Induction and Nonstandard Models of Arithmetic

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Outline

1. Introduction and Preliminaries
2. Nonstandard Models of PA
3. Cuts and Fragments of PA
4. Generalisations of Cuts in RCA_0
5. Summary

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1. Introduction and Preliminaries
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Introduction

- In the view of reverse mathematic, our main question is that which induction axiom are needed to prove theorems of ordinary mathematics. In particular, we consider which theorems of ordinary mathematics are equivalent to Σ_1-induction, over the base theory $\mathrm{RCA_0}^*$.
- In reverse recursion theory, the nonstandard models of PA, especially cuts in them, play a key role in studying the induction axioms in fragments of PA.
- We will review some basic results of nonstandard models of PA. Most important of them, we will prove that the existence of cuts in nonstandard models is an essential characteristic of models which satisfy negative induction.
- At last, we will generalize some results about nonstandard models of first-order arithmetic and cuts to second-order arithmetic. And we can get some disguises of Σ_1-induction by cuts arguments.

Preliminaries

- Let L_A denote the first-order language of arithmetic which consists of the constant symbols 0 and 1, the binary relation symbol $<$, and the two binary function symbols $+$ and $\cdot$.
- Let $L_A{}'$ denote the first-order language arithmetic L_A with adding the two binary function symbol exp.
- The structure $\mathbb{N}$, called the standard model, is the L_A-structure whose domain is the set of non-negative integers, $\{0, 1, 2, ...\}$, where the symbols in L_A are given their obvious interpretation.
- Let L_2 denote the language of second-order arithmetic, which is a two-sorted language with number variables x,y,z,... intended to range over natural numbers and set variables X,Y,Z,... intended to range over sets of natural numbers. In addition, the language includes $+, \cdot$ as operation symbols, $0, 1$ as constants and $<$ as a relation symbol, with adding a binary relation $\in$ to relate the two sorts.

Preliminaries

We will define Peano arithmetic(PA) by a first-order system PA^-, axiomatized by

$$\forall m, n, k((m+n)+k = m+(n+k)),$$
$$\forall m, n(m+n = n+m),$$
$$\forall m, n, k((m \cdot n) \cdot k = m \cdot (n \cdot k)),$$
$$\forall m, n(m \cdot n = n \cdot m),$$
$$\forall m, n, k(m \cdot (n+k) = m \cdot n + m \cdot k),$$
$$\forall m((m+0 = m) \wedge (m \cdot 0 = 0)),$$
$$\forall m(m \cdot 1 = m),$$
$$\forall m, n, k((m < n \wedge n < k) \rightarrow m < k).$$
$$\forall m \neg m < m,$$
$$\forall m, n(m < n \vee n < m \vee m = n),$$
$$\forall m, n, k(m < n \rightarrow m+k < n+k),$$
$$\forall m, n, k(0 < k \wedge m < n \rightarrow m \cdot k < n \cdot k),$$
$$\forall m, n(m < n \rightarrow \exists k m+k = n),$$
$$0 < 1 \wedge \forall m(m > 0 \rightarrow m \geq 1),$$
$$\forall m(m \geq 0),$$

Then PA can be defined by PA^- plus first-order induction axiom

$$(\varphi(0) \wedge \forall n(\varphi(n) \rightarrow \varphi(n+1))) \rightarrow \forall n \varphi(n)$$

where φ is any first-order arithmetical formula.

Preliminaries

- The axioms of Z_2 consists of basic axioms, the induction axiom

$$(0 \in X \wedge \forall n(n \in X \rightarrow n+1 \in X)) \rightarrow \forall n(n \in X),$$

  and the comprehension scheme

$$\exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

  where $\varphi(n)$ is any formula of L_2 in which X does not occur freely.
- The subsystem of Z_2, RCA_0, is based on the schema of recursive comprehension axioms(RCA):

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

  where φ and ψ are Σ_1 and Π_1 respectively.
- We define ${RCA_0}^*$ to be the theory of RCA_0 minus $I\Sigma_1$ (Σ_1-induction) plus $I\Sigma_0$ and the exponentiation axioms: $m^0 = 1$, $m^{n+1} = m^n \cdot m$. (Here, the language of Z_2 should be extended to the language of Z_2 with adding the symbol exp)
- $RCA_0 \equiv {RCA_0}^* + I\Sigma_1$.

Preliminaries

The following lemmas showed that $\mathrm{RCA}_0{}^*$, as a base theory, is strong enough to get some basic theorems which are needed.

Lemma 1.1

The following is provable in $RCA_0{}^*$. Functions can be defined by Kleene's μ-operator.

Lemma 1.2

The following is provable in $RCA_0{}^*$. Functions can be defined by bounded primitive recursion.

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The Existence of Nonstandard Models

It was Skolem who first showed that nonstandard models of all true L_A-sentences exist (1934).

Theorem 2.1

There is an L_A-structure M such that $M \models Th(\mathbb{N})$, but $M \ncong \mathbb{N}$.

Proof.

We now expand our language L_A to L_c by adding to it a new constant symbol c. Consider the following L_c-theory generated by:

$$Th(\mathbb{N}) \cup \{c \neq \underline{n} : n \in \mathbb{N}\}.$$

This theory is consistent. Since for each finite fragment of it is contained in

$$T_m = Th(\mathbb{N}) \cup \{c \neq \underline{n} : n < m\}$$

where $m \in \mathbb{N}$, and clearly T_m is satisfiable. By the compactness theorem $\bigcup_{m \in \mathbb{N}} T_m$ is consistent and has a model M_c. Thus for all $n \in \mathbb{N}$, we have $M_c \models c \neq \underline{n}$. Hence it contains an 'infinite' integer. ❐

End-extension

Lemma 2.2

Let $M \models Th(\mathbb{N})$. Then the map $f : \mathbb{N} \to M$ which sends each $n \in \mathbb{N}$ to n^M is in fact an embedding.

Proof.

Firstly, we check that f is $1-1$, notice that if $n, m \in \mathbb{N}$ with $n \neq m$ then $\mathbb{N} \models \underline{n} \neq \underline{m}$, so the sentence $\underline{n} \neq \underline{m}$ is in $Th(\mathbb{N})$ and hence true in M. Similarly f preserves $<$, $+$ and $\cdot$, since for any n, m, $k \in \mathbb{N}$

$$\begin{aligned} n < m \quad &\text{iff} \quad \mathbb{N} \models \underline{n} < \underline{m} &&\text{iff} \quad M \models \underline{n} < \underline{m}, \\ n + m = k \quad &\text{iff} \quad \mathbb{N} \models \underline{n} + \underline{m} = \underline{k} &&\text{iff} \quad M \models \underline{n} + \underline{m} = \underline{k}, \\ n \cdot m = k \quad &\text{iff} \quad \mathbb{N} \models \underline{n} \cdot \underline{m} = \underline{k} &&\text{iff} \quad M \models \underline{n} \cdot \underline{m} = \underline{k}. \end{aligned}$$

□

End-extension

Furthermore, the embedding $f : \mathbb{N} \to M$ has another important property relating to the order $<$. Notice that $<$ is a linear order on M with a least element 0 and no greatest element. Now for each $n \in \mathbb{N}$ we have

$$M \models \forall x(x < \underline{n} \to (x = \underline{0} \vee x = \underline{1} \vee \ldots \vee x = \underline{n-1})),$$

since this sentence is satisfiable in $\mathbb{N}$. It follows from this that $\mathbb{N}$ is an initial segment of M, and M is an end-extension of $\mathbb{N}$, i.e., the inclusion $\mathbb{N} \subseteq M$ has the property that

for any $n \in \mathbb{N}$, $a \in M$ we have that $M \models a < \underline{n} \to a \in \mathbb{N}$

so that no new elements are added below any given $n \in \mathbb{N}$. Thus, in particular $a \in M$ is nonstandard just in case that $M \models a > \underline{n}$ for each $n \in \mathbb{N}$.

End-extension

Theorem 2.3

Let $M \models PA^-$. Then the map $\mathbb{N} \to M$ given by $n \mapsto \underline{n}^M$ is an embedding sending $\mathbb{N}$ onto an initial segment of M.

The proof of Theorem 2.3 depends on the following four simple lemmas.

Lemma 2.4

If $n, l, k \in \mathbb{N}$ and $n = l + k$, then $PA^- \vdash \underline{n} = \underline{l} + \underline{k}$.

Lemma 2.5

If $n, l, k \in \mathbb{N}$ and $n = l \cdot k$, then $PA^- \vdash \underline{n} = \underline{l} \cdot \underline{k}$.

Lemma 2.6

If $n, k \in \mathbb{N}$ with $n < k$, then $PA^- \vdash \underline{n} < \underline{k}$.

End-extension

Lemma 2.7

For all $n \in \mathbb{N}$, $PA^- \vdash \forall x(x \leq \underline{n} \rightarrow x = \underline{0} \vee x = \underline{1} \vee \ldots \vee x = \underline{n})$.

proof of Theorem 2.3.

Lemma 2.4, 2.5, 2.6 show that the map $n \mapsto \underline{n}^M$ respects $+$, $\cdot$ and $<$. Since $\mathrm{PA}^- \vdash \forall x, y(x < y \rightarrow x \neq y)$, Lemma 2.6 also shows that the map is an embedding. Finally Lemma 2.7 shows that the image $N = \{\underline{n}^M : n \in \mathbb{N}\}$ is an initial segment of M. ❐

Because of Theorem 2.3 we will always identify $\mathbb{N}$ with the smallest initial segment $\{\underline{n}^M : n \in \mathbb{N}\}$ of a model of PA^-. In particular this implies that there will be no confusion if we denote the closed L_A term $\underline{n}$ simply as n.

Σ_1-completeness

The following theorem shows that the class of Δ_0 formulas are absolute with respect to end-extensions.

Theorem 2.8

Let M, N both be L_A-structures, with N an end-extension of M. Then $M \prec_{\Delta_0} N$.

Proof.

Induction on the complexity of Δ_0 formulas. The induction hypothesis is that for all $\varphi(\bar{x}) \in \Delta_0$ with complexity $\leq n$ and for all $\bar{a} \in M$ we have $M \models \varphi(\bar{a}) \Leftrightarrow N \models \varphi(\bar{a})$. □

Σ_1-completeness

Corollary 2.9

Suppose M, N are L_A-structures, with N being an end-extension of M, and $\varphi(\overline{x})$, $\psi(\overline{x})$ are L_A-formulas with $\varphi(\overline{x}) \in \Sigma_1$ and $\psi(\overline{x}) \in \Pi_1$. Then for any $\overline{a} \in M$

$$M \models \varphi(\overline{a}) \Rightarrow N \models \varphi(\overline{a})$$
$$N \models \psi(\overline{a}) \Rightarrow M \models \psi(\overline{a})$$

Σ_1-completeness

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$$M \models \varphi(\overline{a}) \Rightarrow N \models \varphi(\overline{a})$$
$$N \models \psi(\overline{a}) \Rightarrow M \models \psi(\overline{a})$$

Corollary 2.10

$PA^- \vdash \Sigma_1 \cap Th(\mathbb{N})$.

Order-Type of Nonstandard Models of PA^-

If M is an L_A-structure then $M\restriction<$ denote the restrict M in the language $\{<\}$, i.e. the structure with the same domain as M but with only one relation $<$.

It is evident that if $M \models PA^-$, then $M\restriction<$ satisfies the theory $DILO$ of a discrete linear order with first element but no last element, axiomatized by

$$
\begin{aligned}
DILO1: &\quad \forall n \neg n < n;\\
DILO2: &\quad \forall n, m, k(n < m \wedge m < k \rightarrow n < k);\\
DILO3: &\quad \forall n, m(n < m \vee m < n \vee n = m);\\
DILO4: &\quad \forall n(\exists m(m < n) \rightarrow \exists m(m < n \wedge \forall k(k < n \rightarrow (k < m \vee k = m))));\\
DILO5: &\quad \forall n \exists m(n < m \wedge \forall k(n < k \rightarrow m < k \vee m = k));\\
DILO6: &\quad \exists n \forall m(n < m \vee n = m).
\end{aligned}
$$

Order-Type of Nonstandard Models of PA^-

In fact, $DILO$ has $2^{\aleph_0}$ non-isomorphic countable models, for if $(A, <_A)$ is any linearly ordered set then $\mathbb{N} + \mathbb{Z} \cdot A \models DILO$, where the model $\mathbb{N} + \mathbb{Z} \cdot A$ is that with domain $\mathbb{N} \cup (\mathbb{Z} \times A)$ and order $<$ defined by

(a). $n < (z, a)$ for all $n \in \mathbb{N}$ and $(z, a) \in \mathbb{Z} \times A$;

(b). $<$ restricted in $\mathbb{N}$ is the natural order on $\mathbb{N}$;

(c). $(z_1, a_1) < (z_2, a_2)$ iff $(a_1 <_A a_2$ or $(a_1 = a_2)$ and $z_1 < z_2))$ for all $(z_1, a_1), (z_2, a_2) \in \mathbb{Z} \times A$ where $z_1 < z_2$ is the usual order on $\mathbb{Z}$.

We shall now sketch out a proof about the order-type of the models of $DILO$.

Theorem 2.11

All models of $DILO$ are isomorphic to $\mathbb{N} + \mathbb{Z} \cdot A$ for some linearly order set A. Moreover, if $A \not\cong B$ then $\mathbb{N} + \mathbb{Z} \cdot A \not\cong \mathbb{N} + \mathbb{Z} \cdot B$.

Order-Type of Nonstandard Models of PA^-

Proof.

Let $(M, <_M) \models DILO$ and let 0^M be the $<_M$-least element of M and define functions S and P on M by

$$\begin{aligned} S(x) &= \text{the unique } y \in M \ s.t. \\ &\quad M \models x < y \wedge \forall z(x < z \rightarrow (y < z \vee y = z)); \\ P(x) &= \text{the unique } y \in M \ s.t. \\ &\quad M \models y < x \wedge \forall z(z < x \rightarrow (z < y \vee z = y)), \text{ if } x \neq 0^M. \end{aligned}$$

and $P(0^M) = 0^M$. Then we can define an equivalence relation $\sim$ on M by $a \sim b$ iff $a = P^{(n)}(b)$ or $b = S^{(n)}(a)$ for some n. Then it is easy to check the following facts:

(1). $\sim$ is an equivalence relation on M;

(2). $<_M$ induces a linear order $<$ on $M/\sim$, given by $[a] < [b]$ iff $a <_M b$ and $[a] \neq [b]$;

(3). if $A = (M - [0_M])/\sim$ ordered by $<_A$ defined by (2) is a linear order, then $(M, <_M) \cong \mathbb{N} + \mathbb{Z} \cdot A$.

It follows that all models of $DILO$ are of the form $\mathbb{N} + \mathbb{Z} \cdot A$, and by examining the construction above it is easy to see that the ordered set $(A, <_A)$ obtained from $(M, <_M)$ is unique (up to isomorphism).Thus $\mathbb{N} + \mathbb{Z} \cdot A \cong \mathbb{N} + \mathbb{Z} \cdot B$ iff $A \cong B$. ❐

From this we can deduce that, since there are up to isomorphism $2^{\aleph_0}$ countable linearly ordered sets A, there are $2^{\aleph_0}$ countable models of $DILO$.

Order-Type of Nonstandard Models of PA

Let's return to nonstandard models of PA. Surprisingly, if M is countable, there is only one possibility for the order-type of $M \restriction <$.

Theorem 2.12

Let $M \models PA$ be nonstandard. Then $M \restriction < \cong \mathbb{N} + \mathbb{Z} \cdot A$ for some linearly ordered set $(A, <_A)$ satisfying the theory DLO of a dense linear order axiomatized by

$$\begin{aligned}
&DLO1. && \forall n \neg n < n; \\
&DLO2. && \forall n, m, k(n < m \wedge m < k \to n < k); \\
&DLO3. && \forall n, m(n < m \vee m < n \vee n = m); \\
&DLO4. && \forall m, n(n < m \to \exists k(n < k \wedge k < m)); \\
&DLO5. && \forall n \exists m, k(m < n \wedge n < k).
\end{aligned}$$

In particular, if M is countable, then $M \cong \mathbb{N} + \mathbb{Z} \cdot \mathbb{Q}$, where $\mathbb{Q}$ is the set of rationals with its natural order.

Order-type of Nonstandard Models of PA

Proof.

It is a well-known Cantor's theorem that any countable structure $(A, <_A)$ satisfying DLO is isomorphic to $(\mathbb{Q}, <)$. This shows it is sufficient to prove that $M \restriction < \cong \mathbb{N} + \mathbb{Z} \cdot A$ for some $(A, <_A) \models DLO$.

Since the functions $S(x)$ and $P(x)$ defined above are obviously $x + 1$ and $x - 1$, we define a relation $\sim$ by

$$a \sim b \text{ iff } M \models a + n = b \vee a = b + n \text{ for some } n \in \mathbb{N}.$$

Then let $A = (M - 0)/\sim$ with its order $<_A$ induced from $M \restriction <$ by

$$[a] <_a [b] \Leftrightarrow [a] \neq [b] \wedge M \models a < b.$$

Then we verify that $(A, <_A) \models DLO$. ∎

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Cuts

Definition 3.1

A nonempty subset I of a nonstandard model $M \models \mathrm{PA}$ is a cut of M if I is closed under the successor function and downward i.e. $x < y \in I \to x \in I$ and $x \in I \to x + 1 \in I$. And we say I is proper if it is not equal to the domain of M.

Cuts

Definition 3.1

A nonempty subset I of a nonstandard model $M \models \mathrm{PA}$ is a cut of M if I is closed under the successor function and downward i.e. $x < y \in I \rightarrow x \in I$ and $x \in I \rightarrow x + 1 \in I$. And we say I is proper if it is not equal to the domain of M.

Lemma 3.2

Let M be a model of PA, then any proper cut I is undefinable in M.

Proof.

Let $\bar{a} \in M$ and $\varphi(x, \bar{a})$ be an L_A-formula. We suppose that I can be defined by $\varphi(x, \bar{a})$, Then

$$I = \{b \in M | M \models \varphi(b, \bar{a})\}.$$

Since I is nonempty and is closed under the successor function, we have

$$M \models \varphi(0, \bar{a}) \wedge \forall x(\varphi(x, \bar{a}) \rightarrow \varphi(x + 1, \bar{a})),$$

and so by induction in M, $M \models \forall x \varphi(x, \bar{a})$, hence $I = M$ contradicting the assumption that I is proper. ❒

The following overspill theorem, due to Abraham Robinson, essentially says just above lemma.

Theorem 3.3 (Overspill)

Let $M \models PA$ be a nonstandard and let I be a proper cut of M. Suppose $\bar{a} \in M$ and $\varphi(x, \bar{a})$ is a L_A-formula such that for all $b \in I$, $M \models \varphi(b, \bar{a})$. Then there is $c > I$ in M such that $M \models \forall x \leq c\varphi(x, \bar{a})$.

Proof.

Suppose not. Then I would be defined by the L_A-formula

$$\forall y < x\varphi(y, \bar{a}),$$

contradicting the fact that I is undefinable in M. This completes the proof. ❒

Cuts

Corollary 3.4

Let $M \models PA$ be nonstandard and I a proper cut of M. Suppose $\bar{a} \in M$ and $\varphi(x, \bar{a})$ is a L_A-formula, and that for all $x \in I$ there exists $y \in I$ such that $M \models y \geq x \wedge \varphi(y, \bar{a})$. Then for each $c > I$ in M there exists $b \in M$ such that $I < b < c$ and $M \models \varphi(b, \bar{a})$.

Proof.

Apply overspill theorem to the formula $\exists y(x \leq y < c \wedge \varphi(y, \bar{a}))$, Where $c \in M$ is an arbitrarily element satisfying $c > I$. ❐

Intuitively, the corollary says if there are unboundedly many $y \in I$ satisfying $\varphi(y, \bar{a})$, then there are arbitrarily small $b > I$ satisfying $\varphi(b, \bar{a})$. Thus the elements in and beyond I are 'mirror cofinal'.

Theorem 3.5 (Underspill)

Let $M \models PA$ be nonstandard and I a proper cut of M. Suppose $\bar{a} \in M$ and $\varphi(x, \bar{a})$ is a L_A-formula. if for all $c > I$ in M there exists $x > I$ satisfying $M \models \varphi(x, \bar{a}) \wedge x < c$, then for all $b \in I$ there exists $y \in I$ satisfying $M \models y \geq b \wedge \varphi(y, \bar{a})$.

Fragments of PA

- P^- denote the $L_A{}'$-theory which consists of PA$^-$, axioms of exponentiation and Σ_0 induction.
- The Σ_n bounded collection ($B\Sigma_n$) means the scheme:

$$\forall i \exists j(\varphi(i,j)) \rightarrow \forall m \exists n \forall i < m \exists j < n(\varphi(i,j)),$$

  where $\varphi(i,j)$ is any Σ_n formula in which m and n do not occur. Intuitively, it says that every total Σ_n-function onto a proper initial segment has a bounded range.

Theorem 3.6

For all $n \geq 1$, over P^-,

$$I\Sigma_{n+1} \Rightarrow B\Sigma_{n+1} \Rightarrow I\Sigma_n,$$

and the only true implications are the ones indicated.

Lemma 3.7

$P^- \vdash B\Sigma_0 \leftrightarrow B\Sigma_1.$

Proof.

One direction is obvious. Now we just show that $P^- \vdash B\Sigma_0 \rightarrow B\Sigma_1$. Assume $B\Sigma_0$ and $\forall i < m \exists j \varphi(i,j)$, where $\varphi(i,j)$ is Σ_1. Then we suppose $\varphi(i,j)$ is $\exists k \phi(i,j,k)$, where $\phi(i,j,k)$. Thus $\forall i < m \exists j \exists k \phi(i,j,k)$, It is easy to verity that in $P^- + I\Sigma_0$ we can define a bijective function $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, By $B\Sigma_0$, we have that $\forall i < m \exists f(j,k) < n \phi(i,j,k)$, Then $\forall i < m \exists j < n \exists k < n \phi(i,j,k)$, that is, $\forall i < m \exists j < n \varphi(i,j)$, as required. ❐

Theorem 3.8

$P^- + I\Sigma_1 \vdash B\Sigma_1$.

Proof.

Let $M \models P^- + I\Sigma_1$ and suppose $M \models \forall i < n \exists j \varphi(i,j)$, where $\varphi(i,j)$ is Δ_0. Let $\psi(u)$ be the formula $n < u \vee \exists m \forall i < u \exists j < m \varphi(i,j)$. Then $\psi(u)$ is Σ_1 Applying $I\Sigma_1$ to $\psi(u)$. Clearly $M \models \psi(0)$. If $M \models \psi(u)$ with $u < n$ then there are m_1,$m_2 \in M$ such that $M \models \forall i < u \exists j < m_1 \varphi(i,j)$ and $M \models \exists j < m_2 \varphi(u,j)$. Then $m = \max(m_1, m_2)$ clearly satisfies

$$M \models \forall i < u+1 \exists j < m \varphi(i,j),$$

proving the induction step, $M \models \forall u < n(\psi(u) \rightarrow \psi(u+1))$. Thus $M \models \forall u \psi(u)$ by $I\Sigma_1$. And in particular we have $M \models \forall u \leq n(\exists m \forall i < u \exists i < m \varphi(i,j))$, hence $M \models \exists m \forall i < n \exists i < m \varphi(i,j)$. Thus we have that $P^- + I\Sigma_1 \vdash B\Sigma_0$. Thus the aim follows the fact that $P^- \vdash B\Sigma_0 \leftrightarrow B\Sigma_1$ by Lemma 3.7. ❒

Lemma 3.9

Let $M \models P^-$ and let I be a proper initial segment of M closed under $+$ and $\cdot$. Then $I \models P^- + B\Sigma_1^0$.

Proof.

Note that $I \models I\Sigma_0$ since $I \prec_{\Delta_0} M$ and $M \models I\Sigma_0$ and $I\Sigma_0$ is Π_1-axiomatized. Thus by Lemma 3.7. $I \models B\Sigma_1 \leftrightarrow B\Sigma_0$. Then we only need to show that $I \models B\Sigma_0$. Suppose that $I \models \forall i < m \exists j \varphi(i,j)$ where $\varphi(i,j)$ is Σ_0. Then for every $i \in M$ with $i < m$, i is an element of I and so there is $j \in I$ such that $\varphi(i,j)$ is true in I, and hence $\varphi(i,j)$ is also true in M since $I \prec_{\Delta_0} M$. Therefore for all $n \in M - I$, $M \models \forall i < m \exists j < n \varphi(i,j)$. This formula is equivalent to a Σ_0 formula. And by $I\Sigma_0$ we may use the Σ_0 least element principle to deduce that there is a least $n \in M$ such that $M \models \forall i < m \exists j < n \varphi(i,j)$. Clearly this least such n is in I, so for any $i < m$ there exists $j < n$ such that $M \models \varphi(i,j)$. But this j is in I since $n \in I$ and I is a proper initial segment of M. And as $I \prec_{\Delta_0} M$, $I \models \varphi(i,j)$, hence there is $n \in I$ such that $I \models \forall i < m \exists j < n \varphi(i,j)$, that is $I \models B\Sigma_0$. ❒

Lemma 3.10

Let M be a nonstandard model of P^-, then it has a Σ_1-cut if and only if $I\Sigma_1$ fails in M.

Proof.

Immediate from Σ_1 overspill. ❐

The Role of Σ_1-cuts in Fragments of PA

Lemma 3.10

Let M be a nonstandard model of P^-, then it has a Σ_1-cut if and only if $I\Sigma_1$ fails in M.

Proof.

Immediate from Σ_1 overspill. ❐

Theorem 3.11

There exists a model M satisfying $P^- + B\Sigma_1$ which is not a model of $I\Sigma_1$.

Lemma 3.10

Let M be a nonstandard model of P^-, then it has a Σ_1-cut if and only if $I\Sigma_1$ fails in M.

Proof.

Immediate from Σ_1 overspill. □

Theorem 3.11

There exists a model M satisfying $P^- + B\Sigma_1$ which is not a model of $I\Sigma_1$.

Corollary 3.12

Let M be a model of $P^- + B\Sigma_1$, Then M is a $B\Sigma_1$ model if and only if it has a Σ_1-cut.

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Lemma 4.1

$RCA_0{}^*$ proves $B\Sigma_1$.

Proof.

We reason in $RCA_0{}^*$. Suppose $\forall i \exists j \varphi(i,j)$, where $\varphi(i,j)$ is Σ^0_1. Let $\varphi(i,j) = \exists k \psi(i,j,k)$, where $\psi(i,j,k)$ is Σ_0. By definition, $(j,k) = (j+k)^2 + j$. Then we define a function $f : \mathbb{N} \to \mathbb{N}$ by $f(i) =$ least (j,k) such that $\psi(i,j,k)$ holds, with using Lemma 1.1. Then using bounded primitive recursion, by Lemma 1.2, define $g : \mathbb{N} \to \mathbb{N}$ as follows

$$\begin{aligned} g(0) &= f(0), \\ g(m+1) &= \begin{cases} g(m) & \text{if } f(m+1) \leq f(g(m)), \\ m+1 & \text{otherwise.} \end{cases} \end{aligned}$$

Let $h(m) = f(g(m)) + 1$. Then also using Δ_1 comprehension, we can get $h(m) = \max\{f(i) + 1 : i \leq m\}$. Thus $\forall i < m \exists (j,k) < h(m) \psi(i,j,k)$. Let $n = h(m)$, clearly $\forall i < m \exists j < n \varphi(i,j)$. This completes the proof. ❒

Lemma 4.2

Let M be any model of P^- and $B\Sigma_1$. Then there exists a model M' of ${RCA_0}^*$ such that M' has the same first-order part as M.

Generalisations of Cuts in RCA_0

Lemma 4.2

Let M be any model of P^- and $B\Sigma_1$. Then there exists a model M' of $RCA_0{}^*$ such that M' has the same first-order part as M.

Theorem 4.3

The models $RCA_0{}^*$ restricted in the first-order part are precisely the models for $L_A{}'$ which satisfy P^- and $B\Sigma_1$.

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Let M be any model of P^- and $B\Sigma_1$. Then there exists a model M' of ${RCA_0}^*$ such that M' has the same first-order part as M.

Theorem 4.3

The models ${RCA_0}^*$ restricted in the first-order part are precisely the models for ${L_A}'$ which satisfy P^- and $B\Sigma_1$.

Corollary 4.4

Let M be a model of ${RCA_0}^* + \neg I\Sigma_1$. Then there is a Σ_1-cut in M.

Generalisations of Cuts in RCA_0

Using Corollary 4.4, we can get many results which is equivalent to Σ_1-induction within ${\mathrm{RCA}_0}^*$.

Theorem 4.5 (Disguises of Σ_1-induction)

Within ${RCA_0}^*$. The following are equivalent.

(1) Σ^0_1 *induction;*

(2) *The universe of total functions is closed under primitive recursion;*

(3) *Every torsion-free, finitely generated abelian group is free;*

(4) *Every finitely generated vector space over rationals has a basis.*

Outline

1 Introduction and Preliminaries

2 Nonstandard Models of PA

3 Cuts and Fragments of PA

4 Generalisations of Cuts in RCA_0

5 Summary

Summary

- We review some results of nonstandard models of PA, including the order-type of nonstandard models, the relationship between the standard model and nonstandard models.
- We have seen that there are close ties between $I\Sigma_1$ and Σ_1-cuts in nonstandard models of fragments of PA. In other words, the existence of Σ_1-cuts is an essential characteristic of negative $I\Sigma_1$.
- We generalize the idea of cuts to second-order arithmetic, and get some results of disguises of Σ_1 induction by the idea of Σ_1-cuts.

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The End

THANKS!