"Knowing What" as a Normal Modal Logic

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Background

From K_i to Kv_i

Classic epistemic logic (von Wright and Hintikka) studies the inference patterns about propositional knowledge, by using a modal operator K_i to express that agent *i knows that* a proposition is true. One basic idea is that, by formalizing intuitive ideas about knowledge, we can get systems which depicts complex situation w.r.t. knowledge, for example, distributed system and imperfect information games.

From K_i to Kv_i

However, daily knowledge claims include more than propositional knowledge, as is often expressed in terms of knowing the answer to an embedding question:

- **Knowing Whether**: We don't know whether there will be freshman in department of philosophy next semester.
- **Knowing How**: She knows how to cook, but she doesn't know how to wash plates.
- **Knowing Why**: I don't know why we do not have holiday today.
- **Knowing What**: No one knows the password.

From K_i to Kv_i

Technically, the resulting logic are usually *not* normal, in the sense that some usual modal axioms and rules are no longer valid. For example:

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$$Kh_i p \land Kh_i q \to Kh_i(p \land q)$$

From K_i to Kv_i

But this does not mean that we have to abandon the modal route. We can still use Kripke models with different but still intuitive semantics for some non-classic knowledge.

Example: Conditionally Knowing What

For example, the conditional knowing value logic proposed in has the following language $\mathbf{ELKv}^r$:

$$\phi ::= \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid K_i\phi \mid Kv_i(\phi, c)$$

where $Kv_i(\phi, c)$ says i knows the value of c given ϕ.

Example: Conditionally Knowing What

The language is interpreted on first-order Kripke models $\mathcal{M} = \langle S, D, \to, V, V_C \rangle$ where D is a *constant* domain, and $V_{\mathbb{C}}$ assigns to each (non-rigid) $c \in \mathbb{C}$ an element in D on each $s \in S$:

$$\mathcal{M}, s \vDash Kv_i(\phi, c) \iff \text{for any } t_1, t_2: \text{ if } s \to_i t_1, s \to_i t_2, \mathcal{M}, t_1 \vDash \phi \text{ and } \mathcal{M}, t_2 \vDash \phi \text{ then } V_C(c, t_1) = V_C(c, t_2).$$

Example: Conditionally Knowing What

The idea of this semantics is to take $Kv_i(\phi, c)$ as the first-order formula $\exists x K_i(\phi \to c = x)$.

Rich Model, Simple Language

There is always asymmetry between the relatively simple language and the rich models:

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There is always asymmetry between the relatively simple language and the rich models:

This opens the possibility for improvement.

Conditionally Knowing What as Normal Modal Logic

Balancing the Asymmetry

The asymmetry suggests that we can either enrich the language or simplifying the models.

Basic Ideas for MLKv^r

Two hints:

- The values (i.e. elements of the constant domain $\mathbb{D}$) are not explicit in our language. What is essential is the equivalence relation.

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- The values (i.e. elements of the constant domain $\mathbb{D}$) are not explicit in our language. What is essential is the equivalence relation.
- The "$\forall$" expression in the semantics is rather modal-like.

Basic Ideas for MLKv^r

The above ideas can be formalized as:

- **Model**: $\langle S, \{\rightarrow_i \mid i \in \mathbf{I}\}, \{R_i^c \mid i \in \mathbf{I}, c \in \mathbb{D}\}, V\rangle$, where new ternary relations R_i^c replace the constant domain $\mathbb{D}$ and the assignment function $V_{\mathbb{C}}$,
- **Semantics**: $\mathcal{M}, s \vDash \Diamond_i^c \phi \iff \exists t, u$ such that $sR_i^c tu$, $\mathcal{M}, t \vDash \phi$ and $\mathcal{M}, u \vDash \phi$. The semantics is closer to that of arbitrary modal similarity type.

Translation between the Two Languages

We can introduce a translation function $T:$ $\mathbf{ELKv}^r \rightarrow \mathbf{MLKv}^r$ inductively as follows:

- $T(p) = p$,
- $T(\neg\phi) = \neg T(\phi)$,
- $T(\phi \wedge \psi) = T(\phi) \wedge T(\psi)$,
- $T(\Diamond_i \phi) = \Diamond_i T(\phi)$,
- $T(Kv_i(\phi, c)) = \neg\Diamond_i^c T(\phi)$.

Properties of R_i^c

The new model class **MLKv**^{r} has to meet the following conditions:

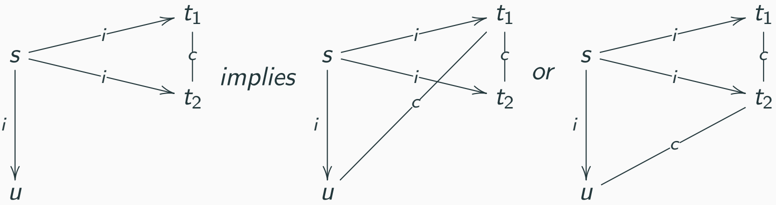
- For each **ELKv**^{r} pointed model $\mathcal{M}, s$, there is an **MLKv**^{r} model $\mathcal{N}, t$ such that $\mathcal{M}, s \vDash \phi \iff \mathcal{N}, t \Vdash T(\phi)$,
- For each **MLKv**^{r} pointed model $\mathcal{N}, t$, there is an **ELKv**^{r} model $\mathcal{M}, s$ such that $\mathcal{M}, s \vDash \phi \iff \mathcal{N}, t \Vdash T(\phi)$,

Properties of R_i^c

We claim that the following three properties are enough:

1. $sR_i^c tu \iff sR_i^c ut$
2. $sR_i^c uv$ only if $s \to_i u$ and $s \to_i v$
3. $sR_i^c tu$ and $s \to_i v$ imply that at least one of $sR_i^c tv$ and $sR_i^c uv$ holds

The Antieculidean Property



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- For each **MLKv**^{r}pointed model $\mathcal{N}, t$, there is an **ELKv**^{r} model $\mathcal{M}, s$ such that $\mathcal{M}, s \vDash \phi \iff \mathcal{N}, t \Vdash T(\phi)$,

The first condition is straightforward: the three condition is necessary for non-equivalent relation.

The second condition requires splitting and tree unravelling of **MLKv**^{r} model $\mathcal{N}, t$.

Splitting

Definition (Splitted Model)

Given $\mathcal{N}$, the splitted model $\mathcal{N}' = \langle S \times \{0,1\}, \{\rightarrow_i' : i \in \mathbf{I}\}, \{P_i^c : i \in \mathbf{I}, c \in C\}, V' \rangle$, where:

- $(u,x) \rightarrow_i' (v,y) \iff u \rightarrow_i v$
- $(u,x) P_i^c (v,y)(w,z) \iff u R_i^c vw$ and $(v,y) \neq (w,z)$
- $V'((u,x)) = V(u)$

Splitting

Fact

For any $u \in \mathcal{N}$*, we have*

$$\mathcal{N}, u \equiv_{\mathbf{LKv}^r} \mathcal{N}', (u, x) \textit{ where } x \in \{0, 1\}$$

Tree Unravelling

Definition (Tree Unravelling)

Given splitted pointed model $\mathcal{N}', s'$, its tree unravelling model is $\mathcal{M}' = \langle W, \{\hookrightarrow_i : i \in I\}, \{Q_i^c : i \in \mathbf{I}, c \in \mathbb{C}\}, U' \rangle$, where:

- $W = \{\langle s', i_1, v_1, \ldots, i_k, v_k \rangle$: there is a path $s' \xrightarrow{i_1} v_1 \ldots \xrightarrow{i_k} v_k$ in $\mathcal{N}'\}$,
- $\langle s', i_1, \ldots, v_k \rangle \hookrightarrow_i \langle s', j_1, \ldots, u_m \rangle$ iff $m = k+1, \langle s', i_1, \ldots, v_k \rangle = \langle s', j_1, \ldots, u_k \rangle, j_m = i$ and $v_k \to_i' u_m$ in $\mathcal{N}'$,
- $\langle s', i_1, \ldots, v_k \rangle Q_i^c \langle s', j_1, \ldots, u_m \rangle \langle s', l_1, \ldots, l_n \rangle$ iff $v_k P_i^c u_m l_n$, $\langle s', i_1, \ldots, v_k \rangle \hookrightarrow_i \langle s', j_1, \ldots, u_m \rangle$ and $\langle s', i_1, \ldots, v_k \rangle \hookrightarrow_i \langle s', l_1, \ldots, l_n \rangle$,
- $U'(\langle s', i_1, \ldots, u \rangle) = V'(u)$.

Tree Unravelling

Fact

$\mathcal{N}', s' \equiv_{\mathbf{MLKv}^r} \mathcal{M}', \langle s' \rangle$

Into ELKv^r Model

Definition

Given $\mathcal{M}'$, we construct a new model $\mathcal{M} = \langle W, \{\hookrightarrow_i : i \in \mathbf{I}\}, D, U, V_{\mathbb{C}} \rangle$ where:

- W and $\{\hookrightarrow_i : i \in \mathbf{I}\}$ are exactly the same as in $\mathcal{M}'$;
- $U = U'$;
- $V_{\mathbb{C}}(d, w) = |(d, w)|_{\sim}$. That is, $V_{\mathbb{C}}(d, w)$ is the equivalence class under the equivalence relation $\sim$ over $\mathbb{C} \times W$ defined as: $\sim = \{\langle (d, u), (e, v) \rangle : d = e, \exists s \exists j : s \hookrightarrow_j u, s \hookrightarrow_j v, \forall w \in W : \neg w Q_j^c uv\} \cup \{\langle (d, u), (d, u) \rangle \mid (d, u) \in \mathbb{C} \times W\}$
- $D = \{|(d, w)|_{\sim} \mid (d, w) \in \mathbb{C} \times W\}$;

Into ELKv^r Model

Fact

For any **ELKv**^{r} *formula* ϕ,

$$\mathcal{M}', \langle s' \rangle \Vdash T(\phi) \iff \mathcal{M}, \langle s' \rangle \vDash \phi \tag{3}$$

System $\mathbb{MLKV}^r$

System $\mathbb{MLKV}^r$

System $\mathbb{MLKV}^r$

Axiom Schemas	
`TAUT`	all the instances of tautologies
`DISTK`	$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
`DISTKv`^{r}	$\Box_i(p \to q) \to (\Box_i^c p \to \Box_i^c q)$
`Kv`$^r\vee$	$\Diamond_i(p \wedge q) \wedge \Diamond_i^c(p \vee q) \to (\Diamond_i^c p \vee \Diamond_i^c q)$

Rules	
`MP`	$\dfrac{\phi, \phi \to \psi}{\psi}$
`NECK`	$\dfrac{\phi}{\Box_i \phi}$
`NECKv`^{r}	$\dfrac{\phi}{\Box_i^c \phi}$
`SUB`	$\dfrac{\phi}{\phi[p/\psi]}$
`RE`	$\dfrac{\psi \leftrightarrow \chi}{\phi \leftrightarrow \phi[\psi/\chi]}$

System MLKv^r

System $\mathbb{ELKV}^r$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$K_i(p \to q) \to (K_i p \to K_i q)$
DISTKv^r	$K_i(p \to q) \to (Kv_i(q, c) \to Kv_i(p, c))$
Kv$^r\bot$	$Kv_i(\bot, c)$
Kv$^r\vee$	$\hat{K}_i(p \wedge q) \wedge Kv_i(p, c) \wedge Kv_i(q, c) \to Kv_i(p \vee q, c)$

Rules

MP	$\dfrac{\phi, \phi \to \psi}{\psi}$
NECK	$\dfrac{\phi}{K_i\phi}$
SUB	$\dfrac{\phi}{\phi[p/\psi]}$
RE	$\dfrac{\psi \leftrightarrow \chi}{\phi \leftrightarrow \phi[\psi/\chi]}$

$\mathbb{MLKV}^r$ and $\mathbb{LKV}^r$

We can show that the two systems are the same under translation T.

- $\texttt{DISTKv}^r$
- $\texttt{DISTKv}^r$
- $\texttt{Kv}^r\bot$ and $\texttt{NECKv}^r$

More importantly, our new system $\mathbf{MLKv}^r$ is normal.

Completeness

Canonical Model

Definition (Canonical Model)

The canonical model of $\mathbb{MLKV}^r$ is a tuple

$$\mathcal{M} = \langle S, \{\rightarrow_i : i \in \mathbf{I}\}, \{R_i^c : i \in \mathbf{I}, c \in \mathbb{C}\}, V\rangle$$

where:

- S is the set of all maximal consistent sets of $\mathbf{LKv}^r$formulas,
- $s \rightarrow_i t \iff \{\phi : \Box_i \phi \in s\} \subseteq t$,
- $sR_i^c tu \iff \{\phi : \Box_i \phi \in s\} \subseteq t \cap u$ and $\{\psi : \Box_i^c \psi \in s\} \subseteq t \cup u$,
- $V(s) = \{p : p \in s\}$.

Canonical Model

- It's easy to verify that $\mathcal{M}$ is an $\mathbf{MLKv}^r$ model.
- The definition is straightforward. This is because the model class we've chosen is simple enough, so that we do not need much additional information to construct the canonical model.

Canonical Model

We need the two Existence Lemma to get the Truth Lemma:

Lemma (Existence Lemma for $\Diamond_i$)

Given a state $s \in S^c$. If $\Diamond_i \phi \in s$, then there exists $t \in S^c$ such that $s \to_i t$ and $\phi \in t$;

Lemma (Existence Lemma for $\Diamond_i^c$)

Given a state $s \in S^c$. If $\Diamond_i^c \psi \in s$, then there exist $t, u \in S^c$ such that $sR_i^c tu$ and $\psi \in t \cap u$.

Extended Language with a Binary Diamond

Extended language $\mathbf{MLKv}^{r+}$

The extended language $\mathbf{MLKv}^{r+}$ is:

$$\phi ::= \top \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box_i \phi \mid \Box_i^c(\phi, \phi)$$

The difference is that we no longer assume the two arguments to be the same here.

Extended language $\mathbf{MLKv}^{r+}$

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The semantics are similar (easier to understand as $\Diamond_i^c(\phi, \psi)$).

Same expressive power

Theorem

For all $\mathbf{MLKv}^{r+}$ *formula* ϕ^+*, there exists an* $\mathbf{ELKv}^{r+}$ *formula* ϕ *such that for any pointed model* $\mathcal{M}, s$*,* $\mathcal{M}, s \vDash \phi^+ \iff \mathcal{M}, s \vDash \psi$*.*

Proof.

We give a reduction function r inductively:

- $r(p) = p$;
- $r(\neg\phi) = \neg r(\phi)$;
- $r(\phi \wedge \psi) = r(\phi) \wedge r(\psi)$;
- $r(\Diamond\phi) = \Diamond r(\phi)$;
- $r(\Diamond_i^c(\phi, \psi)) =$ $(\Diamond_i^c\phi \wedge \Diamond_i\psi) \vee (\Diamond_i^c\psi \wedge \Diamond_i\phi) \vee (\Diamond_i\phi \wedge \Diamond_i\psi \wedge \neg\Diamond_i^c\phi \wedge \neg\Diamond_i^c\psi \wedge \Diamond_i^c(\phi \vee \psi))$.

□

Reduction

A surprise!

But what does this mean?

System $\mathbb{MLKV}^{r+}$

System $\mathbb{MLKV}^{r+}$

Axiom Schemas

TAUT	all the instances of tautologies
DISTK	$\Box_i(p \to q) \to (\Box_i p \to \Box_i q)$
SYM	$\Box_i^c(p, q) \to \Box_i^c(q, p)$
DISTBK	$\Box_i^c(p \to q, r) \to (\Box_i^c(p, r) \to \Box_i^c(q, r))$
INC	$\Diamond_i^c(p, q) \to \Diamond_i p$
ATEUC	$\Diamond_i^c(p, q) \wedge \Diamond_i r \to \Diamond_i^c(p, r) \vee \Diamond_i^c(q, r)$

Rules

MP	$\dfrac{\phi, \phi \to \psi}{\psi}$
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RE	$\dfrac{\psi \leftrightarrow \chi}{\phi \leftrightarrow \phi[\psi/\chi]}$
SUB	$\dfrac{\phi}{\phi[p/\psi]}$

System $\mathbb{MLKV}^{r+}$

The system $\mathbb{MLKV}^{r+}$is normal. Therefore, we can expect that its completeness proof is almost routine.

c-Bisimulation

A binary relation $\mathcal{Z}$ between $\mathcal{M}$ and $\mathcal{N}$ is a c-bisimulation if:

- Inv: $V_1(s_1) = V_2(s_2)$;
- Zig: $s_1 \to_i^1 t_1 \Rightarrow \exists t_2$ such that $s_2 \to_i^2 t_2$ and $t_1 \mathcal{Z} t_2$;
- Zag: $s_2 \to_i^1 t_2 \Rightarrow \exists t_1$ such that $s_2 \to_i^2 t_2$ and $t_1 \mathcal{Z} t_2$;
- Kv-Zig: $s_1 R_i^c t_1 u_1 \Rightarrow \exists t_2, u_2 \in S_2$ such that $t_1 \mathcal{Z} t_2$, $u_1 \mathcal{Z} u_2$ and $s_2 Q_i^c t_2 u_2$;
- Kv-Zag: $s_2 Q_i^c t_2 u_2 \Rightarrow \exists t_1, u_2 \in S_1$ such that $t_1 \mathcal{Z} t_2$, $u_1 \mathcal{Z} u_2$ and $s_1 R_i^c t_1 u_1$.

Compared with logical equivalence

- The similar correspondence between c-bisimulation and logical equivalence is expected to exist.
- For $\mathbf{MLKv}^{r+}$, the proof is quite routine, but not straghtforward for $\mathbf{MLKv}^{r}$
- So our route is: $\mathbf{MLKv}^{r+} \to \mathbf{MLKv}^{r}$

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c-bisimulation and Logical Equivalence

Theorem

Suppose $\mathcal{M}$, $\mathcal{N}$ are finite models. Then $\mathcal{M}, s \leftrightarroweq_{\mathbb{C}} \mathcal{N}, t \iff \mathcal{M}, s \equiv_{\mathbf{LKv}^r} \mathcal{N}, t$.

This shows again some reason why the exteneded language is worth discussion: even if they have same expressive power, yet $\mathbf{MLKv}^{r+}$can express certain structure property simpler and more directly.

Further Work

Completeness for $\mathbb{MLKV}^r$

The completeness result for $S5$ system.

Computability

As the logics are normal, we cane expect to use quite standard computability methods and results in modal logic.

Connection with Other Non-standard Epistemic Logic

Can we compress the "rich models" into relatively simpler modal models?

Questions and Discussions