Sabotage Modal Logic

Finite model property, proof system and expressivity[a]

Liu Jixin, joint work with Prof. Yanjing Wang

March 27, 2016

Department of Philosophy, PKU

[a]The basic idea is from Prof. Yanjing Wang, and this paper is a course paper for Prof. Wang's Logic of knowledge. I appreciate Prof. Wang's guidance very much.

Table of contents

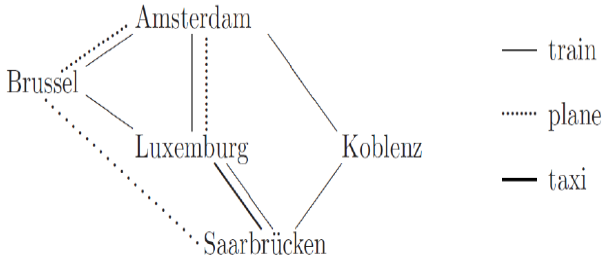
Introduction

Background

Johan van Benthem developed a kind of modal logic, called Sabotage modal logic (SML) in [5]. SML expands the standard modal language with an edge-deletion modality $\blacklozenge\phi$ whose intended reading is “after the deletion of at least one edge in the frame it holds that ϕ”. This logic is related to the sabotage game which is discussed in [2] and [4] .

Background

The diagram below is from [5] to give a instance for SML.



Background

- There have been some technical results about this kind of modal logic.

Background

- There have been some technical results about this kind of modal logic.
- First from Christof Löding and Philipp Rohde's work in [3] we know that this logic is not decidable and does not have the finite model property. But in those proofs it seems that the language has to be binary or ternary, which may not be essential.

Background

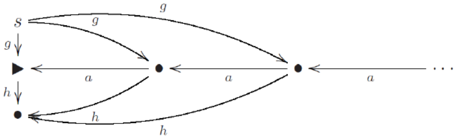
- There have been some technical results about this kind of modal logic.
- First from Christof Löding and Philipp Rohde's work in [3] we know that this logic is not decidable and does not have the finite model property. But in those proofs it seems that the language has to be binary or ternary, which may not be essential.
- Secondly, Guillaume Aucher, Johan van Benthem, and Davide Grossi show that SML has a complete axiomatization in [1]. But their system is a tableau, which seems directly from the semantics and has less intuitions than Hilbert-style systems.

Background

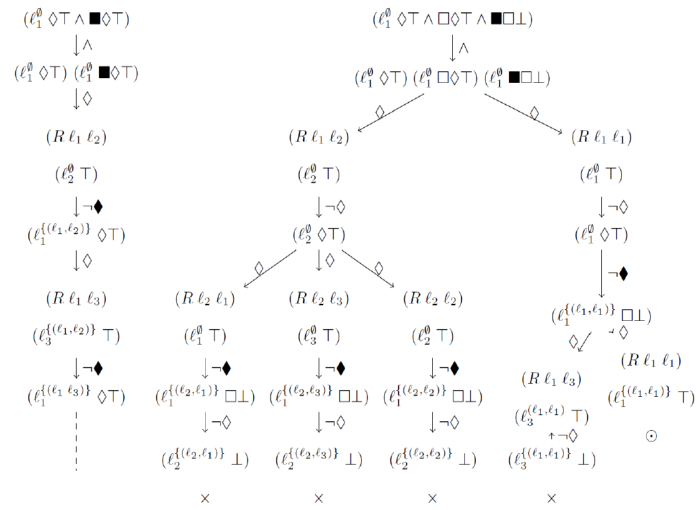
Here is an instance for that SML does not have the finite model property.

Let φ be the following SML-formula:

$$
\begin{aligned}
&\Diamond_g \neg\sigma_a \wedge \diamondminus_g \Box_g \sigma_a \wedge && \text{(M1)}\\
&\Box_g \diamondminus_a \neg\sigma_a \wedge && \text{(M2)}\\
&\Box_g \gamma_h \wedge && \text{(M3)}\\
&\Box_g \Box_a \gamma_h \wedge && \text{(M4)}\\
&\neg\sigma_{\{a,h\}} \wedge \Box_g \neg\sigma_g \wedge && \text{(M5)}\\
&\boxminus_h \Diamond_g (\sigma_h \wedge \Diamond_a \neg\sigma_h) && \text{(M6)}
\end{aligned}
$$



The following is a deduction example of the tableau system.



Goals

- First we will consider an example in unary language which shows that SML does not have finite model property.

Goals

- First we will consider an example in unary language which shows that SML does not have finite model property.
- Another aim of our paper is to find a Hilbert-style proof system, which may give more light on our intuition for this logic, and discuss some problems about the completeness.

Goals

- First we will consider an example in unary language which shows that SML does not have finite model property.
- Another aim of our paper is to find a Hilbert-style proof system, which may give more light on our intuition for this logic, and discuss some problems about the completeness.
- Finally we show that the expressivity between SML and GML is incomparable.

Preliminaries

Syntax

- Let Σ be an index set. The sabotage modal language is defined using proposition letters and two kinds of modal operators $\Diamond_i$ and $\blacklozenge_i$, where $i \in \Sigma$. The well-formed formulas of the basic modal language are given by the rule $\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \Diamond_i\phi \mid \blacklozenge_i\phi$, where p ranges over all proposition letters and $i \in \Sigma$.

Syntax

- Let Σ be an index set. The sabotage modal language is defined using proposition letters and two kinds of modal operators $\Diamond_i$ and $\blacklozenge_i$, where $i \in \Sigma$. The well-formed formulas of the basic modal language are given by the rule $\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \Diamond_i\phi \mid \blacklozenge_i\phi$, where p ranges over all proposition letters and $i \in \Sigma$.
- When we say our language is unary, we mean that $|\Sigma| = 1$, and we will just use $\Diamond$ and $\blacklozenge$. Other Boolean connectives and the the modal operators $\Box_i$ and $\blacksquare_i$ can be defined in the standard way. We will use $\Diamond^n$ and $\blacklozenge^n$ for the abbreviations for n times iteration of $\Diamond$ and $\blacklozenge$.

Semantics

- The models for our language are standard Kripke models $\mathfrak{M} = \langle W, \{R_i\}_{i \in \Sigma}, V \rangle$, where each R_i is the corresponding relation of $\Diamond_i$ and $\blacklozenge_i$.

Semantics

- The models for our language are standard Kripke models $\mathfrak{M} = \langle W, \{R_i\}_{i\in\Sigma}, V\rangle$, where each R_i is the corresponding relation of $\lozenge_i$ and $\blacklozenge_i$.
- The satisfaction relation for our language is defined as usual for the atomic and Boolean cases, and for the standard modalities. For the sabotage modality it is as follows:
  $\langle W, \{R_i\}_{i\in\Sigma}, V\rangle, w \models \blacklozenge_i\phi \iff$
  $\exists(w_1, w_2) \in R_i$ s.t. $\langle W, R_i \setminus \{(w_1, w_2)\}, \{R_j\}_{j\neq i\in\Sigma}, V\rangle, w \models \phi$

Semantics

- The models for our language are standard Kripke models $\mathfrak{M} = \langle W, \{R_i\}_{i \in \Sigma}, V \rangle$, where each R_i is the corresponding relation of $\lozenge_i$ and $\blacklozenge_i$.
- The satisfaction relation for our language is defined as usual for the atomic and Boolean cases, and for the standard modalities. For the sabotage modality it is as follows:
  $\langle W, \{R_i\}_{i \in \Sigma}, V \rangle, w \models \blacklozenge_i \phi \iff$
  $\exists (w_1, w_2) \in R_i$ s.t. $\langle W, R_i \setminus \{(w_1, w_2)\}, \{R_j\}_{j \neq i \in \Sigma}, V \rangle, w \models \phi$
- The above means $\blacklozenge_i \phi$ is satisfied at w iff there are two R_i-related points s.t. if we remove the edge between them, then ϕ holds at w. Other definition are defined as usual.

Finite model property

Finite model property

In this section we will give a counter-example for the f.m.p. of unary sabotage modal logic. Let p, q_1, q_2, q_3 be different propositional letters, where q_1, q_2, q_3 are incompatible. (Let $q_1 = r_1 \wedge \neg r_2 \wedge \neg r_3$, $q_2 = r_2 \wedge \neg r_1 \wedge \neg r_3$, and $q_3 = r_3 \wedge \neg r_1 \wedge \neg r_2$, where r_1, r_2, r_3 are different.)

Finite model property

- Let ϕ be the conjunction of the followings:

Finite model property

- Let ϕ be the conjunction of the followings:
- $\phi_1 = (q_1 \wedge \Diamond p \wedge \blacklozenge\Box\neg p \wedge \Box q_2)$

Finite model property

- Let ϕ be the conjunction of the followings:
- $\phi_1 = (q_1 \wedge \Diamond p \wedge \blacklozenge\Box\neg p \wedge \Box q_2)$
- $\phi_2 = \Box\Diamond q_3 \wedge \Box\blacklozenge\Box\neg q_3$

Finite model property

- Let ϕ be the conjunction of the followings:
- $\phi_1 = (q_1 \wedge \Diamond p \wedge \blacklozenge\Box\neg p \wedge \Box q_2)$
- $\phi_2 = \Box\Diamond q_3 \wedge \Box\blacklozenge\Box\neg q_3$
- $\phi_3 = \Box(p \leftrightarrow \blacklozenge\Box\bot)$

Finite model property

- Let ϕ be the conjunction of the followings:
- $\phi_1 = (q_1 \wedge \Diamond p \wedge \blacklozenge\Box\neg p \wedge \Box q_2)$
- $\phi_2 = \Box\Diamond q_3 \wedge \Box\blacklozenge\Box\neg q_3$
- $\phi_3 = \Box(p \leftrightarrow \blacklozenge\Box\bot)$
- $\phi_4 = \Box(\neg p \to \blacklozenge\blacklozenge\Box\bot) \wedge \Box(\neg p \to \Diamond(q_2 \wedge \Diamond q_3))$

Finite model property

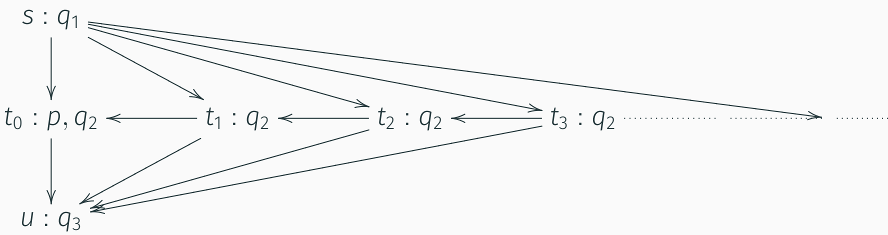
- Let ϕ be the conjunction of the followings:
- $\phi_1 = (q_1 \land \Diamond p \land \blacklozenge\Box\neg p \land \Box q_2)$
- $\phi_2 = \Box\Diamond q_3 \land \Box\blacklozenge\Box\neg q_3$
- $\phi_3 = \Box(p \leftrightarrow \blacklozenge\Box\bot)$
- $\phi_4 = \Box(\neg p \to \blacklozenge\blacklozenge\Box\bot) \land \Box(\neg p \to \Diamond(q_2 \land \Diamond q_3))$
- $\phi_5 = \blacksquare(\Diamond\Box\neg q_3 \to \Diamond(\Diamond q_3 \land \Diamond(q_2 \land \Box\neg q_3)))$

Finite model property

Here ϕ_1 means q_1 is on the original point, say w, each w-successor has q_2, and there is just one w-successor v has p. ϕ_2 means each w-successor can see exact one q_3 point. ϕ_3 says that the p successor v of w has exact one successor. ϕ_4 says every other successors of w has just two successors and one of the two has q_2 and see a q_3 point. Finally ϕ_5 expresses that if we delete a edge which is between one $w-$successor and its q_3 successor, then w can see a point s such that s can see a q_3 point and a q_2 point which has no q_3 successor. We can conclude from ϕ_2, ϕ_3 and ϕ_4 that the p successor has exact one successor which has q_3 and that all other w-successor has exact two successors: one has q_3 and the other has q_2 and can see q_3.

Finite model property

It's easy to see that the model below is an infinite model of ϕ, where ϕ is true at s. We now need to show that ϕ has no finite model.



Finite model property

Lemma

For every model $M, w \models \phi$, each successor of w has a predecessor which is a successor of w.

Proof.

Suppose that there is a successor of w, called v, has no such predecessor. By ϕ_2, v see a q_3 point u. If we delete the relation between v and u, v would be the only successor of w which has no q_3 successor by ϕ_2. By ϕ_5, there must be a successor t of w satisfies $\Diamond q_3 \wedge \Diamond(q_2 \wedge \Box\neg q_3)$, which cannot be the p point or v. But by ϕ_4, other successors of w must still see a $q_2 \wedge \Diamond q_3$ point since they cannot see v, a contradiction. $\Box$

Finite model property

Proposition

For every model $M, w \models \phi$, w has infinitely many successors.

Proof.

Assume that w has only k many successors. Then by ϕ_2 and ϕ_3, w has only $k-1$ many successors which can see a successor of w. Since each successor of w can see at most one successor of w, there are at most $k-1$ many successors of w which is a two-step successor of w. So there must be a successor v of w which is not a two-step successor of w, that is, v has no predecessor which is a successor of w, contradicts the above lemma. $\square$

Expressivity

Graded modal logic

In this section we will compare the expressivity between SML and Graded modal logic(GML). Here we just consider the case for unary language.

Graded modal formulas are built up using propositional letters p, q, ..., the constants $\top$ and $\bot$, Boolean connectives $\neg$, $\wedge$, and the unary modal operators $\Diamond_i$ and $\Box_i$. A model for GML is a Kripke model $M = \langle W, R, V \rangle$, and the satisfaction relation for the modal operators is defined as follows: (other cases are similar as in modal logic)

$$M, w \models \Diamond_i \phi \text{ iff } \exists v_1 \ldots v_i (\bigwedge_{1 \leq j \neq k \leq i} (v_j \neq v_k) \wedge \bigwedge_{1 \leq j \leq i} Rwv_j \wedge \bigwedge_{1 \leq j \leq i} M, v_j \models \phi)$$

and $M, w \models \Box_i \phi$ iff $M, w \models \neg \Diamond_i \neg \phi$.

The above can be read as $\Diamond_i \phi$ says that there are at least i successors which satisfied ϕ.

Expressivity

We have shown that SML can express the property that there are at least n successors. It seems that there are some similarity between the two language, but we will show that their expressivity are incomparable.

Expressivity

We have shown that SML can express the property that there are at least n successors. It seems that there are some similarity between the two language, but we will show that their expressivity are incomparable.

From above we know that GML has the finite model property, which means GML cannot express SML, since we have shown that SML does not have f.m.p. above. Actually SML can express the property that there are at most n edges in the model, by the formula $\blacksquare^n\bot$. For any w in a model M, $M, w \models \blacksquare^n\bot$ iff there are at most n edges in M. If there is a GML formula ϕ which is a translation of $\blacksquare^n\bot$, then any model N, which has exactly n edges, must satisfy ϕ, say $N, v \models \phi$. Let N_1 be any model which has one edge, and N_2 be the disjoint union of N and N_1. From the model equivalence results for GML, we know that $N_2, v \models \phi$, which means N_2 has at most n edges, a contradiction. So from the above observation, GML cannot express SML, even if we only consider finite models.

Two infinite models

Next we will first show that SML cannot express GML by using two infinite models.

Let $M_1 = \langle W_1, R_1, V_1 \rangle$, where
$W_1 = \{a_n \mid n \in \omega\} \cup \{b_n^m \mid m \in \{0,1\} \wedge n \in \omega\} \cup \{c_n \mid n \in \{0,1,2\}\} \cup \{s\}$,
$R_1 = \{(s, a_n) \mid n \in \omega\} \cup \{(s, b_n^0) \mid n \in \omega\} \cup \{(b_n^0, b_n^1) \mid n \in \omega\} \cup \{(s, c_0), (c_0, c_1), (c_1, c_2)\}$, and $V_1(p) = W_1$ for any p.

Let $M_2 = \langle W_2, R_2, V_2 \rangle$, where $W_2 = \{w_n \mid n \in \omega\} \cup \{v_n^m \mid m \in \{0,1\} \wedge n \in \omega\} \cup \{u_n \mid n \in \{0,1,2,3\}\} \cup \{s'\}$,
$R_2 = \{(s', w_n) \mid n \in \omega\} \cup \{(s', v_n^0) \mid n \in \omega\} \cup \{(v_n^0, v_n^1) \mid n \in \omega\} \cup \{(s', u_0), (s', u_1), (u_0, u_2), (u_1, u_2), (u_2, u_3)\}$, and $V_2(p) = W_2$.

we will use l to mark our edges: let $l_0 = (s, c_0)$, $l_1 = (c_0, c_1)$, $l_2 = (c_1, c_2)$, $l_3 = (s', u_0)$, $l_4 = (s', u_1)$, $l_5 = (u_0, u_2)$, $l_6 = (u_1, u_2)$, $l_7 = (u_2, u_3)$.

The two models' diagrams are as follows:

M_1

$$
\begin{tikzcd}
a_0, a_1, \ldots & & & \\
s \arrow[u] \arrow[r] \arrow[d] & c_0 \arrow[r] & c_1 \arrow[r] & c_2 \\
b^0_0, b^0_1, \ldots \arrow[d] & & & \\
b^1_0, b^1_1, \ldots & & &
\end{tikzcd}
$$

M_2

$$\begin{tikzcd}
w_0, w_1, \dots & u_0 \arrow[dr] & & \\
s' \arrow[u] \arrow[ur] \arrow[dr] \arrow[d] & & u_2 \arrow[r] & u_3 \\
v_0^0, v_1^0, \dots \arrow[d] & u_1 \arrow[ur] & & \\
v_0^1, v_1^1, \dots & & &
\end{tikzcd}$$

Obviously, $M_1, s \models \neg\Diamond_2\Diamond\Diamond\top$ and $M_2, s^{'} \models \Diamond_2\Diamond\Diamond\top$. So if we show that M_1, s and $M_2, s^{'}$ are model equivalent in SML, we will know that SML cannot express GML. From [1] we know that if M_1, s and $M_2, s^{'}$ are s-bisimular, then they are model equivalent, so we will give the s-bisimulation relation Z as follows:

Expressivity over infinite models

Obviously, $M_1, s \models \neg\Diamond_2\Diamond\Diamond\top$ and $M_2, s^{'} \models \Diamond_2\Diamond\Diamond\top$. So if we show that M_1, s and $M_2, s^{'}$ are model equivalent in SML, we will know that SML cannot express GML. From [1] we know that if M_1, s and $M_2, s^{'}$ are s-bisimular, then they are model equivalent, so we will give the s-bisimulation relation Z as follows:

Let $Z_0 = \{(a_n, w_n), (b_n^m, v_n^m) \mid m \in \{0,1\} \wedge n \in \omega\} \cup \{(s, s^{'}), (c_0, u_0), (c_0, u_1), (c_1, u_2), (c_2, u_3)\}$ and $Z_1 = (Z_0)^{-}$. We choose Z to be the union of Z_0 and Z_1. Here we define a relation between models: $M = \langle W, R, V\rangle \xrightarrow{\blacklozenge_n} M^{'} = \langle W^{'}, R^{'}, V^{'}\rangle$ iff $W^{'} = W$, $V^{'} = V$ and $R^{'} = R \backslash A$ where $|A| = n \wedge A \subseteq R$.

Expressivity over infinite models

Lemma

Z is a s-bisimulation.

Proof

We only check the conditions between s and $s^{'}$, since it's easy to verify that other $Z-$related points satisfy the s-bisimulation conditions. Obviously s and $s^{'}$ are modal bisimular. So we need to show that for each $n \in \omega$ and $t \in \{1, 2\}$, if $M_t \xrightarrow{\blacklozenge_n} M'_t$ then there is a M'_{3-t} s.t. $M_{3-t} \xrightarrow{\blacklozenge_n} M'_{3-t}$ and $M'_t, s \longleftrightarrow M'_{3-t}, s^{'}$. If in M_1 we delete one edge from the ω part, we only need to delete an arbitrary edge in M_2 from the ω part and Vice Versa. The non-trivial cases are deletions of c_n in M_1 and u_n in M_2. We will use the strategy as follows:

Expressivity over infinite models

Proof.

When we delete l_i, $i \leq 2$ in M_1, we will delete l_7 in M_2 when l_7 is still alive, otherwise we just delete an arbitrary edge in M_2 from the ω part.

When we delete l_i, $3 \leq i \leq 6$ in M_2, there are two cases:

1. After the deletion, there is still a path from $s^{'}$ to u_3: Just delete an edge from the ω part in M_1.

2. Otherwise: if l_0 is alive, delete l_0; if not, Just delete an edge from the ω part in M_1.

When we deltete l_7, if l_0 is alive, delete l_0 and if not, Just delete an edge from the ω part in M_1.

One can easily verify that our strategy will preserve the modal bisimulation in any finite times deletions. $\square$

Expressivity over classes of finite models

From the above instance it's still open whether SML can express GML among finite models. So now we give an instance to show that even in finite models, SML cannot express GML. Our idea is to split the two infinite models into two classes of finite models. But first we need to give a definition for n-bisimulation as in modal logic, which is related to the s-bisimulation in [1].

$m - n-$s-bisimulation

Here we just define $m - n-$s-bisimulations for unary language. Let M and N be models, and let w and v be states of M and N, We define $w \leftrightarroweq_n^m v$ by induction on $m + n$.

$m + n = 0$: $M, w \leftrightarroweq_n^m N, v$ iff $V_M(w) = V_N(v)$;

$m + n = k + 1$: $M, w \leftrightarroweq_n^m N, v$ if the followings hold:

1. Whenever $m > 0$:

$M, w \leftrightarroweq_n^{m-1} N, v$;

If $wRw^{'}$, then there is a $v^{'}$ s.t. $vRv^{'}$ and $M, w^{'} \leftrightarroweq_n^{m-1} N, v^{'}$;

If $vRv^{'}$, then there is a $w^{'}$ s.t. $wRw^{'}$ and $M, w^{'} \leftrightarroweq_n^{m-1} N, v^{'}$.

$m - n-$s-bisimulation

2. Whenever $n > 0$:

$M, w \underline{\leftrightarrow}^m_{n-1} N, v$;

If there is M' s.t. $M, w \rightarrow_{\blacklozenge} M', w$, then there is N' s.t. $N, v \rightarrow_{\blacklozenge} N', v$ and $M', w \underline{\leftrightarrow}^m_{n-1} N', v$;

If there is N' s.t. $N, v \rightarrow_{\blacklozenge} N', v$, then there is M' s.t. $M, v \rightarrow_{\blacklozenge} M', v$ and $M', w \underline{\leftrightarrow}^m_{n-1} N', v$.

Here m correspond to the modal degree, and n correspond to the sabotage degree, as we define below.

Expressivity over classes of finite models

s-depth

Like the modal degree, we define the sabotage degree as follows:

$\mathrm{deg}_s(\top) = \mathrm{deg}_s(p) = 0$;

$\mathrm{deg}_s(\neg\phi) = \mathrm{deg}_s(\phi)$; $\mathrm{deg}_s(\phi \wedge \psi) = MAX(\mathrm{deg}_s(\phi), \mathrm{deg}_s(\psi))$;

$\mathrm{deg}_s(\Diamond\phi) = \mathrm{deg}_s(\phi)$; $\mathrm{deg}_s(\blacklozenge\phi) = \mathrm{deg}_s(\phi) + 1$.

Additional, we need to complete the modal degree on SML:

$\mathrm{deg}(\blacklozenge\phi) = \mathrm{deg}(\phi)$.

Expressivity over classes of finite models

proposition

[for finite many propositional letters]$M, w \leftrightarroweq_n^m N, v \iff M, w \equiv_{SML_n^m} N, v$, where $M, w \equiv_{SML_n^m} N, v$ means for each formula ϕ, if $\deg(\phi) \leq m$ and $\deg_s(\phi) \leq n$, then $M, w \models \phi$ iff $N, v \models \phi$.

Proof.

Similar to the modal version. $\square$

Expressivity over classes of finite models

Here we give two classes of finite models.

Let $M_n = \langle W_n, R_n, V_n \rangle$, where
$W_n = \{a_i \mid i \leq n\} \cup \{b_i^m \mid m \in \{0,1\} \wedge i \leq n+1\} \cup \{c_i \mid i \in \{0,1,2\}\} \cup \{s\}$,
$R_n = \{(s, a_i) \mid i \leq n\} \cup \{(s, b_i^0) \mid i \leq n\} \cup \{(b_i^0, b_i^1) \mid i \leq n\} \cup \{(s, c_0), (c_0, c_1), (c_1, c_2)\}$, and $V_n(p) = W_n$ for any p.

Let $N_n = \langle W_n^{'}, R_n^{'}, V_n^{'} \rangle$, where
$W_n^{'} = \{w_i \mid i \leq n\} \cup \{v_i^m \mid m \in \{0,1\} \wedge i \leq n\} \cup \{u_i \mid i \in \{0,1,2,3\}\} \cup \{s^{'}\}$,
$R_n^{'} = \{(s^{'}, w_n) \mid i \leq n\} \cup \{(s^{'}, v_n^0) \mid i \leq n\} \cup \{(v_n^0, v_n^1) \mid i \leq n\} \cup \{(s^{'}, u_0), (s^{'}, u_1), (u_0, u_2), (u_1, u_2), (u_2, u_3)\}$, and $V_n(p) = W_n$.

Expressivity over classes of finite models

One can check that $M_n, s \leftrightarroweq_n^m N_n, s^{'}$ for any $m \in \omega$, which follows that the two pointed models are equivalent w.r.t. formulas with sabotage depth $\leq n$. The proof is similar to the infinite version.

Expressivity over classes of finite models

One can check that $M_n, s \leftrightarroweq_n^m N_n, s^{'}$ for any $m \in \omega$, which follows that the two pointed models are equivalent w.r.t. formulas with sabotage depth $\leq n$. The proof is similar to the infinite version.

Let $\mathfrak{C}$ $=\{M_n \mid n \in w\}$ and $\mathfrak{B} = \{N_n \mid n \in w\}$. Like the case in the two infinite models, if we choose s and $s^{'}$ to be the original state, $\mathfrak{C}$ and $\mathfrak{B}$ can be distinguished by GML formula $\phi = \Diamond_2\Diamond\Diamond\top$. Actually for all $n \in \omega$, $M_n, s \models \neg\Diamond_2\Diamond\Diamond\top$ and $N_n, s^{'} \models \Diamond_2\Diamond\Diamond\top$. Hence if there is some SML formula ψ express ϕ over finite models, then for each n, $M_n, s \models \neg\psi$ and $N_n, s^{'} \models \psi$. But if ψ has sabotage depth m, it must be the case that $M_m, s \models \psi$ iff $N_m, s^{'} \models \psi$, which is a contradiction.

Axiomatization

Axiomatization

In this section we will try to find a sound and complete Hilbert-style proof system. Our intended system includes the normal modal logic K and the followings[1]:

Axioms:

$\blacksquare(\phi \to \psi) \to (\blacksquare\phi \to \blacksquare\psi)$

$\blacksquare p \leftrightarrow p$

$\blacksquare\neg p \leftrightarrow \neg p$

$\Diamond\top \to \blacklozenge\top$

$\blacklozenge\Diamond\phi \to \Diamond\blacklozenge\phi$

[1]The first idea of these axioms is given by Prof. Wang.

Axiomatization

$\psi_n \to \blacklozenge\psi_{n-1}$ for $n-1 \in \omega$

$\blacklozenge^n\top \wedge \neg\blacklozenge^{n+1}\top \to \blacksquare(\neg\blacklozenge^n\top \wedge \blacklozenge^{n-1}\top)$, for $n-1 \in \omega$

$(\blacklozenge^n\top \wedge \neg\blacklozenge^{n+1}\top) \to \Box(\blacklozenge^n\top \wedge \neg\blacklozenge^{n+1}\top)$, for $n \in \omega$

$\Diamond\blacksquare\phi \to (\psi_n \to \blacksquare(\psi_n \to \Diamond\phi)$, for $n \in \omega$

where $\psi_n = \blacklozenge^n\Box\bot \wedge \blacksquare^{n-1}\Diamond\top$

Rules:

$\blacksquare-$generalization

Axiomatization

Although we seem to find a recursive set of axioms, we need to show that the axioms are sound and complete. One can easily check that the axioms and rules are sound w.r.t. our semantics. But the completeness seems to be more complicated.

Axiomatization

Although we seem to find a recursive set of axioms, we need to show that the axioms are sound and complete. One can easily check that the axioms and rules are sound w.r.t. our semantics. But the completeness seems to be more complicated.

A standard strategy in [6] is as follows:

1. Define an auxiliary semantics $\Vdash$ of SML on Kripke models with $\blacklozenge$−transitions.

2. Find a class $\mathfrak{C}$ of binary Kripke models such that for any SML formula ϕ : $M \models \phi \Longrightarrow \mathfrak{C} \Vdash \phi$.

3. Show that our axioms completely axiomatizes the valid SML formulas on $\mathfrak{C}$ w.r.t. $\Vdash$.

In sum, we proceed as follows (from left to right):

$\mathfrak{M} \models \phi \Longrightarrow \mathfrak{C} \Vdash \phi \Longrightarrow \vdash \phi$.

Problem of the axiomatization

Here the supposed new semantics is defined as follows:

A binary Kripke model $\mathfrak{M}$ is a tuple: $\langle W, R, R_\blacklozenge, V\rangle$, where $R_\blacklozenge$ is a binary relation over W and $\langle W, R, V\rangle$ are just a standard Kripke model.

We will call $\langle W, R, V\rangle$ the kernel of $\langle W, R, R_\blacklozenge, V\rangle$.

The new satisfaction relation $\Vdash$ is defined by(others are the same as for $\models$) :

$M, w \Vdash \blacklozenge\phi$ iff $\exists v : wR_\blacklozenge v$ and $M, v \Vdash \phi$

Here If we want to show $\mathfrak{M} \models \phi \implies \mathfrak{C} \Vdash \phi$, we need to transform a $\Vdash$ model into a $\models$ model. But we cannot just use the kernel of our binary model, which will be showed in the following instance.

Problem of the axiomatization

Consider the following model and the formula $\phi = \blacklozenge^4\top \wedge \neg\blacklozenge^5\top \wedge \blacklozenge^2\Box\bot \wedge \blacksquare\Diamond\top \wedge \Box(\blacklozenge\Box\bot \wedge \Diamond\top \wedge \Box\Box\bot)$ If ϕ is true at the point s w.r.t. the deletion semantics, then the model need to be as the following diagram, where r and t are dead ends.

$$
\begin{tikzcd}
 & \bullet \arrow[r] & r \\
s \arrow[ur] \arrow[dr] & & \\
 & \bullet \arrow[r] & t
\end{tikzcd}
$$

Problem of the axiomatization

But if we use the binary semantics, the model will be like:

$$\begin{tikzcd}
S \arrow[r] \arrow[d, dashed, "\blacklozenge"] & \bullet \arrow[r] \arrow[d, dashed, "\blacklozenge"] & \bullet \\
\bullet \arrow[r] \arrow[d, dashed, "\blacklozenge"] & \bullet & \\
\bullet \arrow[d, dashed, "\blacklozenge"] & & \\
\bullet \arrow[d, dashed, "\blacklozenge"] & & \\
\bullet & &
\end{tikzcd}$$

Problem of the axiomatization

It follows that the kernel of our binary model cannot be enough to make all the SML formulas true. Especially when we consider the canonical model of the binary semantics, there must be some similar problem like here.

The above show that the relation between a $\Vdash$ model and a $\models$ model seems to be complicated, and we need more careful investigation on this. So we will leave the problem here and I hope one can solve it soon.

Further works

Further works

Clearly we need to find a proof for the completeness, and we have two ways to go:

1. Use the original strategy and find a appropriate way to transport $\models$ to $\Vdash$.
2. Consider another strategy for completeness, like defining a canonical model for SML language.

Further works

However, it's a natural idea to consider some restricted version of completeness.

1. Assume that models are all point-generated, which will restrict the arbitrary deletion to a related deletion.
2. Assume that models are all finite, which will be helpful to using our axioms.

Obviously we can just consider finite generated models, which may bring another conjecture: SML can characterize frames w.r.t. isomorphism, which means:

For any finite generated frame $\mathfrak{F}$, there is a formula ϕ s.t. $\mathfrak{F} \models \phi$ and if any finite generated model $\mathfrak{G} \models \phi$, $\mathfrak{F} \cong \mathfrak{G}$.

Ackownledgement

Thanks for Prof. Wang's work(and xy-graphics) on this, which is important for the three counterexample and our intended axioms, and I hope we can find the right axioms later. Additionally, I need to thank Li Kai and Shen Guozhen for their help.

References I

Guillaume Aucher, Johan Van Benthem, and Davide Grossi.
Sabotage modal logic: Some model and proof theoretic aspects.
In *Logic, Rationality, and Interaction*, pages 1–13. Springer, 2015.

Nina Gierasimczuk, Lena Kurzen, and Fernando R Velázquez-Quesada.
Learning and teaching as a game: A sabotage approach.
In *Logic, Rationality, and Interaction*, pages 119–132. Springer, 2009.

Christof Löding and Philipp Rohde.
Model checking and satisfiability for sabotage modal logic.
In *FST TCS 2003: Foundations of Software Technology and Theoretical Computer Science*, pages 302–313. Springer, 2003.

References II

Christof Löding and Philipp Rohde.
Solving the sabotage game is PSPACE-hard.
Springer, 2003.

Johan Van Benthem.
An essay on sabotage and obstruction.
Springer, 2005.

Yanjing Wang and Qinxiang Cao.
On axiomatizations of public announcement logic.
Synthese, 190(1):103–134, 2013.

Thank You!

Questions?