exists x)(\,\varphi z \equiv_z z = x)$$

Whitehead and Russell write (*PM, vol.1,* p. 173):

> It will be found in practice that the scope usually required is the smallest proposition enclosed in dots or brackets in which "$\iota x\varphi x$" occurs. Hence when this scope is to be given to $\iota x\varphi x$, we shall usually omit explicit mention of the scope.

For example, when $f(\iota x\varphi x)$ is assigned as

$$\psi!\,(\iota x\varphi x) \supset p$$

The convention to take the smallest scope yields:

$$[\iota x\varphi x][\,\psi!\,(\iota x\varphi x)] \supset p.$$

The convention is clear : When a *wff* is assigned to the schema $f(\iota x\varphi x)$, we are to take the smallest possible scope.

Principia's examples, however, are infelicitous. When Whitehead and Russell imagine $f(\iota x\varphi x)$ to be assigned to

$$\psi\,(\iota x\varphi x) \supset p$$

They make an *informal* assumption that there are no further embeddings in $\psi(\iota x\varphi x)$ itself. The expression $\psi(\iota x\varphi x)$, however, is no less schematic than $f(\iota x\varphi x)$. The informal assumption is thus in tension with the official position *Principia* adopts governing its notations. That is, Whitehead and Russell's informal assumption (for the sake of their example) that there are no further embedding is in conflict with *Principia's* formal convention on the omission of its scope markers. To be strictly consistent with the formal convention, the shriek (exclamation) is needed. In the above, we have avoided this tension by employing a genuine predicate variable $\psi!$ with the shriek. Genuine predicate variables such as $\psi!$ come with the shriek and this distinguishes them from schematic letters such as ψ. In $\psi!(\iota x\varphi x)$ the letter ψ! is a predicate variable (not a schematic letter) and hence the smallest possible scope is $[\iota x\varphi x][\ \psi!\ (\iota x\varphi x)]$. Thus formally *Principia's* examples should be:

$$\psi!\ (\iota x\varphi x) \supset p \ \ =df\ \ [\iota x\varphi x][\ \psi!(\iota x\varphi x)] \supset p$$
$$p \supset \psi!(\iota x\varphi x) =df\ \ p \supset [\iota x\varphi x][\psi!(\iota x\varphi x)]$$
$$\psi!(\iota x\varphi x) \supset \chi!(\iota x\varphi x) =df\ \ [\iota x\varphi x][\psi!(\iota x\varphi x)] \supset [\iota x\varphi x][\chi!(\iota x\varphi x)].$$

The use of the predicate variables ψ! and χ! force a primary scope for the description in a way that the use of schematic letters ψ and χ cannot.

§2 Descriptions ι*xfx*

Short pieces by Hochberg[3] and by Cassin[4] rekindled a firestorm of worrying, begun by Gödel[5] and Carnap[6], about the order of elimination of incomplete symbols in *Principia.* The worries persist to this day, and every once and a while a new paper emerges expressing some new found anxiety. Identity offers a nice example of the situation. *Principia* has:

$$*13.01\ \ x = y =df\ \ (\varphi)(\ \varphi!x \equiv \varphi!y)$$
$$*13.02\ \ x \neq y =df\ \ \sim\{\ x = y\}.$$

Hochberg wonders how to apply the scope conventions and definitions in cases such as

$$\iota x\psi x = y.$$

He wonders which of the following is intended:

$$[\iota x\psi x][\ \iota x\psi x = y]$$
$$(\varphi)(\ [\iota x\psi x][\varphi!(\iota x\psi x)] \equiv \varphi!y).$$

The second of the above arises from applying the definition *13.01. Hochberg thinks that the scope conventions stated in *Principia* are unclear on this matter and must be clarified.

The worry is groundless. The solution is simply to take seriously the fact that definite descriptions are not genuine terms in *Principia*. The consequence of this is that definitions framed with individual variables (though they are ambiguous with respect to type) apply only to

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[3] Hochberg (1970).
[4] Cassin (1970).
[5] Gödel (1944), p. 126.
[6] Carnap (1947), p. 148.

genuine terms of the formal language and, therefore, they do not apply to definite descriptions. The same point applies to

$$*13.02\ x \neq y =df \sim\{x = y\}.$$

For example, in a case such as

$$\iota x\psi x \neq y$$

elimination proceeds without a hitch, and we arrive at

$$[\iota x\psi x][\iota x\psi x \neq y].$$

This makes perfect sense and is corroborated in the actual practices of the demonstrations in *Principia*. It makes sense of the fact that the following is *not* a theorem

$$\iota x\psi x = \iota x\psi x.$$

The proper elimination yields this:

$$[\iota x\psi x\ ][\iota x\psi x = \iota x\psi x].$$

Applying *14.01, we get

$$(\exists x)(\ \psi z \equiv_z z = x\ .\&.\ x = x),$$

and this is false in some cases of ψ. The very same point applies to the worries of Carnap who, following Gödel, was concerned that

$$\hat{z}\varphi!z = \varphi!\hat{z}$$

$$\hat{z}\varphi!z \neq \varphi!\hat{z}$$

"look like" contradictories when in fact they are not. Since class expressions are incomplete symbols and not genuine terms, definitions *13.01 and *13.02 cannot apply. Hence, the whole matter is a pseudo-problem. The above only look like contradictories to someone unaware of the nature of definitions in *Principia*.

But this certainly does not exhaust the cases one can worry about. Many have noticed that something is not quite right in *Principia's* section *14 with respect to scope.[7] Consider, for example, the theorem schema

$$*14.18\ \ E!(\iota x\varphi x)\ .\supset.\ (x)\psi x \supset \psi(\iota x\varphi x).$$

This has false instances once non-truth-functional contexts are allowed. The issue never arose for *wffs* of the object-language of *Principia*, but when philosophers attempted to apply *Principia's*

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[7] See Smullyan (1983), p.37; Linsky (1983), p. 75.

theory of definite description to modal contexts with *de re* quantification, many difficulties emerged. Given we allow $(x)\ \Box(x = x)$, it is provable that

$$(x)(\ x = y\ \supset_y \Box(x = y)).$$

From this we easily arrive at:

$$(x)(\ \Diamond{\sim}(x = y)\ \supset_y {\sim}(x = y)).$$

But the following would *seem* to be an instance of *14.18:

$$E!(\iota x\varphi x) :\supset: (x)(\ \Diamond{\sim}(x = y)\ \supset_y {\sim}(x = y)) .\supset. \Diamond{\sim}(\iota x\varphi x = y)\ \supset_y {\sim}(\iota x\varphi x = y).$$

The trouble is that the omission of scope markers is supposed to mark a secondary scope for the description $\iota x\varphi x$, so that the last clause is

$$\Diamond{\sim}([\iota x\varphi x][\iota x\varphi x = y])\ \supset_y\ {\sim}([\iota x\varphi x][\ \iota x\varphi x = y]).$$

But this may well be false, in spite of the fact that $E!(\iota x\varphi x)$ and $(x)(\ \Diamond{\sim}(x = y)\ \supset_y {\sim}(x = y))$ are true. The morning star exists. And even though the morning star is identical to Venus it is surely possible that the morning star is not identical to Venus. Allowing *de re* quantification, certain modal contexts violate *14.18 as stated.

One see the patter, we can get a parallel counterexample with *belief* instead of necessity. We can follow Church's analog between *de re* quantification into modal contexts of necessity and *de re* quantification into contexts of belief.[8] As an analog of $(x)\ \Box(x = x)$ Church offers $(x)\ {\sim}\mathrm{bel}^s{\sim}(x = x))$. This yields:

$$(x)(\ x = y\ \supset_y {\sim}\mathrm{bel}^s{\sim}(x = y)).$$

From this, Church arrives at:

$$(x)(\ \mathrm{bel}^s{\sim}(x = y)\ \supset_y\ {\sim}(x = y)).$$

But the following is an apparent instance of *14.18:

$$E!(\iota x\varphi x) :\supset: (x)(\ \mathrm{bel}^s{\sim}(x = y)\ \supset_y\ {\sim}(x = y)) .\supset. \mathrm{bel}^s{\sim}(\iota x\varphi x\ = y)\ \ \supset_y\ {\sim}(\ \iota x\varphi x = y).$$

Paralleling the modal case, the omission of scope markers yields a secondary scope for the last clause:

$$\mathrm{bel}^s{\sim}([\iota x\varphi x][\iota x\varphi x\ = y])\ \ \supset_y\ {\sim}(\ [\iota x\varphi x][\iota x\varphi x = y]).$$

And this again this is clearly false, in spite of the existence of the morning star and the truth of

[8] Church (1988).

(x)(bel$^{s}\sim(x = y) \supset_{y} \sim(x = y)$). Obviously, the fact that the morning star is identical to Venus does not in any way inhibit a person S ignorant of astronomy from believing that the morning star is not identical to Venus.

We can see what has gone awry. Interpreters have been mislead by Principia's inappropriate use of schematic letters in section *14. There are two ways to rectify this. The easiest is to use the genuine predicate variable $\psi!$ rather than the schematic ψ. This yields:

*shriek**14.18 $E!(\iota x\varphi x) .\supset. (x)\ \psi!x \supset \psi!(\iota x\varphi x)$.

The smallest possible scope is, therefore, forced by the shriek to be primary and hence the scope marker can be dropped without trouble. There is another way to rectify the situation. Indeed, it is more likely that what was intended by Whitehead and Russell is this:

*c**14.18 $E!(\iota x\varphi x) .\supset. (x)\ \psi x \supset [\iota x\varphi x][\psi(\iota x\varphi x)]$.

The scope maker forces a primary occurrence of the description while at the same time allowing ψ to remain schematic. Only if ψ is assigned to some *wff* which, by its nature forces a primary scope, can we drop the scope marker under *Principia's* formal conventions for omission and restoration.

The case of *14.22 is also of interest. It too has false instances if certain non-extensional contexts (e.g. *de re* contexts of modality) are allowed. At first blush, it might be thought that

*14.22 $E!(\iota x\varphi x) \supset \varphi(\iota x\varphi x)$

is unassailable. But this arises from a failure to imagine cases where the description has a secondary occurrence in the consequent clause. Such scopes can occur, since φ is used schematically. Consider this

$E!(\iota x)(\psi x \ \&\ \Diamond(\iota y\psi y \neq x)) \ .\supset.$

$\psi\ (\iota x)(\psi x \ \&\ \Diamond(\iota y\psi y \neq x) \ \&\ \Diamond(\iota y\psi y \ \neq(\iota x)(\psi x \ \&\Diamond(\iota y\psi y \neq x)))$.

The omission of scope markers allows secondary occurrences of the descriptions in

$\Diamond(\iota y\psi y \ \neq (\iota x)(\psi x \ \&\ \Diamond(\iota v\psi v \neq x))$

and this asserts that a contradiction is possible. It says

$\Diamond(\exists y)(\ \psi z \equiv_{z} z = y \ .\&.\ (\exists x)(\ \psi z \ \&\ \Diamond(\iota v\psi v \neq z) \,.\equiv_{z}.\ z = x \ \ :\&:\ y \neq x))$.

We can parallel this with belief by replacing the modal operator $\Diamond p$ with bel^{s}p.

To rectify matters, there again two options. As before, we might again think to force primary scope by using a genuine predicate variable (with its shriek). That yields:

*shriek**14.22 $E!(\iota x\varphi!x) \supset \varphi!(\iota x\varphi!x)$.

But again what was likely intended in *Principia* is this:

c *14.22 E!($\iota x\varphi x$) $\supset$ [$\iota x\varphi x$][$\varphi(\iota x\varphi x)$].

The scope marker forces the primary occurrence of the description while allowing φ to remain schematic.

More obviously problematic is *14.21. In the case of *14.18 and *14.22 one might think to blame the oddity on the fact that the formal language of *Principia* does not have contexts of modality and belief. Whitehead and Russell are well aware that without a special assumption of truth-functionality some of the numbers in section *14 have false instances. They offer the example of a case where $f(\iota x\varphi x)$ says that George IV wished to know whether Scott was the author of *Waverley*. They write (*PM*, *vol. 1,* p. 83):

> The inference $(x)fx \supset f(\iota x\varphi x)$ is only valid when E!($\iota x\varphi x$). As soon as we know E!($\iota x\varphi x$) the fact that $\iota x\varphi x$ is an incomplete symbol becomes irrelevant so long as we confine ourselves to truth-functions of whatever propositions is its scope. But even when E!($\iota x\varphi x$), the incompleteness of $\iota x\varphi x$ may be relevant when we pass outside truth-functions.

This shows that primary and secondary occurrences of a description may fail to be equivalent even when E!($\iota x\varphi x$) is assured. But this approach, which excludes all but truth-functional contexts, to salvaging *14.18 and *14.22 cannot work in the case of *14.21. We have seen that officially in *Principia*, $\psi(\iota x\varphi x)$ is schematic for some *wff* ψ in which $\iota x\varphi x$ occurs and we cannot know the scope but only know that is the smallest possible. Given this, it is easy to see that

*14.21 $\psi(\iota x\varphi x) \supset$ E!($\iota x\varphi x$)

as stated, has false instances within the formal language of *Principia* itself. For example,

$\sim(\exists y)(\iota x\varphi x = y) \supset$ E!($\iota x\varphi x$)

is false. What happened?

Clearly, in*14.21 Whitehead and Russell have assumed that the scope of the description is primary. The problem is that such an assumption is manifestly inconsistent with *Principia's* official formal convention on the omission of scope markers—a convention that maintains that the scope be smallest possible in whatever *wff* is assigned to $\psi(\iota x\varphi x)$. As before, one might think to force a primary scope with:

shriek *14.21 ψ!($\iota x\varphi x$) $\supset$ E!($\iota x\varphi x$).

But, as we shall see, offering *shriek**14.21 as a correction will not facilitate the many uses of *14.21 in *Principia.* What is needed is this:

c *14.21 [$\iota x\varphi x$][$\psi(\iota x\varphi x)$] . $\supset$. E!($\iota x\varphi x$).

This preserves the schematic use of ψ and makes sense of the use of *14.21 in *Principia*. At the same time, the presence of the scope marker assures a primary scope for the description in the antecedent clause. To do both of these at once, the scope marker *has* to be explicit. Of course, in cases when $\psi(\iota x\varphi x)$ is assigned to a *wff* that, in particular, forces a primary scope, the scope marker can be omitted under *Principia's* formal convention on omission of scope markers. For example, consider this:

$$\iota x\varphi x = b \supset \mathrm{E}!(\iota x\varphi x).$$

This can be regarded as an instance of *c* *14.21 with its scope marker omitted. The context

$$\iota x\varphi x = b$$

forces a primary scope because the definition of identity at *13.01 *cannot* be applied. In fact, this very example occurs in the proof of *30.5.

How did this happen? The answer is that section *14 adopted an *informal* assumption (as did the infelicitous passages in the Introduction intended to explain the need for a scope marker and the need for an official convention on its omission). The informal assumption is, in fact, incompatible with the formal schematic use of the letters f, ψ, etc. And therefore it is incompatible with the formal convention on the omission of scope markers. The informal assumption is described in the comments after *14.21. Whitehead and Russell write (*PM, vol*. 1, p. 182):

> This proposition shows that if any true statement can be made about $\iota x\varphi x$ then $\iota x\varphi x$ must exist. Its use though out the remainder of the work will be very frequent. When $\iota x\varphi x$ does not exist there are still true propositions in which $\iota x\varphi x$ occurs, but it has, in such propositions, a secondary occurrence in the sense explained in Chapter III of the Introduction, i.e., the asserted proposition is not of the form $\psi(\iota x\varphi x)$, but of the form $f\{\psi(\iota x\varphi x)\}$, in other words, the proposition which is the scope of $\iota x\varphi x$ is only part of the whole asserted proposition.

This informal assumption assures in the intended cases of $\psi(\iota x\varphi x)$ the description has a primary scope. Consider *Principia's* statement: "If any true statement can be made about $\iota x\varphi x$ then $\iota x\varphi x$ must exist." This is an obvious howler which is clearly false. Indeed, Whitehead and Russell demonstrate that it is false in their very next sentence. We must employ the principle of charity in interpreting their intent. And this leads us to their use of the informal assumption.

There are other numbers in section *14 that have to be fixed by making some of the scope markers explicit (or alternatively, by adding shrieks). The following are worth mentioning in particular:

$$c\ {*}14.15\quad \iota x\varphi x = b\ \ .\supset.\ [\iota x\varphi x][\psi(\iota x\varphi x)] \equiv \psi b).$$

c *14.16 $\iota x\varphi x = \iota x\psi x$.⊃. $[\iota x\varphi x][\chi(\iota x\varphi x) \equiv [\iota x\psi x][\chi(\iota x\psi x)]$

c *14.205 $[\iota x\varphi x][\psi(\iota x\varphi x)]$.⊃. $(\exists b)(b = \iota x\varphi x$.&. $\psi b)$.

c *14.23 $E!(\iota x)(\varphi x \,\&\, \psi x)$.⊃. $[(\iota x)(\varphi x \,\&\, \psi x)]\,[\varphi(\iota x)(\varphi x \,\&\, \psi x)]$

c *14.242 $\varphi x \equiv_x x = b$.⊃. $\psi b \equiv [\iota x\varphi x][\psi(\iota x\varphi x)]$

c *14.25 $E!(\iota x\varphi x)$:⊃: $\varphi x \supset_x \psi x$.≡. $[\iota x\varphi x][\psi(\iota x\varphi x)]$

c *14.26 $E!(\iota x\varphi x)$.: ⊃:.

$(\exists x)(\varphi x \,\&\, \psi x) \equiv [\iota x\varphi x][\psi(\iota x\varphi x)]$:&: $(\exists x)(\varphi x \,\&\, \psi x)$.≡. $\varphi x \supset_x \psi x$

c *14.272 $\varphi x \equiv_x \psi x$.⊃. $[\iota x\varphi x][\chi(\iota x\varphi x) \equiv [\iota x\psi x][\chi(\iota x\psi x)]$.

It is amusing to note that in section *14 shortly after dropping the scope markers from *14.1, Whitehead and Russell state that they are following a convention on omission of scope markers. They write (*PM, vol. 1,* p. 176):

> *14.1 $[\iota x\varphi x][\psi(\iota x\varphi x)]$.≡. $(\exists b)(\varphi x \equiv_x x = b$.&. $\psi b)$. *4.2, *14.01
> In virtue of our conventions as to the scope intended when no scope is explicitly indicated, the above proposition is the same as the following
> *14.101 $\psi(\iota x\varphi x)$.≡. $(\exists b)(\varphi x \equiv_x x = b$.&. $\psi b)$. *14.1

But as we have seen, Whitehead and Russell are certainly *not* following their *formal* convention in section *14. The schematic expression $\psi(\iota x\varphi x)$ allows that there may be a yet smaller scope involved once it has been assigned to a *wff*. We shall need

*shriek**14.101 $\psi!(\iota x\varphi x)$.≡. $(\exists b)(\varphi x \equiv_x x = b$.&. $\psi!b)$.

This rectifies the situation.

§3 Class Expressions $\hat{z}\varphi z$

We next come to class expressions $\hat{z}\varphi z$ and definitions framed with free lower-case Greek letters α, β, etc., which stand in for class expressions $\hat{z}\varphi z$. The contextual definitions for a class of individuals (of whatever type), enhanced by the restoration of its scope marker, is given as follows

*20.01 $[\hat{z}\varphi z][f(\hat{z}\varphi z)] =df\ (\exists\psi)(\psi!z \equiv_z \varphi z$.&. $f(\psi!\hat{z}))$

*20.02 $x \in \varphi\hat{x} =df\ \varphi x$

We have restored the scope marker to *20.01 and it is absolutely essential that it be present. Without it, the definition inherits all the same ambiguities, pointed out at length in *Principia*, of offering the definition *14.01 without its scope marker. Indeed, Whitehead and Russell explicitly acknowledge that the marker is needed. They write (*PM, vol.1,* 80):

> As in the case of $f(\iota x\varphi x)$ so in the case of $f(\hat{z}\varphi z)$, there is an ambiguity as to the scope of $\hat{z}\varphi z$ if it occurs in a proposition which itself is part of a larger proposition. But in the case of classes, since we always have the axiom of reducibility, namely
>
> $(\exists\psi)(\ \varphi x \equiv_x \psi!x)$
>
> which takes the place of $E!(\iota x\varphi x)$, it follows that the truth-value of any proposition in which $\hat{z}\varphi z$ occurs is the same whatever scope we may give to $\hat{z}\varphi z$, provided the proposition is an extensional function of whatever function it may contain. Hence, we may adopt the convention that the scope is to be always the smallest proposition enclosed in dots or brackets in which $\hat{z}\varphi z$ occurs. If at any time a larger scope is required, we may indicate it by "$[\hat{z}\varphi z]$" followed by dots, in the same way as we did for $[\iota x\varphi x]$.

The absence of the scope marker at *20.01 has to be regarded as a typo (oversight).

Much neglected in discussions of *Principia* on classes is the fact that *20.01 and *20.02 only provide for the emulation of a typed theory of classes of *individuals* (of whatever type). They do *not* provide for a theory of classes of classes of individuals (of whatever type). For this reason, *Principia* introduces separate clauses for classes of classes of individuals (of whatever type). We find:

*20.08 $[\hat{\alpha}\varphi\alpha][\ f(\hat{\alpha}\varphi\alpha)] =df\ \ (\exists\psi)(\ \psi!\alpha \equiv_\alpha \varphi\alpha\ .\&.\ f(\psi!\hat{\alpha})\ )$

*20.081 $\alpha \in \varphi\hat{\alpha}\ \ =df\ \ \varphi\alpha$

*20.07 $(\alpha)f\alpha\ \ =df\ (\varphi)\ \ f(\hat{z}\varphi!z)$

*20.071 $(\exists\alpha)f\alpha\ \ =df\ (\exists\varphi)\ \ f(\hat{z}\varphi!z)$

At *20.07 and *20.071 we have the introduction of definitions for using bound lower-case Greek and free lower-case Greek. As before, it is essential that the scope marker be restored to *20.08 just as in the case of *20.01.

It also essential, however, that the scope marker be *absent* in *20.07 and *20.071. To see this, first notice that *Principia* is explicit that free use of lower-case Greek α, β etc., just stand in for $\hat{z}\varphi z$ or $\hat{z}\psi z$ etc. See for example, *PM, vol. 1,* p. 80, and comments near *20.33, *20.42, *20.64, *37.231, and *62.33. Note as well that φ and ψ here are schematic letters, and recall that the genuine predicate variables always come with a shriek.

Now the quantification theory of section *10 of *Principia* is typically ambiguous. Hence, it applies to variables of any type. If we restore (suppressed) type indices, we have

*10.1 $(x^t)\theta x^t \supset \theta y^t$,

where y^t is free for free x^t in θ. However, in *Principia*, we also find

*20. 61 $(\alpha)f\alpha \supset f\beta$.

Proof $(\alpha)f\alpha \supset f(\hat{z}\varphi!z)$. *10.1, *20.07.

This is at first surprising since the application of definition *20.07 and replacing the Greek β in *20.61 by $\hat{z}\varphi z$ yields

$$(\psi)(f(\hat{z}\psi!z) \supset f(\hat{z}\varphi z).$$

Universal instantiation of a predicate variable has to be to a predicate variable, never to a schematic letter. Thus, *20.61 does *not* follow immediately from *10.1 and definition *20.07. What follows immediately is this

$$(\psi)(f(\hat{z}\psi!z) \supset f(\hat{z}g!z).$$

To get the result, *Principia* relies tacitly on *12.*n* which assures that

$$(\exists g)(g!x \equiv_x \varphi x).$$

With this in place, we can arrive at any instance of

$$(\psi) f(\hat{z}\psi!z) \supset f(\hat{z}\varphi z).$$

Notice that $f\{\hat{z}\varphi z\}$ represents a possibly secondary scope since f is a schematic letter. Thus, a rigorous proof would have to involve a metalinguistic induction on the length of *wffs*.

It is absolutely essential to the theory of class expressions as incomplete symbols that in $f(\hat{z}\varphi!z)$, and similarly in $f(\hat{z}\varphi z)$ the scope is unknowable until after $f(\ldots)$ is assigned. We only know that the scope is to be the smallest (most secondary) possible. Indeed, the *absence* of scope markers in *20.07 and *20.071, which enable secondary scopes, is extremely important. If they were added to the definitions, the theory of classes of classes would not work. This is easy to see. We have only to go through a proof of the analog, for classes of classes, of

$$*20.3_x \quad x \in \hat{z}\varphi z \equiv \varphi x.$$

The analog is this

$$*20.3_\beta \quad \beta \in \hat{\alpha}\varphi\alpha \equiv \varphi\beta.$$

Expanding the contextual definitions, we see that we are to prove:

$$(\exists\Sigma)(\Sigma!\alpha \equiv_\alpha \varphi\alpha .\&. \Sigma!\beta) \equiv \varphi\beta$$

Now let us replace the lower-case Greek β with $\hat{z}\Gamma z$ to get

$$(\exists\Sigma)(\Sigma!\alpha \equiv_\alpha \varphi\alpha .\&. \Sigma!(\hat{z}\Gamma z)) \equiv \varphi(\hat{z}\Gamma z)$$

The key to the viability of proving this lies in the application of *20.07, so that

$$\Sigma!\alpha \equiv_\alpha \varphi\alpha$$

allows for secondary scopes. By *20.07 we have:

$$[\hat{z}\psi!z][\Sigma!(\hat{z}\psi!z)] \equiv_\psi \varphi(\hat{z}\psi!z).$$

The left-hand side of the biconditional has a primary scope because Σ! is a genuine predicate variable (not a schematic letter), but the right-hand side $\varphi(\hat{z}\psi!z)$ leaves the scope unknown (since the schematic φ is unassigned). Putting all this together, we see that what is wanted is a proof of

$$(\exists\Sigma)([\hat{z}\psi!z][\Sigma!(\hat{z}\psi!z)] \equiv_{\psi} \varphi(\hat{z}\psi!z) \;.\&.\; \Sigma!(\hat{z}\Gamma z)\,) \;\equiv \varphi(\hat{z}\Gamma z)$$

The proof goes through, given the axiom of reducibility (*12*n*). But in stark contrast, if *20.07 had a primary scope—i.e. if one were to have

$$\textit{wrong}{*}20.07\quad (\varphi)([\hat{z}\varphi!z][f(\hat{z}\varphi!z)]$$

then the clause $\Sigma!\alpha \equiv_{\alpha} \varphi\alpha$ would be defined to mean

$$(\varphi)([\hat{z}\psi!z][\Sigma!(\hat{z}\psi!z) \equiv \varphi(\hat{z}\psi!z)])$$

and this makes the proof impossible. Thus, there is conclusive evidence in *Principia* that in $f(\hat{z}\varphi!z)$ it is a secondary scope that is intended—and required.

Thus, the *formal* scope conventions on omission of the scope marker for class expressions, free-lower case Greek and bound lower-case Greek, exactly parallel that which we have adopted for definite descriptions. Smallest scope is intended with $f(\hat{z}\varphi z)$, and we cannot determine the scope until and unless we know what formula the schematic letter *f* gets assigned to. Applying this to definitions, consider

$${*}24.03\quad \exists!\alpha =df\ (\exists x)(\, x \in\alpha).$$

Replacing the free α with the class expression $\hat{z}\varphi z$ yields:

$$\exists!\hat{z}\varphi z. =df\ (\exists x)(\, x \in \hat{z}\varphi z).$$

Here we know the scope, and thus we have:

$$(\exists x)(\, x \in \hat{z}\varphi z) \;\; =df\;\; (\exists x)(\,[\hat{z}\varphi z][x \in \hat{z}\varphi z])$$

Hence to prove

$$\exists!\, \hat{z}\varphi z$$

we would expect *Principia* to proceed as follows

$x \in \hat{z}\varphi z$
$(\exists x)(\, x \in \hat{z}\varphi z)$ *10.24
$\exists!\, \hat{z}\varphi z$ *24.03.

Indeed, this is just what we find in the work.

Principia's proofs embody further perfectly unequivocal evidence in favor of the interpretation that without scope markers, the work uses $f(\hat{z}\varphi z)$ and $f\alpha$ schematically for some unknown secondary scope for the class expression. For example, consider the following:

$${*}22.04\ -\alpha\ =df\ \hat{z}(\,\sim(z \in \alpha))$$

$*22.95\ \ (\alpha)f\alpha \equiv (\alpha)f(-\alpha)$.

In the proof of *22.95 there is the line,

$(\alpha){\sim}f\alpha \equiv (\alpha){\sim}f(-\alpha)$.

Recall that by *20.07 this is

$(\varphi)({\sim}f(\hat{z}\varphi z)) \equiv (\varphi)({\sim}f(-\hat{z}\varphi z))$.

Here is it crystal clear that although there is no dot or brackets for the scope of the tilde, the class expression has secondary scope so that the tilde has wider scope than the class expression. The reason it is crystal clear is that the proof arrives at

$(\exists\alpha)f\alpha \equiv (\exists\alpha)f(-\alpha)$

which, by definition *10.01, namely $(\exists\alpha)(f\alpha)\ =df\ {\sim}(\alpha){\sim}(f\alpha)$, is this:

${\sim}(\alpha){\sim}\, f\alpha \equiv {\sim}(\alpha){\sim}f(-\alpha)$.

Thus, the evidence is conclusive in favor of a secondary scope.

Indeed, there is plenty of evidence of this sort in *Principia*. Consider the following number which is a very often used theorem from *24.03. *Principia* has:

$*24.51\ \ {\sim}\exists!\alpha \equiv \alpha = \Lambda$.

In the proof of this number, we find

${\sim}\exists!\alpha \equiv {\sim}\{(\exists x)(x \in \alpha)\}$.

If we replace α with $\hat{z}\varphi z$ this is

${\sim}\exists!\hat{z}\varphi z \equiv {\sim}(\exists x)(x \in \hat{z}\varphi z)$.

Now *Principia* gets this from

$*24.50\ \ \exists!\alpha \equiv (\exists x)(x \in \alpha)$,

by using the obvious *4.11 $p \equiv q\ .\supset.\ {\sim}p \equiv {\sim}q$ (albeit unstated in the proof). There are no brackets on the left side. But it is clear that for transposition to be applicable, the description must have a secondary scope (so that tilde is the main connective).

There is yet more evidence—if one were to worry that we haven't already seen enough. There is a typo in the proof of the following:

$*35.89\ \ \exists!\beta\ .\supset.\ (\alpha\uparrow\beta) \mid (\beta\uparrow\gamma) = (\alpha\uparrow\gamma)\ :\&:\ {\sim}\exists!\beta\ .\supset.\ (\alpha\uparrow\beta) \mid (\beta\uparrow\gamma) = \Lambda$.

In the proof of the second conjunct of the above, we find the line

$$\sim(\exists!\beta) .\supset. \sim [x\, [(\alpha\uparrow\beta) \mid (\beta\uparrow\gamma)]\, z].$$

Now from this we can easily get

$$\sim(\exists!\beta) .\supset. (x)(z)(\sim [x\, [(\alpha\uparrow\beta) \mid (\beta\uparrow\gamma)]\, z]).$$

And the consequent yields $(\alpha\uparrow\beta) \mid (\beta\uparrow\gamma) = \Lambda$. So it is reasonable to conclude that, in spite of the brackets occurring in the one and the absence of brackets in the other, there is no difference in the scopes involved in $\sim(\exists!\beta)$ and $\sim\exists!\beta$. Once again, note that the free lower-case Greek β is just a stand in for a class expression. Thus, in this number, *Principia* makes no difference between $\sim(\exists!\hat{z}\varphi z)$ and $\sim\exists!\hat{z}\varphi z$. Trivial though it is, it is startling that this typo has not yet been noticed. But it is quite fortuitous. It corroborates our thesis that the use of brackets with the tilde does nothing to confine scope.

§4 Definite Descriptions $\iota\alpha f\alpha$

We saw that our thesis demanding smallest scope in the use of lower-case Greek and class expressions $\hat{z}\varphi z$ holds some important results and surprises. But the surprises are more pronounced when we turn to definite descriptions of the form $\iota\alpha f\alpha$. The first thing to realize is that *Principia* should state analogs of *14.01 and *14.02 as follows

$$*14.01_{\alpha}\ \ [\iota\alpha f\alpha][\psi(\iota\alpha f\alpha)] =\text{df}\ (\exists\alpha)(f\beta \equiv_{\beta} \beta = \alpha .\&.\ \psi\alpha)$$

$$*14.02_{\alpha}\ \ E!(\iota\alpha f\alpha) =\text{df}\ (\exists\alpha)(f\beta\ \equiv_{\beta} \beta = \alpha).$$

These do not follow by typical ambiguity from their counterparts *14.01 and *14.02. Observe that given Reducibility (*12.*n*), we always have $E!(\iota\alpha f\alpha)$ and hence the case of definite descriptions of the form $\iota\alpha f\alpha$, *12.*n* yields the theorem:

$$\iota\alpha f\alpha = \iota\alpha f\alpha$$

for every appropriate f. This is quite unlike the case for $\iota x\varphi x$, where

$$\iota x\varphi x = \iota x\varphi x$$

has many false instances.

Of course, definite descriptions of the form $\iota\alpha f\alpha$ like definite descriptions of the form $\iota x\varphi x$ would not be expected to behave fully like genuine terms x, y etc. But they might have been expected to behave like class expressions $\hat{z}\varphi z$, α, β, etc. We have seen that definitions (such as *13.01 and *13.02) made with individual variables *cannot* apply to definite descriptions $\iota x\varphi x$. The same result *should* hold for definitions (such as *24.03) made with class expressions $\hat{z}\varphi z$, (and definitions made with free-lower-case Greek α, β, etc.) That is, they cannot apply to definite descriptions of the form $\iota\alpha f\alpha$. Telling evidence for this thesis, however, is very difficult to find in *Principia*. It is difficult to find because the work endeavors so hard to make it appear as if definite descriptions for classes behave as if they are genuine terms for classes—i.e., it works hard to make it appear as if what happens with lower-case Greek and class expression can happen for definite descriptions of the form $\iota\alpha f\alpha$.

In some numbers definite descriptions such as $\iota\alpha f\alpha$ are reasonably well-behaved. In others they are rather badly misbehaved. Note that

$\mathrm{D}\ {}^{\backprime}R =\mathrm{df}\ (\iota\alpha)(\alpha\ \mathrm{D}\ R)$.

Consider the following proof in which the description behaves reasonably well:

$*33.15 \quad \vec{R}\ {}^{\backprime}y \subseteq \mathrm{D}\ {}^{\backprime}\ R$

demonstration

$x \in \vec{R}\ {}^{\backprime}y \supset_x xRy \quad *32.18$

$x \in \vec{R}\ {}^{\backprime}y \supset_x (\exists y)(xRy) \quad *10.24$

$x \in \vec{R}\ {}^{\backprime}y \supset_x \mathrm{D}\ {}^{\backprime}\ R \quad *33.13$

Only the last line wants clarification. *Principia* has

$*33.13 \quad x \in \mathrm{D}\ {}^{\backprime}R \equiv_x (\exists y)(xRy).$

Now this is

$x \in (\iota\alpha)(\alpha\ \mathrm{D}\ R) \ \equiv_x (\exists y)(xRy)$

$(\exists\alpha)(\ \beta\ \mathrm{D}\ \mathrm{R} \ \ \equiv_\beta \beta = \alpha\ \ .\&.\ x \in \alpha\ ) \ \equiv_x (\exists y)(xRy) \quad *14.01_\alpha$

This is derived by tacit appeal to *20.57.

Unfortunately, turning to *20.51 we discover same infelicitous informal use of schematic letters as occur in the numbers of section *14. *Principia* has:

$*20.57\ \hat{z}\varphi z = \iota\alpha f\alpha.\ .\supset.\ g(\hat{z}\varphi z) \ \equiv\ g(\iota\alpha f\alpha).$

This is wrong as it stands. We need the scope markers, thus:

$c\ *20\ 57 \quad \hat{z}\varphi z = \iota\alpha f\alpha.\ .\supset.\ [\ \hat{z}\varphi z][g(\hat{z}\varphi z)] \ \equiv\ [\iota\alpha f\alpha][\ g(\iota\alpha f\alpha)]\ .$

The following, which omits scope markers under the formal convention of smallest scope possible, is an instance of c*20 57:

$\hat{z}(\exists y)(z\ R\ y) = (\iota\alpha)(\alpha\ \mathrm{D}\ R)\ .\supset.\ x \in \hat{z}(\exists y)(z\ R\ y) \ \equiv\ x \in (\iota\alpha)(\alpha\ \mathrm{D}\ R).$

The smallest scope possible in the consequent clause makes this an abbreviation for

$\hat{z}(\exists y)(z\ R\ y) = (\iota\alpha)(\alpha\ \mathrm{D}\ R)\ .\supset.$

$[\hat{z}(\exists y)(z\ R\ y)][\ x \in \hat{z}(\exists y)(z\ R\ y)] \ \equiv\ [(\iota\alpha)(\alpha\ \mathrm{D}\ R)\ [[x \in (\iota\alpha)(\alpha\ \mathrm{D}\ R)].$

This happens because definition *20.01 $x \in \varphi =df\ \varphi x$ cannot apply to incomplete symbols.

Principia advises its readers to note that appeals to *20.57 will be tacit in many of its proofs (*PM, vol. 1,* p. 244). But this tacit use hides cases where the description is very misbehaved. Consider the case of the demonstration *51.161. Recall that *Principia* has:

$\iota‘x$ =df $(\iota\alpha)(\alpha\ \iota\ x)$.

Now consider this:

*51.161 $\exists!\ \iota‘x$

demonstration

$x \in \iota‘x$ *51.16

$(\exists x)(x \in \iota‘x)$ *10.24

$\exists!\iota‘x$

We are justly bewildered by the demonstration (which gives no annotation for the last line). One cannot apply *24.03 to $(\exists x)(x \in \iota‘x)$ in order to get the result $\exists!\iota‘x$. Instead we have

$(\exists x)(x \in \iota‘x)$

$(\exists x)(\exists\alpha)(\beta\ \iota\ x \ \equiv_\beta \beta = \alpha\ .\&.\ x \in \alpha)$ $*14.01_\alpha$

We need to get

$(\exists\alpha)(\beta\ \iota\ x \ \equiv_\beta \beta = \alpha\ .\&.\ (\exists x)(x \in \alpha))$

To understand what has happened, we have to trace the derivation of *51.16 back to *51.11 which is an identity of the form $\hat{z}\varphi z = \iota\alpha f\alpha$. The proper way to understand the demonstration, therefore, lies it use of the following theorem:

$(\iota\alpha)(\alpha\ \iota\ x) = \hat{z}(z = x)\ .\supset.\ \exists!\hat{z}(z = x) \equiv \exists!(\iota\alpha)(\alpha\ \iota\ x)$.

Let us rewrite the demonstration as follows:

$\iota‘x = \hat{z}(z = x)$ *51.11

$\hat{z}(z = x) = \iota‘x\ .\supset.\ x \in \hat{z}(z = x) \ \equiv\ x \in \iota‘x$ *20.57

$x \in \hat{z}(z = x) \ \equiv\ x \in \iota‘x$

$x \in \iota‘x$ *51.16

$x \in \hat{z}(z = x)$

$(\exists x)(x \in \hat{z}(z = x))$. *10.24

$\exists!\hat{z}(z = x)$ *24.03

$\iota‘x = \hat{z}(z = x)\ .\supset.\ \exists!\hat{z}(z = x) \equiv \exists!\iota‘x$

$\exists!\hat{z}(z = x) \equiv \exists!\iota‘x$

$\exists!\iota‘x$

The key move then is the employment of general theorem:

**20.571 $\hat{z}\varphi z = (\iota\alpha)(f\alpha)\ .\supset.\ \exists!\hat{z}\varphi z \equiv \exists!(\iota\alpha)(f\alpha)$

We have given it a number for easy reference. Unfortunately, for all its many theorems, *Principia* doesn't pause to prove **20.571. Perhaps Whitehead and Russell thought it to be immediate from *20.57. Let's see. Our theorem **20.571 is this:

$$\hat{z}\varphi z = (\iota\alpha) f\alpha \,.\supset.\, (\exists x)(\,[\hat{z}\varphi z][\,x \in \hat{z}\varphi z]) \;\equiv\; [(\iota\alpha) f\alpha\,][(\exists x)(x \in (\iota\alpha) f\alpha)\,].$$

What follows from *20.5, however, is this

$$\hat{z}\varphi z = (\iota\alpha) f\alpha \,.\supset.\; [\hat{z}\varphi z][\,x \in \hat{z}\varphi z] \;\equiv\; [(\iota\alpha) f\alpha\,][x \in (\iota\alpha) f\alpha].$$

Now from this one can readily get:

$$\hat{z}\varphi z = (\iota\alpha) f\alpha \,.\supset.\, (\exists x)(\,[\hat{z}\varphi z][\,x \in \hat{z}\varphi z]) \;\equiv\; (\exists x)(\;[(\iota\alpha) f\alpha\,][x \in (\iota\alpha) f\alpha]).$$

By definition *24.03 this yields:

$$(\iota\alpha)(\alpha\,\iota\,x) = \;\hat{z}(z = x)\;\;.\supset.\; \exists!\hat{z}(z = x) \equiv (\exists x)(\;[(\iota\alpha) f\alpha\,][x \in (\iota\alpha) f\alpha]).$$

But this is not yet our theorem **20.571. What is needed is:

$$(\exists x)(\,[(\iota\alpha) f\alpha\,][x \in (\iota\alpha) f\alpha]) \equiv\; [(\iota\alpha)(\alpha\,\iota\,x)][(\exists x)(\,x \in (\iota\alpha)(\alpha\,\iota\,x))\,]\,.$$

This is easily proved by $*14.01_{\alpha}$ we have:

$$\exists!(\iota\alpha)f\alpha =df\;\; [(\iota\alpha) f\alpha\,][(\exists x)(x \in (\iota\alpha) f\alpha)\,].$$

Thus we have our theorem **20.571.

There are odd cases in which the use of a theorem **20.571 is implicit. Consider the following derivation:

$*33.4\;\; D\,‘R = \hat{z}(\exists!\overleftarrow{R}\,z)$

demonstration

$x \in D\,‘R \equiv (\exists y)(xRy)$ *33.13

$x \in D\,‘R \equiv (\exists y)(y \in \overleftarrow{R}\,‘x)$ *32.181

$x \in D\,‘R \equiv \exists!\overleftarrow{R}\,‘x$ *24.5

$x \in D\,‘R \equiv_x x \in \hat{z}(\exists!\overleftarrow{R}\,‘z)$ *20.33

$D\,‘R = \hat{z}(\exists!\overleftarrow{R}\,z)$

Now $(\exists y)(xRy)$ yields $(\exists y)(y \in \hat{z}(\,z\,\overleftarrow{R}\,x))$ and by *20.57 this in turn yields $(\exists y)(y \in \overleftarrow{R}\,‘x)$. Also by *24.5 $(\exists y)(y \in \hat{z}(\,z\,\overleftarrow{R}\,x))$ yields $\exists!\hat{z}(\,z\,\overleftarrow{R}\,x)$. But we don't yet have

$$\exists!\overleftarrow{R}\,‘x.$$

Appeal to our **∃!20.57 solves the problem, we have

$$\hat{z}\,(\,z\,\overleftarrow{R}\;\,x) = \;\overleftarrow{R}\,‘x\;\;.\supset.\;\; \exists!\hat{z}(\,z\,\overleftarrow{R}\,x) \equiv \exists!\overleftarrow{R}\,‘x.$$

By *32.111 we have $\hat{z}\,(\,z\,\overleftarrow{R}\;\,x) = \;\overleftarrow{R}\,‘x$, and we have $\exists!\hat{z}(\,z\,\overleftarrow{R}\,x)$. Hence, we can arrive at $\exists!\overleftarrow{R}\,‘x$. It is by means of **20.571 that it appears as if definition *24.03, which is formulated for class expressions, applies to descriptions of the form $(\iota\alpha) f\alpha$.

The derivation of our theorem**20.571 is useful since it applies in several places in *Principia*. Consider for example the following:

*60.31 $\exists!\,\mathrm{Cl}\,‘\alpha$ [*60.3, *10.24].

Now *60.3 is $\Lambda \in \mathrm{Cl}\,‘\alpha$ and by the analog for lower-case Greek of *10.24 we have $(\exists\beta)(\beta \in \mathrm{Cl}\,‘\alpha)$, but we cannot apply the analog of *24.03,

*24.03$_\alpha$ $\exists!\alpha^{(t)}$ =df $(\exists\beta^t)(\beta^t \in \alpha^{(t)})$,

to arrive at $\exists!\,\mathrm{Cl}\,‘\alpha$. Instead we need **20.57$_\alpha$ in the form

$\hat{\beta}(\beta \subseteq \alpha) = \mathrm{Cl}\,‘\alpha\ .\supset.\ \exists!\hat{\beta}(\beta \subseteq \alpha) \equiv \exists!\,\mathrm{Cl}\,‘\alpha.$

In this case, we have to track the derivation back through several numbers to arrive at the antecedent

*60.12 $\mathrm{Cl}\,‘\alpha = \hat{\beta}(\beta \subseteq \alpha)$.

But likely this is the sort of proof that was intended. A similar case must be made for

*103.13 $\exists!\mathrm{N_0c}\,‘\alpha$ [*103.12, *10.24].

Recall that here we have a homogeneous cardinal. That is:

$\mathrm{N_0c}\,‘\alpha^t =df\, (\iota\sigma^{(t)})(\sigma^{(t)}\ \overrightarrow{sm}\ \alpha^t)$

$\sigma^{(t)}\ \overrightarrow{sm}\ \alpha^t\ .\equiv.\ \sigma^{(t)} = \hat{\beta}^t(\beta^t\ sm\ \alpha^t)$.

The number *103.12 is $\alpha^t \in \mathrm{N_0c}\,‘\alpha^t$ and so *10.24 yields $(\exists\beta^t)(\beta^t \in \mathrm{N_0c}\,‘\alpha^t)$. But we cannot apply *24.03$_\alpha$. Once again, what is needed is **20.571$_\alpha$ so that we have:

$\hat{\beta}^t(\beta^t\ sm\ \alpha^t) = \mathrm{N_0c}\,‘\alpha^t\ .\supset.\ \exists!\hat{\beta}^t(\beta^t\ sm\ \alpha^t) \equiv \exists!\,\mathrm{N_0c}\,‘\alpha^t.$

Then from

*100.1 $\hat{\beta}^t(\beta^t\ sm\ \alpha^t) = \mathrm{N_0c}\,‘\alpha^t$

the proof of *103.13 follows.

The demonstrations in cases such as *33.4, *60.13 and *103.13 seem to be more of a failing than *Principia's* proof of $\exists!\iota‘x$, which simply omits the annotation of the central step. But matters are worse when it comes to the demonstration *Principia* offers at

*37.46 $x \in R\,‘‘\alpha \supset_x \exists!\alpha \cap \overleftarrow{R}\,‘x$

$x \in R\,‘‘\alpha \supset (\exists y)(y \in \alpha\ \&\ xRy)$ *37.1

$y \in \overleftarrow{R}\,‘x \equiv yRx$ *32.181.

This demonstration suggests that we can arrive at $\exists!\alpha \cap \overleftarrow{R}\,‘x$ rather immediately from

$(\exists y)(y \in \alpha\ \&\ y \in \overleftarrow{R}\,‘x)$.

But the definitions

*24.03 $\exists!\alpha$ =df $(\exists x)(x \in \alpha)$

*22.02 $\alpha \cap \beta$ =df $\hat{z}(z \in \alpha \ \& \ z \in \beta)$

cannot be applied to $\exists!\alpha \cap \overleftarrow{R}\,‘x$. The definite description $\overleftarrow{R}\,‘x$ which is $(\iota\beta)(\beta \overleftarrow{R}\, x)$ must be eliminated before we can apply definitions *22.04 and *24.03. Thus $\exists!\alpha \cap \overleftarrow{R}\,‘x$ is

$$[\overleftarrow{R}\,‘x][\,(\exists y)(y \in \alpha \ \& \ y \in \overleftarrow{R}\,‘x)\,]$$

which by *14.01$_{\alpha}$ is

$$(\exists\beta)(\gamma \overleftarrow{R}\, x \equiv_{\gamma} \gamma = \beta \,.\&.\, \exists!\alpha \cap \beta).$$

Perhaps the best way to flush out the demonstration at *37.46 is to proceed as follows:

1. $x \in R‘‘\alpha \supset (\exists y)(y \in \alpha \ \& \ xRy)$ *37.1
2. $[\overleftarrow{R}\,‘x][y \in \overleftarrow{R}\,‘x] \equiv yRx$ *32.181.
3. $(\exists y)(y \in \alpha \ \& \ [\overleftarrow{R}\,‘x][y \in \overleftarrow{R}\,‘x]) \supset (\exists y)(y \in \alpha \ \& \ [\overleftarrow{R}\,‘x][y \in \overleftarrow{R}\,‘x])$
4. $(\exists y)(y \in \alpha \ \& \ [\overleftarrow{R}\,‘x][y \in \overleftarrow{R}\,‘x]) \supset [\overleftarrow{R}\,‘x][\,(\exists y)(y \in \alpha \ \& \ y \in \overleftarrow{R}\,‘x)\,]$
5. $x \in R‘‘\alpha \supset [\overleftarrow{R}\,‘x][\,(\exists y)(y \in \alpha \ \& \ y \in \overleftarrow{R}\,‘x)\,]$
6. $x \in R‘‘\alpha \supset \exists!\alpha \cap \overleftarrow{R}\,‘x$ *22.02, *24.03

The key to the demonstration is:

$$(\exists y)(y \in \alpha \ \& \ [\overleftarrow{R}\,‘x][y \in \overleftarrow{R}\,‘x]) \supset [\overleftarrow{R}\,‘x][\,(\exists y)(y \in \alpha \ \& \ y \in \overleftarrow{R}\,‘x)\,].$$

This, of course, is quite easily proved.

We must take care not be misled by *Principia*'*s* efforts to hide the fact that definite descriptions are not genuine terms. Many of *Principia's* demonstration *pretend* that definitions made with class expressions $\hat{z}\varphi z$ and lower-case Greek can be applied directly to definite descriptions. No *errors* can be charged to these demonstrations precisely because they are demonstrations. They are not intended to be regarded as detailed proofs. However, there are cases where we cannot be quite so charitable. Consider the following:

*51.511 $\breve{\iota}\,‘\iota‘x = x$ [*51.51 $\frac{\iota‘x}{\alpha}$. *20.2].

Now *Principia* is suggesting that we substitute into *51.51 as follows

$$\alpha = \iota\,‘x \,.\!\equiv\!.\, x = \breve{\iota}\,‘\alpha \quad \frac{\iota‘x}{\alpha} \quad .$$

But $\iota‘x$ abbreviates $(\iota\beta)(\beta \ \iota\, x)$, and hence it is not properly substitutable for a class expression α in the above. Perhaps the best way to rectify the demonstration at *51.511 is to put:

$$\alpha = \iota\,‘x \,.\!\equiv\!.\, x = \breve{\iota}\,‘\alpha \quad \frac{\hat{z}(z=x)}{\alpha}$$
$$\hat{z}(z = x) = \iota\,‘x \,.\!\equiv\!.\, x = \breve{\iota}\,‘\hat{z}(z = x)$$
$$\hat{z}(z = x) = \iota\,‘x \quad \text{*51.1}$$
$$x = \breve{\iota}\,‘\hat{z}(z = x)$$
$$\breve{\iota}\,‘\hat{z}(z = x) = \breve{\iota}\,‘\iota‘x$$
$$\breve{\iota}\,‘\iota‘x = x$$

For this approach we need the theorem:

$$\hat{z}(z = x) = \iota\ `x \ .\supset.\ \breve{\iota}\ `\hat{z}(z = x) = \breve{\iota}\ `\iota`x.$$

This follows from the general theorem:

$$\alpha = \iota\beta f\beta \ .\supset.\ \breve{R}\ `\alpha = \breve{R}\ `\iota\beta f\beta \ .$$

And this is readily derived from

$$\alpha = \beta \ \ .\supset.\ (R)(\ R\ `\alpha = R`\beta\ )$$

which comes from *c**20.57. The same odd illicit substitution occurs at *51.52. The source of the misstep probably derives from *Principia's Introduction* which explains some notations by introducing informal definitions. Whitehead and Russell write (*PM.vol. 1*, p. 36):

> We denote by $\iota`x$ the class whose only member is x. Thus,
> $\iota`x = \hat{z}\ (z = x)$ Df..
> *i.e*, "$\iota`x$" means "the class of objects which are identical with x".

The same convenience employed in the *Introduction* in asserting

$$\breve{R} = \hat{x}\hat{y}(x\ Ry)$$
$$\mathrm{Cnv}\ `R = \breve{R}.$$

And here Whitehead and Russell hasten to add: "The second of these is not a formally correct definition since we ought to define "Cnv" and deduce the meaning of Cnv 'R. But it is not worthwhile to adopt this plan in our present introductory account, which aims at simplicity rather than formal correctness" (*PM, vol. 1., p. 32).* Perhaps, Whitehead and Russell had forgotten about the formal definitions when they wrote the inappropriate substitutions at *51.511 and *51.52. Thus, they are not straightforward errors.

This, however, brings us to something which can only be interpreted as a straightforward error. This is the curious case of $\exists!\iota\alpha f\alpha$ and $\sim\exists!\iota\alpha f\alpha$. According to our thesis, these differ significantly from $\exists!\alpha$ and $\sim\exists!\alpha$ which, as we have conclusively shown in our §2, involve secondary scopes for α, i.e., $\hat{z}\varphi z$. Consider $\exists!\ \vec{R}\ `y$ and $\sim\exists!\ \vec{R}\ `y$ which are alternatively written as $\exists!\ (\iota\alpha)(\ \alpha\vec{R}\ y)$ and $\sim\exists!\ (\iota\alpha)(\ \alpha\vec{R}\ y)$ respectively. Now *24.03 cannot apply to $\exists!\ \vec{R}\ `y$ and hence the scope is forced to be primary. This yields:

$$\exists!\ \vec{R}\ `y =df\ \ [\vec{R}\ `y][\exists!\ \vec{R}\ `y].$$

Otherwise put, this is:

$$\exists!\ (\iota\alpha)(\ \alpha\vec{R}y) \ =df\ \ [(\iota\alpha)(\ \alpha\vec{R}y)][\exists!\ (\iota\alpha)(\ \alpha\vec{R}y)]$$

Eliminating the definite description then yields:

$$(\exists\alpha)(\ \beta\vec{R}y \equiv_{\beta} \beta = \alpha\ .\&.\ \exists!\alpha)\ .$$

Its contradictory (both syntactic and semantic) is $\sim\exists!\ \vec{R}$ ‘y. We have:

$\exists!\ \vec{R}$ ‘y $=df$ $\sim[\vec{R}$ ‘$y][\exists!\ \vec{R}$ ‘$y]$.

Otherwise put this is:

$\sim\exists!\ (\iota\alpha)(\alpha\vec{R}y)$ $=df$ $\sim[(\iota\alpha)(\alpha\vec{R}y)][\exists!\ (\iota\alpha)(\alpha\vec{R}y)]$.

Be this as it may, strange comments follow *32.121. *Principia* writes (*PM*, p. 244):

> *32.12 E! $\vec{R}$‘y
>
> *32.121 E! $\overleftarrow{R}$‘y
>
> “E! $\vec{R}$‘y” must not be confounded with “$\exists!\vec{R}$‘y.” The former means that there is a class as $\vec{R}$‘y, which, as we have just seen, is always true; the latter means that $\vec{R}$‘y is not null, which is only true if y is a term to which some other term has the relation R. Note that by *14.21 both $\exists!\ \vec{R}$‘y and $\sim\exists!\ \vec{R}$‘y imply E! $\vec{R}$‘y. The contradictory of $\exists!\vec{R}y$ is not $\sim\exists!\ \vec{R}$‘y but $\sim\{[\vec{R}$‘$y]$. $\exists!\ \vec{R}$‘$y\}$. This last would not imply E! $\vec{R}$‘y but for the fact that E!$\vec{R}$‘y is always true.

Principia’s comments after *32.121 are bewildering! In a stunning nonsequitur, *Principia* says that by *14.21 we see that $\sim\exists!\ \vec{R}$‘y is not the contradictory of $\exists!\vec{R}y$. We have seen some typos and *faux pas*, but this is beyond the pale!

Now we have seen that E! $\vec{R}$‘y is proved at *32.12 and this is involves the following instance of of *14.21_{α} with the scope markers dropped:

$\hat{z}(z\ \mathrm{R}\ y) = \vec{R}$‘$y \supset$ E! $\vec{R}$‘y .

There is also little wonder that

$\sim\exists!\ \vec{R}$‘$y \supset$ E! $\vec{R}$‘y

Since E! $\vec{R}$‘y is always true, this readily follows from E! $\vec{R}$‘y together with,

*2.02 $q .\supset. p \supset q$.

In any event, $\sim\exists!\ \vec{R}$‘y is the contradictory of $\exists!\vec{R}y$. Whitehead and Russell are correct that they both entail E! $\vec{R}$‘y. They are wrong, however, to say that they are not contradictories. Indeed, it should be noted, that since Reducibility assures

E!$(\iota\alpha)(\alpha\,\vec{R}\,y)$,

if follows that

$[(\iota\alpha)(\alpha\,\vec{R}\,y)][\ \exists!(\iota\alpha)(\alpha\,\vec{R}\,y)]$

$[(\iota\alpha)(\alpha\,\vec{R}\,y)][\ \sim\exists!(\iota\alpha)(\alpha\,\vec{R}\,y)]$

are *also* contradictories! To be sure, these are not *syntactic* contradictories. Eliminating the descriptions they are (respectively):

$$(\exists\alpha)(\beta\,\vec{R}y \equiv_\beta \beta = \alpha. .\&. \exists!\alpha)$$

$$(\exists\alpha)(\beta\,\vec{R}y \equiv_\beta \beta = \alpha. .\&. {\sim}\exists!\alpha).$$

But given $E!(\iota\alpha)(\alpha\,\vec{R}\,y)$, they are *semantic* contradictories. The former asserts there is an α that has some member; the latter asserts that this same α has no member.

Principia comments after *32.121 are just mistaken. What happened? Observe that because of Reducibility it turns out that

$$\exists!\,\vec{R}\text{‘}y = \Lambda$$
$$\exists!\,\vec{R}\text{‘}y \neq \Lambda$$

are *not* contradictories. Eliminating the descriptions, they are (respectively):

$$(\exists\alpha)(\,\beta\vec{R}y \equiv_\beta \beta = \alpha .\&. \alpha = \Lambda)$$
$$(\exists\alpha)(\,\beta\vec{R}y \equiv_\beta \beta = \alpha .\&. \alpha \neq \Lambda).$$

Applying *20.07 and *20.071, these yield:

$$(\exists\varphi)(\hat{z}\psi!z\,\vec{R}y \equiv_\psi \hat{z}\psi!z = \hat{z}\varphi!z \;.\&.\; \hat{z}\varphi!z = \Lambda)$$
$$(\exists\varphi)(\hat{z}\psi!z\,\vec{R}y \equiv_\psi \hat{z}\psi!z = \hat{z}\varphi!z \;.\&.\; \hat{z}\varphi!z \neq \Lambda).$$

Both of these are true because we have

$$\hat{z}\varphi!z = \Lambda \;\; =df \;\; (\exists\Gamma)(\Gamma!z \equiv_z \varphi!z .\&. (\exists\theta)(\theta!z \equiv_z z \neq z .\&. \Gamma!\hat{z} = \theta!\hat{z}))$$

$$\hat{z}\varphi!z \neq \Lambda \;\; =df \;\; (\exists\Gamma)(\Gamma z \equiv_z \varphi z .\&. (\exists\theta)(\theta z \equiv_z z \neq z .\&. \Gamma!\hat{z} \neq \theta!\hat{z})).$$

That is, we can find θ and θ*, both unexemplified, which are such that $\theta!\hat{z} \neq \theta^*!\hat{z}$.

Now from the fact that $\exists!\,\vec{R}\text{‘}y = \Lambda$ and $\exists!\,\vec{R}\text{‘}y \neq \Lambda$ it by no means follows that ${\sim}\exists!\,\vec{R}\text{‘}y$ is not the contradictory of $\exists!\vec{R}y$. Nonetheless, in the summary of section *32 Whitehead and Russell write (*PM vol.* 1, p. 243):

> Thus by *14.21 we always have $E!\,\vec{R}\,\text{‘}y$ and $E!\,\overleftarrow{R}\text{‘}x$. …Thus taking R to be the relation of parent to child, $\vec{R}\,\text{‘}y$ = the parents of *y* and $\overleftarrow{R}\text{‘}x$ = the children of *x*. Thus, $\overleftarrow{R}\text{‘}x = \Lambda$ i.e.,${\sim}\exists!\,\overleftarrow{R}\text{‘}x$, when *x* is childless and $\vec{R}\,\text{‘}y = \Lambda$, i.e. ${\sim}\exists!\,\vec{R}\,\text{‘}y$ when *y* is Adam or Eve. The two existences $E!\,\vec{R}\,\text{‘}y$ and $\exists!\,\vec{R}\,\text{‘}y$ can both be *significantly* be predicated of $\vec{R}\,\text{‘}y$, because “ $\vec{R}\,\text{‘}y$” is a descriptive function whose value is a class; and the same applies to $\overleftarrow{R}\text{‘}x$. it will be seen that (by *14.21) $\exists!\,\vec{R}\,\text{‘}y \supset E!\,\vec{R}\,\text{‘}y$, but the converse implication does not hold in general.

Whitehead and Russell have misused the theorem:

*24.51 $\sim\exists!\alpha \equiv \alpha = \Lambda$.

Now the following is indeed provable because of Reducibility:

$\sim\exists!(\iota\alpha)(f\alpha) \equiv (\iota\alpha)(f\alpha) = \Lambda$.

But the scope of lower-case Greek α in $\exists!\alpha$ is not the same as that of $\exists!\iota\alpha f\alpha$. This is the likely source of the error in the comments after *32.121. Happily, the error does not show up in proofs.

The number *14.21 looms large in many existence proofs of the form E! $\iota\alpha f\alpha$. Salient examples in *Principia's* volume 1 are these:

*31.13 E! *Cnv*‘*P*
*32.12 E! $\vec{R}$‘*y*
*33.12 E!D ‘*R*
*33.12 E! ᗡ ‘*R*
*33.112 E! C ‘*R*
*72.16 E!*p* ‘*k*
*80.12 E!P_{Δ} ‘*k*

In each of these cases, the annotation of the proof employs *14.21 referring to

$\psi(\iota x\varphi x) \supset E!(\iota x\varphi x)$.

We saw, however, that this violates the formal scope convention on omission of scope markers. The markers have to be restored. Hence, we have:

*c**14.21 $[\iota x\varphi x[\psi(\iota x\varphi x)] \supset E!(\iota x\varphi x)$.

But even this is not what is needed for these proofs. What is needed is this:

c_{α} *14.21 $[\iota\alpha f\alpha][\psi(\iota\alpha f\alpha)] \supset E!(\iota\alpha f\alpha)$.

Theorem *c**14.21 is certainly typically ambiguous. Nonetheless, c_{α}*14.21 is not an instance of it by employing typical ambiguity. Lower-case Greek is defined, and classes cannot be understood as individuals of a given type. Theorem *c* *14.21_{α} is involved in the proofs of the above. Each of the proofs employ an instance of the following:

$(\iota\alpha)(f\alpha) = \hat{z}\varphi z \,.\supset.\, E!\,(\iota\alpha)(f\alpha)$.

Here the scope marker can be omitted from the antecedent clause because the *wff* is known and indeed it is known to afford a primary scope. For example, consider the proof of

*32.12 E! $\vec{R}$‘*y*.

Note that by definition *30.01 we have the following:

$\vec{R}$ ‘*y* =df $(\iota\alpha)(\alpha\vec{R}y)$.

The proof of *32.12 employs the following instance of *c* *14.21_{α} with its scope marker omitted for convenience:

$\hat{x}(x \text{ R } y) = \vec{R}\text{ ‘}y \supset \text{E! } \vec{R}\text{‘}y.$

Then by appeal to

*32.11 $\hat{x}(x \text{ R } y) = \vec{R}\text{ ‘}y$

we have the theorem. Of course, *Principia* first offers a proof of *32.11 itself. That leads us to another oddity. The proof offers

*30.3 $x = \vec{R}\text{ ‘}y \,.\equiv.\, z \text{ R } y \equiv_z z = x$.

The trouble is that

*30.3$_\alpha$ $\hat{x}(x \vec{R}\, y) = \vec{R}\text{ ‘}y \,.\equiv.\, \beta \text{ R } y \equiv_\beta \beta = \hat{x}(x \vec{R}\, y)$.

is not an instance of *30.3. *Principia* is just caviler about these differences.

Searching for a better understanding of the use of *14.21 in *Principia* reveals a demonstration with a typo. We find a problem in the annotation for

*33.125 $\gamma \text{C} R \equiv \gamma = \text{C ‘}R.$

The annotation wrongly cites *32.123 which is a *non-existent* number. The proper citation should be to

*33.122 E!C ‘R

which is proved by an appeal to *14.21. Oddly, in the 260-leaf catalogue "Props Where Used", there is an entry that says that *33.123 is used at *33.125. This is odd, since the proper number used at *33.125 should clearly be *33.122 which concerns the campus (field) of R. Deleted propositions got listed at a certain stage, Russell never listed a *32.123 though there may once have been such a number. [9]

§5 The missing *shriek*

The mistake in the comments at *32.121 draws attention to *14.21. And *14.21 returns our attention to the questions surrounding the many issues surrounding truth-functionality and extensionality. We noted that *Principia*'s object-language does not allow contexts of modality or belief. But its formal language is not extensional. Its allows non-truth functional contexts made when the identity sign is flanked by predicate variables. Even with Reducibility, primary and

[9] This discovery was made in conversation by Kenneth Blackwell of the Bertrand Russell Research Center (McMaster University, Hamilton, Ontario, Canada).

secondary scopes of class expressions may well not be equivalent when the context is non-extensional. Whitehead and Russell were well aware of this. They write (*PM, vol. 1*, p. 84):

> It will be observed that $\theta!\hat{z} = \psi!\hat{z}$ is not an extensional function of $\psi!\hat{z}$. Thus the scope of $\hat{z}\varphi z$ is relevant in interpreting the product
>
> $\hat{z}\varphi z = \psi!\hat{z} \,.\, \hat{z}\varphi z = \chi!\hat{z}$.
>
> …We may say generally that the fact that $\hat{z}\varphi z$ is an incomplete symbol is not relevant so long as we confine ourselves to extensional functions of function, but is apt to become relevant for other functions of functions.

One can even imagine a case where a primary scope does not entail a secondary scope. Observe that

$$[\hat{z}\varphi!z][\sim\{\hat{z}\varphi!z = \varphi!\hat{z}\}] \supset \sim\{[\hat{z}\varphi!z][\hat{z}\varphi!z] = \varphi!\hat{z}\}$$

is false. This may be something of a surprise, since we may be inclined to think that the primary occurrence should entail the secondary.[10] It does not. We can see this when we apply *20.01 to arrive at

$$(\exists\Gamma)(\Gamma!z \equiv_z \varphi!z \,.\&.\, \sim\{\Gamma!\hat{z} = \varphi!\hat{z}\}) \supset \sim\{(\exists\Gamma)(\Gamma!z \equiv_z \varphi!z \,.\&.\, \Gamma!\hat{z} = \varphi!\hat{z})\}.$$

The consequent is false since quite clearly $\varphi!\hat{z} = \varphi!\hat{z}$. The antecedent, however, is true.

Even with Reducibility (the axiom assuring classes) primary and secondary scopes of a description of the form $\iota\alpha f\alpha$ may well not be equivalent when the context is non-extensional. The non-extensionality cannot show up in the formal language with contexts involving only $\iota x\varphi x$, but it can show up with contexts involving descriptions of the form $\iota\alpha f\alpha$ and with predicate variables in contexts such as $\sim\{\iota\alpha f\alpha = \varphi\hat{z}\}$. Just as in the case of class expressions we find that

$$[\iota\alpha\varphi!\alpha][\sim\{\iota\alpha\varphi!\alpha = \varphi!\hat{z}\}] \supset \sim\{[\iota\alpha\varphi!\alpha][\iota\alpha\varphi!\alpha] = \varphi!\hat{z}\},$$

is false. Consider the analog of *14.18 for descriptions of the form $\iota\alpha\varphi\alpha$. We have:

$$E!(\iota\alpha\varphi\alpha) \,.\supset.\, (\alpha)\,\psi\alpha \supset \psi(\iota\alpha\varphi\alpha),$$

By *20.07 we can see that this is:

$$E!(\iota\alpha\varphi\alpha) \,.\supset.\, (\Gamma)\,\psi(\hat{z}\Gamma!z) \supset \psi(\iota\alpha\varphi\alpha).$$

This adds a new dimension because $\psi(\hat{z}\Gamma!z)$ secures a secondary scope for the class expression $\hat{z}\Gamma!z$. But this still does not avoid the problem that it has false instances in certain non-extensional contexts. Once again we need:

$$c_\alpha\text{*14.18} \quad E!(\iota\alpha\varphi\alpha) \,.\supset.\, (\alpha)\,\psi\alpha \supset [\iota\alpha\varphi\alpha][\psi(\iota\alpha\varphi\alpha)],$$

---

[10] It should surprise Linsky who correctly maintains that *Principia* allows non-truth functional contexts. See Linsky (1983), p. 160.

Similarly we shall need:

c_α *14.22 $E!(\iota\alpha\varphi\alpha) .\supset. [\iota\alpha\varphi\alpha][\varphi(\iota\alpha\varphi\alpha)]$.

With this in place, all is well.

This assumption of truth-functionality is important, but so also is the assumption of extensionality. In explaining the issue of truth-functionality we find the following:

c*14.3

$(p \equiv q .\supset_{p,q} . fp \equiv fq\ ) \ \&\ E!(\iota x\varphi x) .\supset.\ f\{[\iota x\varphi x][\chi(\iota x\varphi x)]\} \equiv [\iota x\varphi x][f\{\chi(\iota x\varphi x)\}]$

Whitehead and Russell comment (PM, vol. 1, p.185):

> In this proposition, however, the use of propositions as apparent variables involves an apparatus not require elsewhere, and we have therefore not used this proposition in subsequent proofs.

They go on to say that *14.3 "… involves propositions (*p*, *q* namely) as apparent variables, which we have not done elsewhere, and cannot do legitimately without the explicit introduction of the hierarchy of propositions with a reducibility axiom such as *12.1". Now of course, *14.3 is wholly outside of the formal grammar of *Principia*. It is therefore not part of the formal theory. Nonetheless, Whitehead and Russell would be correct to point out every instance of the following is a theorem:

c *14.3 $E!(\iota x\varphi x) .\supset.\ f\{[\iota x\varphi x][\chi(\iota x\varphi x)]\} \equiv [\iota x\varphi x][f\{\chi(\iota x\varphi x)\}]$,

where χ is truth-functional. Extensionality cannot come into play. For definite descriptions of the form $\iota\alpha f\alpha$, however, the analog is this:

c_α *14.3 $E!(\iota\alpha f\alpha) .\supset.\ f\{[\iota\alpha f\alpha][\chi(\iota\alpha f\alpha)]\} \equiv [\iota\alpha f\alpha][f\{\chi(\iota\alpha f\alpha)\}]$,

where f is truth-functional and χ is extensional. This parallels the situation of class expressions:

$c_{\alpha\alpha}$*14.3 $E!(\hat{z}\varphi z) .\supset.\ f\{[\hat{z}\varphi z][\chi(\hat{z}\varphi z)]\} \equiv [\hat{z}\varphi z][f\{\chi(\hat{z}\varphi z)\}]$,

where f is truth-functional and χ is extensional. In the comments after *14.3 Whitehead and Russell observe that this cannot be proved in general [without employing mathematical induction on the length of a *wff* of *Principia's* object-language]. In this section (and in section *9) they did not want to employ mathematical induction, since mathematical induction is precisely the sort of theorem that a demonstration of Logicism is expected to prove. [11]

There are yet a few more typos before we can rest—that is, rest our case in this paper. We find that *Principia* has

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[11] This is made explicit in *Principia's* comments on proofs in the quantification theory of section *9. See PM, p. 129.

*51.59 $\psi(\breve{\imath}\text{‘}\hat{z}\varphi z) \equiv \psi(\iota x\varphi x)$.

Once again this violates the formal convention on omission of scope markers. We can rectify this by adding a shriek to form

*shriek**51.59 $\psi!(\breve{\imath}\text{‘}\hat{z}\varphi z) \equiv \psi!(\iota x\varphi x)$.

But as in other cases from section *14, more likely what is intended is

*c**51.59 $[\breve{\imath}\text{‘}\hat{z}\varphi z][\psi(\breve{\imath}\text{‘}\hat{z}\varphi z)] \equiv [\iota x\varphi x][\psi(\iota x\varphi x)]$.

From this one can arrive at *shriek**51.59 as a special case in which scope markers may be omitted because the predicate variable $\psi!$ forces a primary occurrence of the definite description.

This brings us to an alarming derivation at *14.17. Whitehead and Russell take themselves to have proved:

*14.15 $\iota x\varphi x = b \,.\supset.\, \psi(\iota x\varphi x) \equiv \psi b$.

Then by universal generalization (*10.1) and elementary logic, they arrive at

*14.17 $\iota x\varphi x = b \,.\supset.\, \psi!(\iota x\varphi x) \equiv_{\psi} \psi! b$.

This universal generalization illicit since ψ in *14.15 is a schematic letter. *Principia* is quite explicit that letters without the shriek cannot be generalized (*PM*, *vol. 1*, p. 165). Properly understood, what has been derived at *14.15 is this

*c**14.15 $\iota x\varphi x = b \,.\supset.\, [\iota x\varphi x][\psi(\iota x\varphi x)] \equiv \psi b$.

Now an instance of this is

*shriek**14.15 $\iota x\varphi x = b \,.\supset.\, \psi!(\iota x\varphi x) \equiv \psi! b$.

Since $\psi!$ is a predicate variable, the scope marker can be dropped for the smallest scope possible is the primary scope. It is from *shriek**14.15 that *14.17 is properly derived by universal generalization. A similar situation arises *20.19 which offers the misleading appearance that it is arrived at by a universal generalization on a schematic letter *f* at *20.18. That is we have

c *20.18 $\hat{z}\varphi z = \hat{z}\psi z \,.\supset.\, [\hat{z}\varphi z][\psi(\hat{z}\varphi z)] \equiv [\hat{z}\psi z][\psi(\hat{z}\psi z)]$.

An instance of this is the following:

*shriek**20.18 $\hat{z}\varphi z = \hat{z}\psi z .\supset. \psi!(\hat{z}\varphi z) \equiv \psi!(\hat{z}\psi z)$.

It is from *shriek* *20.18 that the universal generalization at *20.19 is derived. This is the problem of the missing shriek (exclamation). And with that, we have exhausted, so far as I know, the known types of typos concerning incomplete symbols in *Principia*.

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