On the Finite Model Property of S4 Logics with Finite Width

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For transitive logics, we have many results in modal logic: concerning finite model property in modal logic.

- (Segerberg, 1971) and (Bull and Segerberg, 1984) shows that any transitive logic with finite depth has the f.m.p.
- (Fine, 1974) shows that any transitive logic with finite width is complete.
- (Bull, 1966) and (Fine, 1971) shows that any normal extension of $S4.3$ has the f.m.p.
- (Xu, 2002) and (Xu, 2013) show that any normal extension of $G.3$ and a class of normal extension of $K4.3$ has the f.m.p. and is finitely axiomatizable.
- (Li, 2011) shows that any any normal extension of $K4.3z$ has the f.m.p. and is finitely axiomatizable.

In this slides we mainly concern finite model property of reflective
Finite Width

**Definition**

Finite width $\mathbf{S4}$ logic is a logic containing following formulas $I_n(n > 0)$ and $\mathbf{S4}$ where:

$$I_n = \bigwedge_{i=0}^{n} \lozenge p_i \rightarrow \bigvee_{0 \leq i \neq j \leq n} \lozenge (p_i \land (p_j \lor \lozenge p_j)).$$
Finite Width

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\]

Let \( \mathcal{F} = (W, R) \) be any frame, let \( w, u \in W \) and let \( A \subseteq W \). \( w \) and \( u \) are \((R-)incomparable\) if neither \( wRu \) nor \( uRw \). \( A \) is a \( cluster \) if \( A \neq \emptyset \) and for all \( w, u \in A \), \( wRu \) and \( uRw \). \( A \) is an \( anti-chain \) if for all \( w, u \in A \), \( w \neq u \) only if \( w \) and \( u \) are incomparable.
Finite Width

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$$I_n = \bigwedge_{i=0}^{n} \Diamond p_i \rightarrow \bigvee_{0 \leq i \neq j \leq n} \Diamond (p_i \land (p_j \lor \Diamond p_j))$$

Let $\mathfrak{F} = (W, R)$ be any frame, let $w, u \in W$ and let $A \subseteq W$. $w$ and $u$ are (R-)incomparable if neither $wRu$ nor $uRw$. $A$ is a cluster if $A \neq \emptyset$ and for all $w, u \in A$, $wRu$ and $uRw$. $A$ is an anti-chain if for all $w, u \in A$, $w \neq u$ only if $w$ and $u$ are incomparable.

Fact

Any frame $\mathfrak{F}$ of a finite width (containing $I_n$) S4 logic is reflective, transitive and of finite width, i.e., any generated subframe of $\mathfrak{F}$ has no anti-chain longer than $n$. 
Definition ($p$-morphism)

Let $\mathcal{F} = (W, R)$ and $\mathcal{F}' = (W', R')$ be two frames. A function $f : W \rightarrow W'$ is a $p$-morphism from $\mathcal{F}$ to $\mathcal{F}'$ if

1. $f$ is a surjection from $W$ to $W'$,
2. for all $w, u \in W$, $wRu$ implies $f(w)R'f(u)$,
3. for all $w \in W$ and $u' \in U$, $f(w)Ru'$ implies $wRu$ for some $u \in W$ such that $f(u) = u'$.

$\mathcal{F}'$ is a $p$-morphic image of $\mathcal{F}$ if there is a $p$-morphism from $\mathcal{F}$ to $\mathcal{F}'$. 
**Definition (p-morphism)**

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$\mathcal{F}'$ is a p-morphic image of $\mathcal{F}$ if there is a p-morphism from $\mathcal{F}$ to $\mathcal{F}'$.

**Fact**

*If $\mathcal{F}$ is a frame of some logic and $\mathcal{F}'$ is a p-morphic image of $\mathcal{F}$, then $\mathcal{F}'$ is also a frame of this logic.*
Chains

Definition
Let $\mathcal{F} = (W, R)$ be any frame. A sequence of points $w_1, w_2, \ldots, w_n \in W$ is an $R$-chain if $w_{i+1}Rw_i$ for each $i$ with $0 < i \leq n$. We use $C, C', \ldots$ for $R$-chains, and we abuse the notation $w \in C$, $C \cap C'$ and $C \subseteq A$ for $w$ is an element in this sequence, the set consisting of the common elements of $C$ and $C'$, and every element of $C$ is in $A$. $R$-chain $w_1, w_2, \ldots, w_n$ is strict if not $w_iRw_{i+1}$ for all $i$ with $0 < i \leq n$. $\mathcal{F}$ is Noetherian if $\mathcal{F}$ is transitive and there is no infinite strict $R$-chain.
### Chains

#### Definition

Let $\mathcal{F} = (W, R)$ be any frame. A sequence of points $w_1, w_2, \ldots, w_n \in W$ is an $R$-chain if $w_{i+1} Rw_i$ for each $i$ with $0 < i \leq n$. We use $C, C', \ldots$ for $R$-chains, and we abuse the notation $w \in C, C \cap C'$ and $C \subseteq A$ for $w$ is an element in this sequence, the set consisting of the common elements of $C$ and $C'$, and every element of $C$ is in $A$. $R$-chain $w_1, w_2, \ldots, w_n$ is strict if not $w_i Rw_{i+1}$ for all $i$ with $0 < i \leq n$. $\mathcal{F}$ is Notherian if $\mathcal{F}$ is transitive and there is no infinite strict $R$-chain.

#### Theorem (Completeness Result by Fine)

*Any logic contains $\text{I}_n (n > 0)$ and $\text{S4}$ is characterized by a class of Notherian frames.*
Witness Set

**Definition (witness set)**

Let $\mathcal{M} = (\mathcal{W}, R, V)$ be any model and let $\alpha$ be any formula satisfiable on $\mathcal{M}$. We use $\text{final}(\alpha)$ for the set of $R$-maximal points in $\{w \in \mathcal{W} | M,w \models \alpha\}$, i.e., for each $w \in \mathcal{W}$, $w \in \text{final}(\alpha)$ iff $M, w \models \alpha$ and for each $u \in \mathcal{W}$ such that $M, u \models \alpha$, $wRu$ implies $uRw$. Furthermore we use $\text{sub}(\alpha)$ for the set of all subformulas of $\alpha$. The **witness set** of $\alpha$ (w.r.t. $\mathcal{M}$) is $\bigcup_{\beta \in \text{sub}(\alpha)} \text{final}(\beta)$. 
**Witness Set**

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**Fact**

Let $\mathcal{M} = (W, R, V)$ be any model and let $\alpha$ be any formula satisfiable on $\mathcal{M}$. If $\mathcal{M}$ is of finite width, then the witness set of $\alpha$ is finite.
Let $\mathcal{M} = (W, R, V)$ be any Notherian model and let $\alpha$ be any formula satisfiable on $\mathcal{M}$. If there is a $p$-morphism $f$ from $(W, R)$ to $\mathcal{F} = (W', R')$, the witness set $A$ of $\alpha$ is a subset of $W$ and the $f$ restricted to $A$ is an isomorphism, then $\alpha$ is satisfiable in $\mathcal{F}$. 

**Definition.**

A logic has the finite model property (f.m.p.) if it is characterized by a class of finite model.
Let $\mathcal{M} = (W, R, V)$ be any Notherian model and let $\alpha$ be any formula satisfiable on $\mathcal{M}$. If there is a $p$-morphism $f$ from $(W, R)$ to $\mathcal{F} = (W', R')$, the witness set $A$ of $\alpha$ is a subset of $W$ and the $f$ restricted to $A$ is an isomorphism, then $\alpha$ is satisfiable in $\mathcal{F}$.

**Definition**

A logic has the finite model property (f.m.p.) if it is characterized by a class of finite model.
Let \( M = (W, R, V) \) be any Notherian model and let \( \alpha \) be any formula satisfiable on \( M \). If there is a \( p \)-morphism \( f \) from \( (W, R) \) to \( \mathcal{F} = (W', R') \), the witness set \( A \) of \( \alpha \) is a subset of \( W \) and the \( f \) restricted to \( A \) is an isomorphism, then \( \alpha \) is satisfiable in \( \mathcal{F} \).

**Lemma**

**Definition**

A logic has the finite model property (f.m.p.) if it is characterized by a class of finite model.

Let \( L \) be any logic. In order to show that \( L \) has the f.m.p, we want to prove that:

for any formula \( \alpha \) consistent with \( L \) and any model \( \langle \mathcal{F}, V \rangle \) satisfying \( \alpha \) there is a finite model \( \langle \mathcal{F}', V' \rangle \) satisfying \( \alpha \) and \( \mathcal{F}' \) is a \( p \)-morphic image of \( \mathcal{F} \).
Interval and Substructure

**Definition**

Let $\mathcal{F} = (W, R)$ be any frame, and let $A \subseteq W$. $A$ is an *interval* if for all $w, u \in A$ and each $v \in W$, $wRvRu$ only if $v \in A$. We use $A \uparrow_R$ for the set $\{ w \in W | uRw \text{ for some } u \in A \}$, and $w \uparrow_R$ for $w \uparrow_R$ for.
Interval and Substructure

Definition

Let $\mathcal{F} = (W, R)$ be any frame, and let $A \subseteq W$. $A$ is an interval if for all $w, u \in A$ and each $v \in W$, $wRvRu$ only if $v \in A$. We use $A \uparrow_R$ for the set $\{w \in W | uRw$ for some $u \in A\}$, and $w \uparrow_R$ for $w \uparrow_R$ for.

Definition

Let $\mathcal{F} = \langle W, R \rangle$ be any frame. Frame $\mathcal{G} = \langle U, S \rangle$ is a subframe of $\mathcal{F}$ if:

- $U \subseteq W$,
- $S = R \cap (U \times U)$.

Let $B \subseteq W$. $\mathcal{G} = \langle U, S \rangle$ is the subframe of $\mathcal{F}$ restricted to $B$ if $U = B$ and $\mathcal{G}$ is a subframe of $\mathcal{F}$. $\mathcal{G} = \langle U, S \rangle$ is a generated subframe of $\mathcal{F}$ from $B$ if $U = B \uparrow_R$. The submodel, generated submodel, point generated subframe and point generated submodel is defined as usual.
Interval Cuts

**Definition (Interval Cuts)**

Let \( \mathcal{M} = (W, R, V) \) be any model, let \( \alpha \) be any formula satisfiable on \( \mathcal{M} \) and let \( A \) be the witness set of \( \alpha \). The *interval cuts* of \( \mathcal{M} \) w.r.t. \( \alpha \) is a sequence of anti-chains \( C_1, C_2, \ldots, C_n \) such that \( C_1 \) is the set of all \( R \)-maximal points in \( \mathcal{M} \). For each \( k + 1 \), \( C_{k+1} \) is a maximal anti-chain containing the \( R' \)-maximal elements of \( A \) in the submodel \( \mathcal{M}' = \langle W', R', V' \rangle \) of \( \mathcal{M} \) restricted to \( W - C_k \uparrow R \).
Interval Cuts

Definition (Interval Cuts)

Let $\mathcal{M} = (W, R, V)$ be any model, let $\alpha$ be any formula satisfiable on $\mathcal{M}$ and let $A$ be the witness set of $\alpha$. The interval cuts of $\mathcal{M}$ w.r.t. $\alpha$ is a sequence of anti-chains $C_1, C_2, \ldots, C_n$ such that $C_1$ is the set of all $R$-maximal points in $\mathcal{M}$. For each $k + 1$, $C_{k+1}$ is a maximal anti-chain containing the $R'$-maximal elements of $A$ in the submodel $\mathcal{M}' = \langle W', R', V' \rangle$ of $\mathcal{M}$ restricted to $W - C_k \uparrow_R$.

Lemma

Let $\mathcal{M} = (W, R, V)$ be any Noetherian model of finite width and let $\alpha$ be any formula satisfiable on $\mathcal{M}$. Then the interval cuts of $\mathcal{M}$ w.r.t. $\alpha$ is a finite sequence.
Iner-connected Intervals

Definition
Let $\mathcal{F} = (W, R)$ be any frame, let $w, u \in W$ and let $A \subseteq W$. $w$ is **tough** if either there are incomparable points in $u, v \in W$ such that $w$ is an $R$-maximal point to see both $u$ and $v$, (i.e., $wRu$ and $wRv$, and for each $w' \in W$, $w'Ru$, $w'Rv$ and $wRw'$ only if $w'Rw$) or $w$ is an $R$-maximal point in $W$, (i.e., for each $u \in W$, $wRu$ only if $uRw$).
Iner-connected Intervals

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Let $\mathfrak{F} = (W, R)$ be any frame, let $w, u \in W$ and let $A \subseteq W$. $w$ is \textit{tough} if either there are incomparable points in $u, v \in W$ such that $w$ is an $R$-maximal point to see both $u$ and $v$, (i.e., $wRu$ and $wRv$, and for each $w' \in W, w'Ru, w'Rv$ and $wRw'$ only if $w'Rw$) or $w$ is an $R$-maximal point in $W$, (i.e., for each $u \in W, wRu$ only if $uRw$).

An interval $B$ is \textit{iner-connected} if it is an $R$-chain and for each $w, u \in B$, if $w$ is tough, then $uRw$. An iner-connected interval $B$ is maximal w.r.t. $A$ if $B \subseteq A$ and there is no iner-connected interval $B'$ such that $B \subset B' \subseteq A$. 
Iner-connected Intervals

**Definition**

Let $\mathcal{F} = (W, R)$ be any frame, let $w, u \in W$ and let $A \subseteq W$. $w$ is *tough* if either there are incomparable points in $u, v \in W$ such that $w$ is an $R$-maximal point to see both $u$ and $v$, (i.e., $wRu$ and $wRv$, and for each $w' \in W$, $w'Ru$, $w'Rv$ and $wRw'$ only if $w'Rw$) or $w$ is an $R$-maximal point in $W$, (i.e., for each $u \in W$, $wRu$ only if $uRw$).

An interval $B$ is *iner-connected* if it is an $R$-chain and for each $w, u \in B$, if $w$ is tough, then $uRw$. An iner-connected interval $B$ is maximal w.r.t. $A$ if $B \subseteq A$ and there is no iner-connected interval $B'$ such that $B \subset B' \subseteq A$. 
Theorem (p-morphism and Iner-connected Intervals)

Let $\mathcal{M} = (W, R, V)$ be any Noetherian $\mathsf{S}4$-modal, let $\alpha$ be any formula satisfiable on $\mathcal{M}$ and let $C_1, C_2, \ldots, C_n$ be an interval cuts of $\mathcal{M}$ w.r.t. $\alpha$. Let $W' \subseteq W$ be the set such that $w \in W'$ iff $w$ is an $R$-maximal point in an iner-connected interval $B$ is maximal w.r.t. $C_{k+1} \uparrow_R - C_k \uparrow_R$, then the submodel restricted to $W'$ is a $p$-morphic image of $\mathcal{M}$. 
## Tough Chains

### Definition

\(w_1, w_2, \ldots, w_n\) is an *(R-)tough chain* if it is a strict *R*-chain and \(w_i\) is tough for all \(i\). *R*-chain (\(R\)-tough chain) \(C\) is *maximal with respect to an interval* \(A\) if \(C \subseteq A\), and there is no longer *R*-chain (\(R\)-tough chain) in \(A\) contains every elements in \(C\), note that maximal implies filled). *R*-chain (\(R\)-tough chain) \(w_1, w_2, \ldots, w_n\) is *filled* if for each \(w \in W\) (that is tough), \(w_{i+1} R w_i\) for some \(i < n\) only if either \(w R w_i\) or \(w_{i+1} R w\). Sequences \(w_1, w_2, \ldots, w_n\) and \(u_1, u_2, \ldots, u_m\) are conjugate if \(w_1 = u_1\) and \(w_n = u_m\). A sequence of *R*-chains are conjugate if any two of these chains are conjugate. A sequence of *R*-chains \(C_1, C_2, \ldots\) is *anti-chain generable* if they are distinct, pairwise conjugate and for each \(i\) such that \(1 < i < n\) where \(n\) is the length of \(C_1\), \(w_i\) is incomparable to any element.
Generating Infinite Anti-chain

Lemma

Let $\mathcal{F} = (W, R)$ be any frame without infinite tough chain. Suppose there is an infinite sequence $C_1, C_2, \ldots$ of distinct, filled and conjugate tough chains. Then there is an infinite sequence $S = (C'_1, C'_2, \ldots)$ of filled and anti-chain generable tough chains such that each $C'_i$ is a subchain of $C_j$ for some $j \in \omega$. 
Generating Infinite Anti-chain

Proof.

Let \( w_1, w_2, \ldots, w_n \) be \( C_1 \). Without losing any generality, suppose \( n > 2 \). Then there is an infinite sub-sequence of \( S: C_{i_1}, C_{i_2}, C_{i_3}, \ldots \) such that \( C_{i_1} = C_1 \) and for all \( j > 1 \) \( C_{i_1} \cap C_{i_j} = C_{i_1} \cap C_{i_2} \). (because \( C_1 \) is finite, \( \{ C_1 \cap C_i \mid i \in \omega \} \) is finite, recall that each \( C_i \) is distinct.) \( C_{i_1} \cap C_{i_2} = C_{i_1} \), for otherwise \( C_{i_1} \) is a subchain of \( C_{i_2} \), contrary to our presupposition that they are filled and conjugate.

Consider any \( w_k \in C_{i_1} - (C_{i_1} \cap C_{i_2}) \) and any \( j > 1 \). Let \( C_{i_j} = (u_1, u_2, \ldots, u_l) \). Without losing any generality, suppose \( w_{k-1}, w_{k+1} \in C_{i_1} \cap C_{i_2} \) and \( w_{k-1} = u_n, w_{k+1} = u_m \). Then \( m \neq n + 1 \), for otherwise \( u_mRw_kRu_n \), contrary to that \( C_{i_j} \) is filled. Obviously \( w_k \) and \( u_{k'} \) are incomparable for each \( k' \) such that \( n < k' < m \).
Generating Infinite Anti-chain

Lemma

Let $\mathcal{F} = (W, R)$ be any frame without infinite tough chain, let $A$ be an interval. Then there is no infinite sequence of distinct and maximal though chains.
Lemma

Let \( \mathcal{F} = (W, R) \) be any frame without infinite tough chain, let \( A \) be an interval. Then there is no infinite sequence of distinct and maximal though chains.

Proof.

Suppose there are infinitely many such though chains. We prove that there is an infinite anti-chain.

There is an infinite sequence \( S = (C_1, C_2, \ldots) \) of distinct and conjugate though chains. This is because any two distinct first elements of these though chains maximal w.r.t. \( A \), say \( C \) and \( C' \), are incomparable, for otherwise \( C \) or \( C' \) is not maximal w.r.t. \( A \). The same goes for the last elements.
Generating Infinite Anti-chain

Proof.
Hence by finite width, if there is no such $S_1$, there is an infinite anti-chain.

We construct an infinite anti-chain as follows:
Using Lemma 18, we have an infinite sequence $S_1 = (C_1^1, C_2^1, C_3^1, \ldots)$ of filled and anti-chain generable though chains such that each $C_i^1$ is a subchain of $C_j$ for some $j \in \omega$. Let $w_1$ be the second element of $C_1^1$. 

\[ \square \]
Generating Infinite Anti-chain

Proof.

If we have sequence $S_n = (C^n_1, C^n_2, C^n_3, \ldots)$ and $w_n$, using Lemma 18 on $C^n_2, C^n_3, C^n_3, \ldots$ we obtain an infinite sequence $S_{n+1} = (C^{n+1}_1, C^{n+1}_2, C^{n+1}_3, \ldots)$ of filled and anti-chain generable though chains such that each $C^{n+1}_i$ is a subchain of $C^n_j$ for some $j \in \omega$ with $j > 1$. Let $w_{n+1}$ be the second element of $C^{n+1}_1$. \qed
Generating Infinite Anti-chain

Proof.

Now we claim that the sequence \( w_1, w_2, w_3, \ldots \) is an anti-chain. Consider any nonzero \( i < j \in \omega \). \( w_i \) and \( w_j \) are the second element of \( C^i_1 \) and \( C^j_1 \) respectively. An easy induction can show that \( C^j_1 \) is a subchain of \( C^i_k \) for some \( k \in \omega \) with \( k > 1 \). Furthermore by the definition of anti-chain generable, \( C^j_1 \) has at least three elements, we can get that \( w_j \) is neither the first nor the last element of \( C^i_k \), and then \( w_i \) and \( w_j \) are incomparable.
Theorem

Let $L$ be any finite width S4 logic without infinite though chain. Then $L$ has the f.m.p.
Generating Infinite Anti-chain

**Theorem**

Let $\mathbf{L}$ be any finite width $\mathbf{S4}$ logic without infinite though chain. Then $\mathbf{L}$ has the f.m.p.

**Proof.**

Consider any $\mathbf{L}$-consistent formula $\alpha$. We know that there is a point generated and Notherian $\mathbf{L}$-model $\mathcal{M} = \langle W, R, V \rangle$ such that $\alpha$ is true at the root of $\mathcal{M}$. Let $C_1, C_2, \ldots, C_n$ be an interval cuts of $\mathcal{M}$ w.r.t. $\alpha$ and let $\mathcal{M}' = \langle W', R', V' \rangle$ be the submodel of $\mathcal{M}$ such that $W' = \{ w \in W | w \text{ is tough} \} \cup \bigcup_{0 < i \leq n} C_i$. We have $W'$ is finite. We only need to show that there is a $p$-morphism $f$ from $\langle W, R \rangle$ to $\langle W', R' \rangle$ and $f$ restricted to $W'$ is an isomorphism.
Theorem (f.m.p. for finite width S4 logic without infinite though chain)

Let $L$ be any finite width S4 logic without infinite though chain. Then $L$ has the f.m.p.

Proof.

It is easy to check that for each $w \in W$, $w \in W'$ iff $w$ is an $R$-maximal point in an interconnected interval maximal w.r.t. $C_{k+1}^{\uparrow_R} - C_k^{\uparrow_R}$. 


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