A Brief Introduction to Justification Logic
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1 Outline

- History - from constructive proof to justification
- Elements of justification logics
  - Syntax
  - Semantics
  - Realization of modal logics
- Selected topics
  - Joint systems with modal logics
  - Self-referentiality
- References

2 History

2.1 Origin - Artemov's explicit provability

- Brouwer: Truth in intuitionistic maths means the existence of a proof
- Heyting 1930: Axiomatization of intuitionistic propositional logic IPC
- Heyting and Kolmogorov (independently): An informal stipulation - "BHK semantics"
  - A proof of $\alpha \wedge \beta$ consists of a proof of α and a proof of β
  - A proof of $\alpha \vee \beta$ is given by presenting either a proof of α or a proof of β
  - A proof of $\alpha \rightarrow \beta$ is a construction transforming proofs of α into proofs of β
  - $\perp$ has no proof, and $\neg\alpha$ abbreviates $\alpha \rightarrow \perp$

- Gödel 1933: (proposed) Formalize intuitionistic truth in terms of classical provability
  - A classical propositional logic with provability operator:
    - Read $\Box\phi$ as "ϕ is provable"
    - Essentially $\mathsf{S4}$
  - An embedding of IPC into $\mathsf{S4}$: $(\cdot)^G$
    - Prefixing each subformula with a "$\Box$" - truth $\mapsto$ provability
    - $\mathsf{IPC} \vdash \phi$ implies $\mathsf{S4} \vdash \phi^G$
    - Opposite direction by McKinsey, Tarski 1948
    - Alternative embeddings exist
  - Does not work well when interpreted as $\exists x Proof(x, \ulcorner\phi\urcorner)$:

$$\begin{array}{ll} \Box\bot \to \bot & \mathsf{t}\text{-axiom / reflection principle} \\ \Box(\Box\bot \to \bot) & \\ \Box\neg\Box\bot & \end{array}$$

  - Approach 1 - Insisting on $\exists x Proof(x, \ulcorner\phi\urcorner)$, the desired logic is GL
    - Taking $\Box(\Box\alpha \to \alpha) \to \Box\alpha$ instead of $\Box\alpha \to \alpha$
    - Completeness by Solovay 1976
  - Approach 2 - Find an appropriate interpretation of $\Box$, and formalize a provability semantics for $\mathsf{S4}$
    - Gödel's 1938 lecture offers an idea - first published 1995
    - Artemov 1994 LP - the first justification logic
- Artemov's LP - the logic of proofs
  - conference presentation 1994, technical report 1995, journal publishment 2001
  - Explicit provability
    - Instead of $\Box\phi$, employ $t{:}\phi$
      - Intuitively, t is a proof of ϕ
      - $t{:}\phi \to \phi$ is provably valid:
        if $t{:}\phi$ holds, then ϕ has a concrete proof;
        otherwise, since $Proof$ is recursive, PA sees $\neg t{:}\phi$
    - t is a term, with an inductive structure
      - Constant: proof of axiom
      - Variable: proof of hypothesis
      - Operator: operations on proofs
- Formalization of provability semantics of $\mathsf{S4}$ and IPC
  - Gödel's modal embedding: interpret IPC in $\mathsf{S4}$

- Artemov's realization: interpret $\mathsf{S4}$ in LP
  - Realizer: a map that replaces $\Box$-occurrences by terms
  - $\mathsf{S4} \vdash \phi$ iff there is a realizer $(\cdot)^r$ s.t. $\mathsf{LP} \vdash \phi^r$
- Artemov's completeness: interpret LP as a provability logic, e.g., of PA
  - Arithmetical interpretation $(\cdot)^*$:
    $(t:\phi)^*$ is defined as $Proof(t^*, \ulcorner \phi^* \urcorner)$
  - Let $\mathcal{CS}$ be a constant specification (defined later):
    $\mathsf{LP}(\mathcal{CS}) \vdash \phi$ iff $\forall (\cdot)^*$ that admits $\mathcal{CS}$, $\mathsf{PA} \vdash \phi^*$

2.2 Generalization - explicit modal logics

- LP can be seen as an explicit version of $\mathsf{S4}$
  - $\Box\phi$ - some (concrete) proof proves ϕ
  - $t:\phi$ - t proves ϕ
  - Explicit provability is normal, reflexive and transitive
  - Necessitation is simulated by a meta theorem (internalization)
- Other modal logics can also be explicitized
  - Keynote ideas
    - Read negative $\Box$ as input and positive $\Box$ as output
      - e.g., $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$
    - Assign operators to model operations from input to ouput
      - in this case, binary operator "$\cdot$"
    - Correspond each modal axiom with a justification axiom
      - in this case, $t_1:(\alpha \to \beta) \to (t_2:\alpha \to t_1 \cdot t_2:\beta)$
    - Offer completeness while maintaining intended realization
  - Explicit versions have been found for modal logics:
    - $\mathsf{S4}$ - by Artemov 1995,2001
    - $\mathsf{K}, \mathsf{D}, \mathsf{T}, \mathsf{K4}, \mathsf{D4}$ - by Brezhnev 2000
    - $\mathsf{S5}$ - independently by Pacuit 2005 and Rubtsova 2005
    - several logics with $\mathsf{k}, \mathsf{d}, \mathsf{t}, \mathsf{4}, \mathsf{b}, \mathsf{5}$-axioms - by Pacuit, Artemov, Goetschi, and Kuznets 2005 2012
    - $\mathsf{K4}^3, \mathsf{S4.2}$ - by Fitting 2014 (not included in this talk)

3 Elements of justification logics

ML, JL denote a modal logic, justification logic, resp.

3.1 Syntax

- Justification languages
  - Terms $t ::= c^i_j \mid x_k \mid t \cdot t \mid t{+}t \underbrace{\mid !t \mid ?t \mid \bar{?}t}_{optional}$
    - c^i_j - j-th constant in series i (less formally, $a, b, c, ...$)
    - x_k - k-th variable (less formally, $x, y, z, ...$)
    - operators:
      - $\cdot$ - application
      - $+$ - sum
      - ! - positive introspection
      - ? - negative introspection
      - $\bar{?}$ - weak negative introspection
  - Formulas $\phi ::= \bot \mid p \mid \phi \to \phi \mid t{:}\phi$
    - $\neg, \wedge, \vee$ defined as usual
    - $t{:}\phi$ intuitively means "t is an evidence for ϕ"

- Axiomatization
  - Primary axiom schemes:
    - Finite complete propositional axiom schemes
    - $(\mathsf{app}) t_1{:}(\phi \to \psi) \to (t_2{:}\phi \to t_1 \cdot t_2{:}\psi)$
      - cf. $(\mathsf{k}) \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$ in modal logics
    - $(\mathsf{sum}_0) t_1{:}\phi \to t_1{+}t_2{:}\phi$
    - $(\mathsf{sum}_1) t_2{:}\phi \to t_1{+}t_2{:}\phi$
  - Rule schemes:
    - (MP)
    - $\dfrac{}{c^i_n{:}c^i_{n-1}{:}\ ...\ {:}c^i_1{:}A}(AN)$, where A is an axiom
  - The logic as above is J, the minimal justification logic
  - Optional axiom schemes:
    - $(\mathsf{jd}) t{:}\bot \to \bot$
      - cf. $(\mathsf{d}) \Box\bot \to \bot$
    - $(\mathsf{jt}) t{:}\phi \to \phi$
      - cf. $(\mathsf{t}) \Box\phi \to \phi$
    - $(\mathsf{j4}) t{:}\phi \to !t{:}t{:}\phi$
      - cf. $(\mathsf{4}) \Box\alpha \to \Box\Box\alpha$
    - $(\mathsf{j5}) \neg t{:}\phi \to ?t{:}\neg t{:}\phi$
      - cf. $(\mathsf{5}) \Diamond\alpha \to \Box\Diamond\alpha$

◇ (jb)$\phi \to \bar{?}t : \neg t : \neg\phi$

▷ cf. (b)$\alpha \to \Box\Diamond\alpha$

- All justification logics are extensions of J, and are named by their optional schemes
  - E.g., JD45 is the logic with optional schemes (jd, j4, j5)
  - Exception: omit letter D if (jt) presents
  - JT4 is historically called LP
  - This gives $3 \times 2^3 = 24$ justification logics
  - In logics with (j4), it is sufficient to have only the first constant in each series, and use $\dfrac{}{c : A}\,(AN_4)$ instead
- An example proof of $x : \alpha \vee y : \beta \to (a \cdot !x) + (b \cdot y) : (x : \alpha \vee \beta)$ in LP

1.	$a : (x : \alpha \to x : \alpha \vee \beta)$	AN
2.	$a : (x : \alpha \to x : \alpha \vee \beta) \to (!x : x : \alpha \to a \cdot !x : (x : \alpha \vee \beta))$	app
3.	$!x : x : \alpha \to a \cdot !x : (x : \alpha \vee \beta)$	$MP, 2, 1$
4.	$x : \alpha \to !x : x : \alpha$	j4
5.	$a \cdot !x : (x : \alpha \vee \beta) \to (a \cdot !x) + (b \cdot y) : (x : \alpha \vee \beta)$	sum
6.	$x : \alpha \to (a \cdot !x) + (b \cdot y) : (x : \alpha \vee \beta)$	classical logic, 4, 3, 5
7.	$b : (\beta \to x : \alpha \vee \beta)$	AN
8.	$b : (\beta \to x : \alpha \vee \beta) \to (y : \beta \to b \cdot y : (x : \alpha \vee \beta))$	app
9.	$y : \beta \to b \cdot y : (x : \alpha \vee \beta)$	$MP, 8, 7$
10.	$b \cdot y : (x : \alpha \vee \beta) \to (a \cdot !x) + (b \cdot y) : (x : \alpha \vee \beta)$	sum
11.	$y : \beta \to (a \cdot !x) + (b \cdot y) : (x : \alpha \vee \beta)$	classical logic, 9, 10
12.	$x : \alpha \vee y : \beta \to (a \cdot !x) + (b \cdot y) : (x : \alpha \vee \beta)$	classical logic, 6, 11

  - Cf. theorem $\Box\alpha \vee \Box\beta \to \Box(\Box\alpha \vee \beta)$ in S4
- Constant specification
  - Intuitively, a justification logic breaks evidences to atomic ones, variables and constants
    - ◇ a constant specification assigns to constants axioms they support
  - A constant specification (notation: $\mathcal{CS}$) is a "downward closed" set of formulas of the type $c^i_n : c^i_{n-1} : \ldots : c^i_1 : A$
    - ◇ downward closure means, for every $n > 0$ $c^i_{n+1} : c^i_n : \ldots : c^i_1 : A \in \mathcal{CS}$ implies $c^i_n : \ldots : c^i_1 : A \in \mathcal{CS}$
    - ◇ formulas of this type are those introducible by (AN)
  - JL($\mathcal{CS}$) is JL with (AN) restricted to $\mathcal{CS}$
    - ◇ JL is JL($\mathcal{CS}$) with the full $\mathcal{CS}$
  - A constant specification $\mathcal{CS}$ has property:
    - ◇ axiomatically appropriate - if for every axiom A and every number n there exist a series of constants $c_1, \ldots, c_n$ s.t. $c^i_n : c^i_{n-1} : \ldots : c^i_1 : A \in \mathcal{CS}$
    - ◇ schematic - if whenever $c^i_n : c^i_{n-1} : \ldots : c^i_1 : A \in \mathcal{CS}$ and A, B are instances of a same axiom scheme, $c^i_n : c^i_{n-1} : \ldots : c^i_1 : B \in \mathcal{CS}$

  - self-referential - if $c^i_1 : A \in \mathcal{CS}$ while c^i_j occurs in A

- Some meta-theorems
  - J$\mathsf{L}\mathcal{CS}$ enjoys deduction theorem
    - The standard proof works
  - J$\mathsf{L}\mathcal{CS}$ enjoys uniform substitution (variable/term, atom/formula), provided $\mathcal{CS}$ is schematic
    - $\mathcal{CS}$ should to be schematic, since when dealing with (AN), we need to use same constants provided by $\mathcal{CS}$
  - (Internalization) if $\mathcal{CS}$ is axiomatically appropriate and $\alpha_1, ..., \alpha_n \vdash \beta$, then there is a term $t(x_1, ..., x_n)$ s.t. $t_1 : \alpha_1, ..., t_n : \alpha_n \vdash t(t_1, ..., t_n) : \beta$ holds for any terms $t_1, ..., t_n$
    - induction on the original derivation
    - axiomatically appropriateness provides constants to treat axioms and (AN)'s
    - hypothesis is cared by variables
    - (app) helps in dealing with (MP)

3.2 Semantics

Note: we will not mention semantics for logics with jb, since no completeness for them has been formally claimed, as far as I know...

- Evidence function
  - Firstly appears in Mkrtychev 1997
  - $E ::= Term \mapsto 2^{Formula}$
    - specify a set of formulas to whom t serves as an evidence
  - Closure conditions (depending on the logic)
    - if $\alpha \to \beta \in E(t_1)$ and $\alpha \in E(t_2)$ then $\beta \in E(t_1 \cdot t_2)$
    - $E(t_1) \subseteq E(t_1 + t_2)$ and $E(t_2) \subseteq E(t_1 + t_2)$
    - $\phi \in E(t)$ implies $t : \phi \in E(!t)$ (when j4 is adopted)
    - $\phi \notin E(t)$ implies $\neg t : \phi \in E(?t)$ (when j5 is adopted)
  - If $\phi \in E(c)$ holds for every $c : \phi \in \mathcal{CS}$, then E is a $\mathcal{CS}$-evidence function

- Fitting model
  - A Fitting model is $\mathfrak{M} = (W, R, E, V)$ where
    - (W, R, V) is a Kripke model
      - R must be (serial, reflexive, transitive, euclidean) if (jd, jt, j4, j5) presents
    - $\mathcal{E}$ is an evidence assignment that assigns to each u an evidence function $\mathcal{E}(u)$
      - If j4 presents, then monotonicity is required: uRv implies $\mathcal{E}(u) \subseteq \mathcal{E}(v)$

- A Fitting-model is a $\mathcal{CS}$-model if for every $u \in W$, $\mathcal{E}(u)$ is a $\mathcal{CS}$-evidence function
- $\mathfrak{M}, u \vDash t\!:\!\psi$ iff
  (1) $\mathfrak{M}, v \vDash \psi$ for every v s.t. uRv and (2) $\psi \in \mathcal{E}(u)(t)$
  - If $\mathsf{j5}$ presents, then clause (1) is dropped (this is called "strong evidence") (alternative setting exists)
- Fitting Completeness (Fitting 2005, Artemov 2008, etc.)
  - $\mathsf{JL}(\mathcal{CS}) \vdash \phi$ iff
    for every $\mathcal{CS}$-model of JL and state w, $\mathfrak{M}, w \Vdash \phi$ holds
  - when JL has scheme jd, then we need a further requirement that $\mathcal{CS}$ is axiomatically appropriate
    - this is so because internalization is employed in the completeness proof
  - Proved usually by canonical model construction
    - $W = \{\Gamma \,|\, \Gamma \text{ is a maximal } \mathsf{JL}(\mathcal{CS} \text{ consistent set})\}$
    - $R = \{(\Gamma, \Delta) \,|\, \Gamma^\sharp \subseteq \Delta\}$, where $\Gamma^\sharp = \{\phi \,|\, (\exists t) t\!:\!\phi \in \Gamma\}$
    - $\mathcal{E}(\Gamma)(t) = \{\phi \,|\, t\!:\!\phi \in \Gamma\}$
    - $V(p) = \{\Gamma \in W \,|\, p \in \Gamma\}$

3.3 Realization of modal logics

- A justification logic is an explicit version of the modal logic linked to it by realization
- Realization theorem says:
  $\mathsf{ML} \vdash \phi$ iff $\exists$ realizer r to the language of JL s.t. $\mathsf{JL} \vdash \phi^r$
  - If this holds, we say that ML realizes to JL
    in notation: $\mathsf{ML} \hookrightarrow \mathsf{JL}$
- Some existing realizations:
  - Artemov 1995,2001: $\mathsf{S4} \hookrightarrow \mathsf{LP}$ (i.e., $\mathsf{JT4}$)
  - As expected, Brezhnev 2000 shows:
    $\mathsf{K} \hookrightarrow \mathsf{J}$, $\mathsf{D} \hookrightarrow \mathsf{JD}$, $\mathsf{T} \hookrightarrow \mathsf{JT}$, $\mathsf{K4} \hookrightarrow \mathsf{J4}$, $\mathsf{D4} \hookrightarrow \mathsf{JD4}$
  - Similarly for other modal logics with $\mathsf{k}, \mathsf{d}, \mathsf{t}, \mathsf{4}, \mathsf{5}, \mathsf{b}$ axioms, except that:
    - $\mathsf{KB5}$ realizes to each of $\mathsf{JB4}$, $\mathsf{JB5}$, and $\mathsf{JB45}$
    - $\mathsf{S5}$ realizes to each of $\mathsf{JT5}$, $\mathsf{JTB5}$, $\mathsf{JDB5}$, $\mathsf{JT45}$, $\mathsf{JTB45}$, $\mathsf{JDB45}$, $\mathsf{JTB4}$, $\mathsf{JDB4}$
    - Both modal logics have multiple axiomatizations, each modal scheme has a distinct corresponding justification scheme, some even with distinct operators
    - Thus 15 modal logics have 24 justification counterparts
    - Goetschi 2012 offers an embedding w.r.t. which all justification counterparts of a same modal logic are pairwise equivalent
- There are various methods in proving realization

- ∘ Artemov 1995,2001: induction on cut-free sequent proofs for S4
  - ◇ read sequents as derivations
  - ◇ for most rules, similar to the conversion of sequent proofs to axiomatical proofs (deduction theorem employed)
  - ◇ for $\dfrac{\Box\Theta \Rightarrow \eta}{\Box\Theta, \Gamma \Rightarrow \Delta, \Box\eta}(R\Box)$, employ internalization on the premise-derivation to compute an LP-term to replace the principal $\Box$
- ∘ Brezhnev 2000: transplant to K, D, T, K4, D4, as each also enjoys a cut-free sequent calculus
- ∘ Fitting 2009: a sophisticated algorithmic proof, some properties with prices
- ∘ Goetschi Kuznets 2012: employ nested sequent calculi
  - ◇ capable of realizing modal logics with b, 5 axioms
- ∘ Other methods in Fitting 2005, Wang 2011, etc.

4 Selected topics

4.1 Joint systems with modal logics

- The logic GLA Nogina 2006
  - ∘ A mixed language of GL and LP
  - ∘ Axiom and rule schemes of both, together with:
    - ◇ $t{:}\phi \to \Box\phi$
    - ◇ $\neg t{:}\phi \to \Box\neg t{:}\phi$
    - ◇ $t{:}\Box\phi \to \phi$ [cf. $\Box\phi \to \phi$]
    - ◇ $\dfrac{\vdash \Box\phi}{\vdash \phi}$ reflection
  - ∘ Completeness w.r.t. a mixed provability semantics
  - ∘ Goris 2006
    - ◇ The collection of GL-theorems realizable in LP is exactly $\mathsf{GL} \cap \mathsf{S4}$
    - ◇ An axiomatization of $\mathsf{GL} \cap \mathsf{S4}$
- The logic S4LP Fitting 2008
  - ∘ A mixed language of S4 and LP
  - ∘ Axiom and rule schemes of both, together with:
    - ◇ $t{:}\phi \to \Box\phi$
  - ∘ Justification takes care of both accessibility and evidence function, while modality takes care of only accessibility
  - ∘ Local realization:

- ◇ If $\mathfrak{M}, u \Vdash \phi$, then for some realization ψ of ϕ, $\mathfrak{M}, u \Vdash \psi$
  - Completeness w.r.t. to models that meets local realizability condition, with some proviso

- Epistemic reading
  - Logic with both knowledge and justification
  - Using different accessible relations for $\Box$ and t:

4.2 Self-referentiality

- In justification language, $t{:}\phi$ is a legal formula even if t occurs in ϕ
  - t is a justification of a propositional ϕ about t itself
  - Atomic case: $c{:}A(c)$ (arithmetical reading)
- Employ constant specification to control
  - Recall: $\mathcal{CS}$ is self-referential, if $c^i_1{:}A \in \mathcal{CS}$ while c^i_j occurs in A
  - In LP, as $\mathsf{j4}$ presents, we can take the reduced form: $\mathcal{CS}$ is self-referential if $c^i{:}A \in \mathcal{CS}$ while c^i occurs in A
- What happens if $\mathcal{CS}$ is restricted to be non-self-referential?
  - Not interesting for completeness, which has $\mathcal{CS}$ as a parameter
  - Realization?
- Kuzents 2006,2008:
  - Each K or D theorem can be realized (in J or JD) without using self-referential $\mathcal{CS}$
  - In each of $\mathsf{T}, \mathsf{K4}, \mathsf{S4}$, there is a theorem whose realization necessarily calls for self-referential $\mathcal{CS}$
    - ◇ As an instance: $\Box\neg(p\to\Box p)\to\Box\bot$
- Yu 2014:
  - There are IPC-theorems whose all images (in $\mathsf{S4}$) under Gödel-style embeddings each requires self-referential $\mathcal{CS}$ to be realized (in LP)
  - For example: $\neg\neg\alpha$ where α is intuitionistic invalid tautology
  - As an example in $\mathsf{IPC}_{\to}$: $((((p\to q)\to p)\to p)\to q)\to q$
- Yu 2015:
  - Non-self-referential realizable fragments of modal logics $\mathsf{T}, \mathsf{K4}, \mathsf{S4}$ are closed under MP
  - There are modal theorems self-referential in a smaller logic but non-self-referential in a larger one

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