On the Finite Model Property of S4 Logics with Finite Width

李楷

北大哲学系, 2016.1.29

For transitive logics, we have many results in modal logic: concerning finite model property in modal logic.

- (Segerberg, 1971) and (Bull and Segerberg, 1984) shows that any transitive logic with finite depth has the f.m.p.
- (Fine, 1974) shows that any transitive logic with finite width is complete
- (Bull, 1966) and (Fine, 1971) shows that any normal extension of $\mathbf{S4.3}$ has the f.m.p.
- (Xu, 2002) and (Xu, 2013) show that any normal extension of $\mathbf{G.3}$ and a class of normal extension of $\mathbf{K4.3}$ has the f.m.p.and is finitely aximotizable,
- (Li, 2011) shows that any any normal extension of $\mathbf{K4.3z}$ has the f.m.p.and is finitely aximotizable,

In this slides we mainly concern finite model property of reflective

Finite Width

Definition

Finite width $\mathbf{S4}$ logic is a logic containing following formulas $\mathbf{I}_n (n > 0)$ and $\mathbf{S4}$ where:

$$\mathbf{I}_n = \bigwedge_{i=0}^{n} \Diamond p_i \to \bigvee_{0 \leqslant i \neq j \leqslant n} \Diamond (p_i \wedge (p_j \vee \Diamond p_j))。$$

Finite Width

Definition

Finite width $\mathbf{S4}$ logic is a logic containing following formulas $\mathbf{I}_n (n > 0)$ and $\mathbf{S4}$ where:

$$\mathbf{I}_n = \bigwedge_{i=0}^{n} \Diamond p_i \to \bigvee_{0 \leqslant i \neq j \leqslant n} \Diamond (p_i \wedge (p_j \vee \Diamond p_j))。$$

Let $\mathfrak{F} = (W, R)$ be any frame, let $w, u \in W$ and let $A \subseteq W$. w and u are *(R-)incomparable* if neither wRu nor uRw. A is a *cluster* if $A \neq \emptyset$ and for all $w, u \in A$, wRu and uRw. A is an *anti-chain* if for all $w, u \in A$, $w \neq u$ only if w and u are incomparable.

Finite Width

Definition

Finite width $\mathbf{S4}$ logic is a logic containing following formulas $\mathbf{I}_n (n > 0)$ and $\mathbf{S4}$ where:

$$\mathbf{I}_n = \bigwedge_{i=0}^{n} \Diamond p_i \to \bigvee_{0 \leqslant i \neq j \leqslant n} \Diamond (p_i \wedge (p_j \vee \Diamond p_j))\text{。}$$

Let $\mathfrak{F} = (W, R)$ be any frame, let $w, u \in W$ and let $A \subseteq W$. w and u are *(R-)incomparable* if neither wRu nor uRw. A is a *cluster* if $A \neq \emptyset$ and for all $w, u \in A$, wRu and uRw. A is an *anti-chain* if for all $w, u \in A$, $w \neq u$ only if w and u are incomparable.

Fact

Any frame $\mathfrak{F}$ of a finite width (containing $\mathbf{I}_n$) $\mathbf{S4}$ logic is reflective,

p-morphism

Definition (p-morphism)

Let $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$ be two frame. A function $f : W \to W'$ is a *p-morphism* from $\mathfrak{F}$ to $\mathfrak{F}'$ if

1. f is a surjection from W to W',
2. for all $w, u \in W$, wRu implies $f(w)R'f(u)$,
3. for all $w \in W$ and $u' \in U$, $f(w)Ru'$ implies wRu for some $u \in W$ such that $f(u) = u'$.

$\mathfrak{F}'$ is a *p-morphic image* of $\mathfrak{F}$ if there is a p-morphism from $\mathfrak{F}$ to $\mathfrak{F}'$.

p-morphism

Definition (p-morphism)

Let $\mathfrak{F} = (W, R)$ and $\mathfrak{F}' = (W', R')$ be two frame. A function $f : W \to W'$ is a *p-morphism* from $\mathfrak{F}$ to $\mathfrak{F}'$ if

1. f is a surjection from W to W',
2. for all $w, u \in W$, wRu implies $f(w)R'f(u)$,
3. for all $w \in W$ and $u' \in U$, $f(w)Ru'$ implies wRu for some $u \in W$ such that $f(u) = u'$.

$\mathfrak{F}'$ is a *p-morphic image* of $\mathfrak{F}$ if there is a p-morphism from $\mathfrak{F}$ to $\mathfrak{F}'$.

Fact

If $\mathfrak{F}$ is a frame of some logic and $\mathfrak{F}'$ is a p-morphic image of $\mathfrak{F}$, then $\mathfrak{F}'$ is also a frame of this logic.

Chains

Definition

Let $\mathfrak{F} = (W, R)$ be any frame. A sequence of points $w_1, w_2, \ldots, w_n \in W$ is an *R-chain* if $w_{i+1} R w_i$ for each i with $0 < i \leq n$. We use $C, C', \ldots$ for R-chains, and we abuse the notation $w \in C$, $C \cap C'$ and $C \subseteq A$ for w is an element in this sequence, the set consisting of the common elements of C and C', and every element of C is in A. R-chain $w_1, w_2, \ldots, w_n$ is *strict* if not $w_i R w_{i+1}$ for all i with $0 < i \leq n$. $\mathfrak{F}$ is *Notherian* if $\mathfrak{F}$ is transitive and there is no infinite strict R-chain.

Chains

Definition

Let $\mathfrak{F} = (W, R)$ be any frame. A sequence of points $w_1, w_2, \ldots, w_n \in W$ is an *R-chain* if $w_{i+1} R w_i$ for each i with $0 < i \leq n$. We use $C, C', \ldots$ for R-chains, and we abuse the notation $w \in C$, $C \cap C'$ and $C \subseteq A$ for w is an element in this sequence, the set consisting of the common elements of C and C', and every element of C is in A. R-chain $w_1, w_2, \ldots, w_n$ is *strict* if not $w_i R w_{i+1}$ for all i with $0 < i \leq n$. $\mathfrak{F}$ is *Notherian* if $\mathfrak{F}$ is transitive and there is no infinite strict R-chain.

Theorem (Completeness Result by Fine)

Any logic contains $\mathbf{I}_n (n > 0)$ and $\mathbf{S4}$ is characterized by a class of Notherian frames.

Witness Set

Definition (witness set)

Let $\mathfrak{M} = (W, R, V)$ be any model and let α be any formula satisfiable on $\mathfrak{M}$. We use $final(\alpha)$ for the set of R-maximal points in $\{w \in W | \mathfrak{M}, w \models \alpha\}$, i.e., for each $w \in W$, $w \in final(\alpha)$ iff $\mathfrak{M}, w \models \alpha$ and for each $u \in W$ such that $\mathfrak{M}, u \models \alpha$, wRu implies uRw. Furthermore we use $sub(\alpha)$ for the set of all subformulas of α. The *witness set* of α (w.r.t. $\mathfrak{M}$) is $\bigcup_{\beta \in sub(\alpha)} final(\beta)$.

Witness Set

Definition (witness set)

Let $\mathfrak{M} = (W, R, V)$ be any model and let α be any formula satisfiable on $\mathfrak{M}$. We use *final*(α) for the set of R-maximal points in $\{w \in W | \mathfrak{M}, w \models \alpha\}$, i.e., for each $w \in W$, $w \in$ *final*(α) iff $\mathfrak{M}, w \models \alpha$ and for each $u \in W$ such that $\mathfrak{M}, u \models \alpha$, wRu implies uRw. Furthermore we use *sub*(α) for the set of all subformulas of α. The *witness set* of α (w.r.t. $\mathfrak{M}$) is $\bigcup_{\beta \in sub(\alpha)} final(\beta)$.

Fact

Let $\mathfrak{M} = (W, R, V)$ be any model and let α be any formula satisfiable on $\mathfrak{M}$. If $\mathfrak{M}$ is of finite width, then the witness set of α is finite.

Notherian

Lemma

Let $\mathfrak{M} = (W, R, V)$ be any Notherian model and let α be any formula satisfiable on $\mathfrak{M}$. If there is a p-morphism f from (W, R) to $\mathfrak{F} = (W', R')$, the witness set A of α is a subset of W and the f restricted to A is an isomorphism, then α is satisfiable in $\mathfrak{F}$.

Notherian

Lemma

Let $\mathfrak{M} = (W, R, V)$ be any Notherian model and let α be any formula satisfiable on $\mathfrak{M}$. If there is a p-morphism f from (W, R) to $\mathfrak{F} = (W', R')$, the witness set A of α is a subset of W and the f restricted to A is an isomorphism, then α is satisfiable in $\mathfrak{F}$.

Definition

A logic has the finite model property (f.m.p.) if it is characterized by a class of finite model.

Notherian

Lemma

Let $\mathfrak{M} = (W, R, V)$ be any Notherian model and let α be any formula satisfiable on $\mathfrak{M}$. If there is a p-morphism f from (W, R) to $\mathfrak{F} = (W', R')$, the witness set A of α is a subset of W and the f restricted to A is an isomorphism, then α is satisfiable in $\mathfrak{F}$.

Definition

A logic has the finite model property (f.m.p.) if it is characterized by a class of finite model.

Let $\mathbf{L}$ be any logic. In order to show that $\mathbf{L}$ has the f.m.p, we want to prove that:

for any formula α consistent with $\mathbf{L}$ and any model $\langle\mathfrak{F}, V\rangle$ satisfying α there is a finite model $\langle\mathfrak{F}', V'\rangle$ satisfying α and $\mathfrak{F}'$ is a

Interval and Substructure

Definition

Let $\mathfrak{F} = (W, R)$ be any frame, and let $A \subseteq W$. A is an *interval* if for all $w, u \in A$ and each $v \in W$, $wRvRu$ only if $v \in A$. We use $A{\uparrow}_R$ for the set $\{w \in W | uRw \text{ for some } u \in A\}$, and $w{\uparrow}_R$ for $w{\uparrow}_R$ for.

Interval and Substructure

Definition

Let $\mathfrak{F} = (W, R)$ be any frame, and let $A \subseteq W$. A is an *interval* if for all $w, u \in A$ and each $v \in W$, $wRvRu$ only if $v \in A$. We use $A{\uparrow}_R$ for the set $\{w \in W | uRw \text{ for some } u \in A\}$, and $w{\uparrow}_R$ for $w{\uparrow}_R$ for.

Definition

Let $\mathfrak{F} = \langle W, R \rangle$ be any frame. Frame $\mathfrak{G} = \langle U, S \rangle$ is a subframe of $\mathfrak{F}$ if:

- $U \subseteq W$,
- $S = R \cap (U \times U)$。

Let $B \subseteq W$. $\mathfrak{G} = \langle U, S \rangle$ is the subframe of $\mathfrak{F}$ restricted to B if $U = B$ and $\mathfrak{G}$ is a subframe of $\mathfrak{F}$. $\mathfrak{G} = \langle U, S \rangle$ is a generated subframe of $\mathfrak{F}$ from B if $U = B{\uparrow}_R$. The submodel generated

Interval Cuts

Definition (Interval Cuts)

Let $\mathfrak{M} = (W, R, V)$ be any model, let α be any formula satisfiable on $\mathfrak{M}$ and let A be the witness set of α. The *interval cuts* of $\mathfrak{M}$ w.r.t. α is a sequence of anti-chains $C_1, C_2, \ldots, C_n$ such that C_1 is the set of all R-maximal points in $\mathfrak{M}$. For each $k+1$, C_{k+1} is a maximal anti-chain containing the R'-maximal elements of A in the submodel $\mathfrak{M}' = \langle W', R', V' \rangle$ of $\mathfrak{M}$ restricted to $W - C_k{\uparrow}_R$.

Interval Cuts

Definition (Interval Cuts)

Let $\mathfrak{M} = (W, R, V)$ be any model, let α be any formula satisfiable on $\mathfrak{M}$ and let A be the witness set of α. The *interval cuts* of $\mathfrak{M}$ w.r.t. α is a sequence of anti-chains $C_1, C_2, \ldots, C_n$ such that C_1 is the set of all R-maximal points in $\mathfrak{M}$. For each $k+1$, C_{k+1} is a maximal anti-chain containing the R'-maximal elements of A in the submodel $\mathfrak{M}' = \langle W', R', V' \rangle$ of $\mathfrak{M}$ restricted to $W - C_k{\uparrow}_R$.

Lemma

Let $\mathfrak{M} = (W, R, V)$ be any Notherian model of finite width and let α be any formula satisfiable on $\mathfrak{M}$. Then the interval cuts of $\mathfrak{M}$ w.r.t. α is a finite sequence.

Iner-connected Intervals

Definition

Let $\mathfrak{F} = (W, R)$ be any frame, let $w, u \in W$ and let $A \subseteq W$. w is *tough* if either there are incomparable points in $u, v \in W$ such that w is an R-maximal point to see both u and v, (i.e., wRu and wRv, and for each $w' \in W$, $w'Ru$, $w'Rv$ and wRw' only if $w'Rw$) or w is an R-maximal point in W, (i.e., for each $u \in W$, wRu only if uRw).

Iner-connected Intervals

Definition

Let $\mathfrak{F} = (W, R)$ be any frame, let $w, u \in W$ and let $A \subseteq W$. w is *tough* if either there are incomparable points in $u, v \in W$ such that w is an R-maximal point to see both u and v, (i.e., wRu and wRv, and for each $w\prime \in W$, $w\prime Ru$, $w\prime Rv$ and $wRw\prime$ only if $w\prime Rw$) or w is an R-maximal point in W, (i.e., for each $u \in W$, wRu only if uRw). An interval B is *iner-connected* if it is an R-chain and for each $w, u \in B$, if w is tough, then uRw. An iner-connected interval B is maximal w.r.t. A if $B \subseteq A$ and there is no iner-connected interval B' such that $B \subset B' \subseteq A$.

Iner-connected Intervals

Definition

Let $\mathfrak{F} = (W, R)$ be any frame, let $w, u \in W$ and let $A \subseteq W$. w is *tough* if either there are incomparable points in $u, v \in W$ such that w is an R-maximal point to see both u and v, (i.e., wRu and wRv, and for each $w\prime \in W$, $w\prime Ru$, $w\prime Rv$ and $wRw\prime$ only if $w\prime Rw$) or w is an R-maximal point in W, (i.e., for each $u \in W$, wRu only if uRw). An interval B is *iner-connected* if it is an R-chain and for each $w, u \in B$, if w is tough, then uRw. An iner-connected interval B is maximal w.r.t. A if $B \subseteq A$ and there is no iner-connected interval B' such that $B \subset B' \subseteq A$.

p-morphism and Iner-connected Intervals

Theorem (p-morphism and Iner-connected Intervals)

Let $\mathfrak{M} = (W, R, V)$ be any Notherian $\mathbf{S4}$-modal, let α be any formula satisfiable on $\mathfrak{M}$ and let $C_1, C_2, \ldots, C_n$ be an interval cuts of $\mathfrak{M}$ w.r.t. α. Let $W' \subseteq W$ be the set such that $w \in W'$ iff w is an R-maximal point in an iner-connected interval B is maximal w.r.t. $C_{k+1}{\uparrow}_R - C_k{\uparrow}_R$, then the submodel restricted to W' is a p-morhic image of $\mathfrak{M}$.

Tough Chains

Definition

$w_1, w_2, \ldots, w_n$ is an *(R-)tough chain* if it is a strict R-chain and w_i is tough for all i. R-chain (R-tough chain) C is *maximal with respect to an interval* A if $C \subseteq A$, and there is no longer R-chain (R-tough chain) in A contains every elements in C(, note that maximal implies filled). R-chain (R-tough chain) $w_1, w_2, \ldots, w_n$ is *filled* if for each $w \in W$ (that is tough), $w_{i+1}RwRw_i$ for some $i < n$ only if either wRw_i or $w_{i+1}Rw$. Sequences $w_1, w_2, \ldots, w_n$ and $u_1, u_2, \ldots, u_m$ are conjugate if $w_1 = u_1$ and $w_n = u_m$. A sequence of R-chains are conjugate if any two of these chains are conjugate. A sequence of R-chains $C_1, C_2, \ldots$ is *anti-chain generable* if they are distinct, pairwise conjugate and for each i such that $1 < i < n$ where n is the length of C_1, w_i is incomparable to any element

Generating Infinite Anti-chain

Lemma

Let $\mathfrak{F} = (W, R)$ be any frame without infinite tough chain. Suppose there is an inifinite sequence $C_1, C_2, \ldots$ of distinct, filled and conjugate though chains. Then there is an inifinite sequence $S = (C'_1, C'_2, \ldots)$ of filled and anti-chain generable though chians such that each C'_i is a subchain of C_j for some $j \in \omega$.

Generating Infinite Anti-chain

Proof.

Let $w_1, w_2, \ldots, w_n$ be C_1. Without losing any generality, suppose $n > 2$. Then there is an infinite sub-sequence of S: $C_{i_1}, C_{i_2}, C_{i_3}, \ldots$ such that $C_{i_1} = C_1$ and for all $j > 1$ $C_{i_1} \cap C_{i_j} = C_{i_1} \cap C_{i_2}$. (because C_1 is finite, $\{C_1 \cap C_i | i \in \omega\}$ is finite, recall that each C_i is distinct.) $C_{i_1} \cap C_{i_2} = C_{i_1}$, for otherwise C_{i_1} is a subchain of C_{i_2}, contrary to our presupposition that they are filled and conjugate.

Consider any $w_k \in C_{i_1} - (C_{i_1} \cap C_{i_2})$ and any $j > 1$. Let $C_{i_j} = (u_1, u_2, \ldots, u_l)$. Without losing any generality, suppose $w_{k-1}, w_{k+1} \in C_{i_1} \cap C_{i_2}$ and $w_{k-1} = u_n, w_{k+1} = u_m$. Then $m \neq n + 1$, for otherwise $u_m R w_k R u_n$, contrary to that $C_{i_j}$is filled. Obviously w_k and $u_{k'}$ are incomparable for each k' such that $n < k' < m$.

Generating Infinite Anti-chain

Lemma

Let $\mathfrak{F} = (W, R)$ be any frame without infinite tough chain, let A be an interval. Then there is no infinite sequence of distinct and maximal though chains.

Generating Infinite Anti-chain

Lemma

Let $\mathfrak{F} = (W, R)$ be any frame without infinite tough chain, let A be an interval. Then there is no infinite sequence of distinct and maximal though chains.

Proof.

Suppose there are inifinitely many such though chains. We prove that there is an infinite anti-chain.

There is an infinite sequence $S = (C_1, C_2, \ldots)$ of distinct and conjugate though chains. This is because any two distinct first elements of these though chains maximal w.r.t. A, say C and C', are incomparable, for otherwise C or C' is not maximal w.r.t. A. The same goes for the last elements. □

Generating Infinite Anti-chain

Proof.

Hence by finite width, if there is no such S_1, there is an infinite anti-chain.

We constuct an infinite anti-chain as follows:

Using Lemma 18, we have an inifinite sequence $S_1 = (C_1^1, C_2^1, C_3^1, \ldots)$ of filled and anti-chain generable though chians such that each C_i^1 is a subchain of C_j for some $j \in \omega$. Let w_1 be the second element of C_1^1. □

Generating Infinite Anti-chain

Proof.

If we have sequence $S_n = (C_1^n, C_2^n, C_3^n, \ldots)$ and w_n, using Lemma 18 on $C_2^n, C_3^n, C_3^n, \ldots$ we obtain an inifinite sequence $S_{n+1} = (C_1^{n+1}, C_2^{n+1}, C_3^{n+1}, \ldots)$ of filled and anti-chain generable though chians such that each C_i^{n+1} is a subchain of C_j^n for some $j \in \omega$ with $j > 1$. Let w_{n+1} be the second element of C_1^{n+1}. □

Generating Infinite Anti-chain

Proof.

Now we claim that the sequence $w_1, w_2, w_3, \ldots$ is an anti-chain. Consider any nonzero $i < j \in \omega$. w_i and w_j are the second element of C_1^i and C_1^j respectively. An easy induction can show that C_1^j is a subchain of C_k^i for some $k \in \omega$ with $k > 1$. Furthermore by the definition of anti-chain generable, C_1^j has at least three elements, we can get that w_j is neither the first nor the last element of C_k^i, and then w_i and w_j are incomparable. □

Generating Infinite Anti-chain

Theorem

Let $\mathbf{L}$ *be any finite width* $\mathbf{S4}$ *logic without infinite though chain.*

Then $\mathbf{L}$ *has the f.m.p.*

Generating Infinite Anti-chain

Theorem

Let $\mathbf{L}$ be any finite width $\mathbf{S4}$ logic without infinite though chain. Then $\mathbf{L}$ has the f.m.p.

Proof.

Consider any $\mathbf{L}$-consistent formula α. We know that there is a point generated and Notherian $\mathbf{L}$-model $\mathfrak{M} = \langle W, R, V \rangle$ such that α is true at the root of $\mathfrak{M}$. Let $C_1, C_2, \ldots, C_n$ be an interval cuts of $\mathfrak{M}$ w.r.t. α and let $\mathfrak{M}' = \langle W', R', V' \rangle$ be the submodel of $\mathfrak{M}$ such that $W' = \{w \in W | w \text{ is tough}\} \cup \bigcup_{0<i\leq n} C_i$. We have W' is finite. We only need to show that there is a p-morphism f from $\langle W, R \rangle$ to $\langle W', R' \rangle$ and f restricted to W' is an isomorphism. □

f.m.p. for logics without infinite though chain

Theorem (f.m.p. for finite width $\mathbf{S4}$ logic without infinite though chain)

Let $\mathbf{L}$ be any finite width $\mathbf{S4}$ logic without infinite though chain. Then $\mathbf{L}$ has the f.m.p.

Proof.

It is easy to check that for each $w \in W$, $w \in W'$ iff w is an R-maximal point in an inerconnected interval maximal w.r.t. $C_{k+1}{\uparrow}_R - C_k{\uparrow}_R$.

□

Bull, R. A. (1966). That all normal extensions of S4.3 have the finite model property [J]. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 12:341–344.

Bull, R. A. and Segerberg, K. (1984). Basic modal logic [A]. In Gabbay, D. M. and Guenthner, F., editors, *Handbook of Philosophical Logic*, volume 2, pages 1–88. Reidel, Dordrecht.

Fine, K. (1971). The logics containing S4.3 [J]. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 17:371–376.

Fine, K. (1974). Logics containing K4, part I [J]. *The Journal of Symbolic Logic*, 39:229–237.

Li, K. (2011). Normal extensions of K4.3Z [Z]. Manuscript, August 2011, Department of Philosophy, Wuhan University.

Segerberg, K. (1971). *An Essay in Classical Modal Logic [D]*, volume 13. Filosofiska Föreningen och Filosofiska Institutionen vid Uppsala Universitet, Uppsala.

Xu, M. (2002). Normal extensions of G.3 [J]. *Theoria*, 68(2):170–176.

Xu, M. (2013). Some normal extensions of K4.3 [J]. *Studia Logica*, 101 (3):583–599.