Conditional Ought, a Game Theoretical Perspective

Xin Sun

Department of Philosophy, Tsinghua University,
Beijing, China
shinksun@gmail.com

Abstract. This paper presents a new consequentialist deontic logic in which the relation of preference over sets of possible worlds and the relation of conditional dominance are both transitive. This logic validate the principle that absolute ought can be derived from conditional ought whenever the conditional statement is the agent's absolute ought. Ought about conditionals is not implied by conditional ought in this logic.

Keywords: consequentialist deontic logic, transitivity, conditional dominance.

1 Introduction

Lucy is playing the matching pennies game with Lily. They choose, simultaneously, whether to show the head or the tail of a coin. If they show the same side, Lucy receives one dollar; if they show different sides, Lucy receive 0 dollar. This situation can be depicted in Figure 1:(The letters φ and ψ are not referred in current discussion, but are involved in the proof of proposition 21.)

		Lily: head	Lily: tail
Lucy	head	φ w_1 1 ψ	φ w_2 0
	tail	w_3 0 γ ψ	w_4 1

Fig. 1.

It is obvious that under the condition of Lily showing head, Lucy ought to see to it that she shows head. Our question is, does Lucy ought to see to it that if Lily shows head, then she shows head? The answer seems to be positive at first sight. But here we claim the answer is negative. The conclusion can be achieved by following reasoning: Denote the situation in which Lucy shows tail and Lily shows head as situation γ. First note that to Lucy, both showing head and showing tail are optimal, which implies Lucy is permitted to show tail. Since Lucy showing tail may lead to situation γ, Lucy is permitted to lead to situation

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γ. Next, assume the answer is positive, that is, Lucy ought to see to it that if Lily shows head, then Lucy shows head. This means Lucy ought to prohibit the the following outcome of the game: Lily shows head but Lucy shows tail. Hence Lucy ought to prohibit situation γ. Contradiction.

We refer sentences of the form "under the condition of φ, agents ought to see to it that ψ" as conditional ought; and we use ought about conditionals to name sentences of the form "agents ought to see to it that if φ then ψ". [1] suggests that our desired theory of conditional obligation should include the principle that conditional ought implies ought about conditionals. In [2], this principle is involved in the strongly normal system **G**. But our matching pennies example indicates that this principle is doubtable. In this paper, we are going to present a consequentianlist deontic logic in which ought about conditionals is not implied by conditional ought.

Our consequentianlist deontic logic is based on the logic of [5], [7], [6] and [11]. [5] is a notable book, which represents a major advance in the field of stit-based deontic logic. However, there are some technical mistakes which are derived from an inappropriate definition of *preference over sets of possible worlds* and causes several problems when discussing the properties of conditional ought. [7], inspired by [5], developed a consequentianlist deontic logic which can be used to analysis moral conflicts between different groups of agents with different moral codes. [6] adds conditional ought to consequentianlist deontic logic. Although the semantic of conditional ought in [6] is slightly different from that of [5], it potentially bears similar problems as [5]. In our new consequentianlist deontic logic, most mis-proved theorems in [5] can be proved and the potential problems of conditional ought in [6] are avoided. Furthermore, as we have already mentioned, this new logic can distinguish conditional ought and ought about conditionals. Consequentianlist deontic logic is in some sense a kind of preference based deontic logic, related work in this field include [12] and [3].

The structure of this paper is as follows: Section 2 is an introduction to our new consequentianlist deontic logic, including the language and semantics. In Section 3 we analyze some principles of conditional ought using our new logic. Section 4 is conclusion and future work. The proves for propositions are listed in the Appendix.

2 A New Consequentianlist Deontic Logic

2.1 Language

The language of consequentianlist deontic logic is built from a finite set A of agents and a countable set P of atomic propositions. We use p and q as variables for atomic propositions in P, use F and G, where $F, G \subseteq A$, as groups of agents. The consequentianlist deontic language $\mathfrak{L}$ is given by the following Backus-Naur Form:

$$\varphi \ ::= \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid [G]\varphi \mid \odot_G^F \varphi \mid \odot_G^F(\varphi/\varphi)$$

Intuitively, $\Diamond\varphi$ can be read as " It is possible that φ". $[G]\varphi$ can be read as "Group G sees to it that φ". $\bigodot_G^F \varphi$ can be read as "In the interest of group F, group G ought to see to it that φ". $\bigodot_G^F(\varphi/\psi)$ can be read as "In the interest of group F, group G ought to see to it that φ under the condition of ψ". We use $\mathbb{P}_G^F\varphi$ as an abbreviation of $\neg\bigodot_G^F\neg\varphi$, which can be read as 'In the interest of group F, group G is permitted to lead to a situation in which φ is true."

2.2 Consequentialist Frame

The semantics of consequentianlist deontic logic is based on consequentianlist frames. Similar to [6], our definition of consequentianlist frame is as follows:

Definition 1 (consequentianlist frame). A *consequentianlist frame* $\mathfrak{F}$ is a quadruple $\langle W, A, Choice, \{Value_F\}_{F\subseteq A}\rangle$, where W is a nonempty set of possible worlds, A is a finite set of agents, *Choice* is a choice function, and $Value_F$, represents the preference of some group of agents $F \subseteq A$, is a function from W to the set of real numbers $\mathbb{R}$. Formally, $Value_F : W \to \mathbb{R}$.

The choice function *Choice* is a function from the power set of A to the power set of the power set of W, *i.e.* $Choice : \wp(A) \mapsto \wp(\wp(W))$. *Choice* is built from the individual Choice function $IndChoice$: $A \mapsto \wp(\wp(W))$. $IndChoice$ must satisfy the following three conditions: (1) for each agent $i{\in}A$ it holds that $IndChoice(i)$ is a partition of W; (2) for each selection function s that assigning to each agent $i \in A$ a set of possible worlds $s(i) \in IndChoice(i)$, it holds that $\bigcap_{i\in A} s(i)$ in nonempty; (3) for each $i \in A$, the set $IndChoice(i)$ is finite. Let *Select* be the set of all selection functions, then

$$Choice(G) = \{\bigcap_{i\in G} s(i) : s \in Select\}$$

if G is nonempty. Otherwise, $Choice(G) = \{W\}$. For any two world w and w', if there exist a $K \in Choice(G)$ such that $w \in K$ and $w' \in K$, we denote it as $w \sim_G w'$. Intuitively, $w \sim_G w'$ means the choice of group G cannot sperate w and w'.

Take the Prisoner's Dilemma in [9] as an example:

		player β	
		quiet	fink
player α	quiet	w_1 3, 3	w_2 0, 4
	fink	w_3 4, 0	w_4 1, 1

Fig. 2.

In this example, A=$\{\alpha\ \beta\}$,$W = \{w_1, w_2, w_3, w_4\}$, $IndChoice\ (\alpha)$=$\{\{w_1, w_2\},\{w_3,w_4\}\}$, $IndChoice(\beta)$ =$\{\{w_1, w_3\},\{w_2,w_4\}\}$. Apparently both

$IndChoice(\alpha)$ and $IndChoice(\beta)$ are partitions of W. And there are four selection functions, $Select = \{s_1, s_2, s_3, s_4\}$, where:

$$s_1(\alpha) = \{w_1, w_2\},\ s_1(\beta) = \{w_1, w_3\}$$

$$s_2(\alpha) = \{w_1, w_2\},\ s_2(\beta) = \{w_2, w_4\}$$

$$s_3(\alpha) = \{w_3, w_4\},\ s_3(\beta) = \{w_1, w_3\}$$

$$s_4(\alpha) = \{w_3, w_4\},\ s_4(\beta) = \{w_2, w_4\}$$

So we have for each $s \in Select$, $\bigcap_{i \in A} s(i)$ is not empty. Therefore the two conditions of individual choice are both satisfied. Then we have $Choice(A) = \{\bigcap_{i \in A} s(i) : s \in Select\} = \{\{w_1\}, \{w_2\}, \{w_3\}, \{w_4\}\}$.

Having defined consequentianlist frames, we are able to define *preferences over sets of worlds*. In [5], the definition is like following:

Definition 2′. Let $X \subseteq W$, $Y \subseteq W$ be two sets of worlds, F a group of agents from a consequentialist frame. Then $X \leq_F Y$ if and only if for each $w \in X$, for each $w' \in Y$, $Value_F(w) \leq Value_F(w')$.

As we have mentioned, it is this definition that causes mistakes. According to this definition, for any set $X \subseteq W$, $X \leq_F \emptyset$ is true and for any set $Y \subseteq W$, $\emptyset \leq_F Y$ is also true. Hence this preference relation can't be transitive. But intuitively preference relations should be transitive and in fact lots of theorems in [5] are based on the transitivity of this preference relation. So we must modify the definition to make it transitive. Our attempt is following:

Definition 2 (preferences over sets of worlds; $\leq_F$, $<_F$). Let $X \subseteq W$, $Y \subseteq W$ be two sets of worlds, F a group of agents from a consequentialist frame. Then $X \leq_F Y$(Y is *weakly preferred* to X) if and only if (1)for each $w \in X$, for each $w' \in Y$, $Value_F(w) \leq Value_F(w')$ and (2) there exist some $v \in X$, $v' \in Y$, $Value_F(v) \leq Value_F(v')$; $X <_F Y$ (Y is *stongly preferred* to X) if and only if $X \leq_F Y$ and it is not the case that $Y \leq_F X$.

Given Definition 2, we have some useful lemma and propositions as follows:

Lemma 3. Let X and Y be two sets of worlds, F a group of agents from a consequentialist frame. Then $X \leq_F Y$ if and only if $Value(w) \leq Value(w')$ for each $w \in X$, for each $w' \in Y$ and $X \neq \emptyset$, $Y \neq \emptyset$.

Proposition 4. Let X and Y be two sets of worlds, F a group of agents from a consequentialist frame. Then $X <_F Y$ if and only if (1) $Value_F(w) \leq Value_F(w')$ for each $w \in X$, for each $w' \in Y$, and (2) $Value_F(w) < Value_F(w')$ for some $w \in X$, for some $w' \in Y$.

Proposition 5. Let X and Y be sets of worlds, F a group of agents from a consequentialist frame. Then:

1. If $X \leq_F Y$ and $Y \leq_F Z$, then $X \leq_F Z$.
2. If $X \leq_F Y$ and $Y <_F Z$, then $X <_F Z$.
3. If $X <_F Y$ and $Y \leq_F Z$, then $X <_F Z$.
4. If $X <_F Y$ and $Y <_F Z$, then $X <_F Z$.

Proposition 5 states that the relation of preference over sets of worlds is transitive. This property is the foundation of our semantics. Only with this transitive relation, we can properly define the concept of dominance and optimal.

Definition 6 (dominance relation; $\leq_G^F$). Let M be a consequentialist frame. Let F, $G \subseteq A$ and K, $K' \in Choice(G)$. Then
$K \leq_G^F K'$ iff for all $S \in Choice(A - G)$, $K \cap S \leq_F K' \cap S$.

$K \leq_G^F K'$ can be read as "in the interest of group G, K' weakly dominates K". From a game theoretical perspective, $K \leq_G^F K'$ means no matter how other agents act, the agent's payoff of choosing K' is no less then that of choosing K. We use $K <_G^F K'$ as an abbreviation of $K \leq_G^F K'$ but $K' \leq_G^F K$ does not hold. If $K <_G^F K'$, then we say K' strongly dominate K.

Proposition 7. Let F, G be groups of agents from a consequentialist frame, and let K, $K' \in Choice(G)$. Then $K <_G^F K'$ if and only if (1) $K \cap S \leq_F K' \cap S$ for each state $S \in Choice(A - G)$, and (2) $K \cap S <_F K' \cap S$ for some state $S \in Choice(A - G)$.

Proposition 8. Let F, G be groups of agents from a consequentialist frame, and let K, K', $K'' \in Choice(G)$. Then:

1. If $K \leq_G^F K'$ and $K' \leq_G^F K''$, then $K \leq_G^F K''$.
2. If $K \leq_G^F K'$ and $K' <_G^F K''$, then $K <_G^F K''$.
3. If $K <_G^F K'$ and $K' \leq_G^F K''$, then $K <_G^F K''$.
4. If $K <_G^F K'$ and $K' <_G^F K''$, then $K <_G^F K''$.

Proposition 8 states that the dominance relation is transitive. This transitivity is actually still true even if we replace our Definition 2 by Definition 2′. Because the definition of choice function ensures that for any $K \in Choice(G)$, for any $S \in Choice(A - G)$, $K \cap S \neq \emptyset$. However, the transitivity of conditional dominance (Definiton 10) do rely on Definition 2. The conditional dominance is defined on restricted choice sets.

Definition 9 (restricted choice sets). Let F, G be groups of agents from a consequentialist frame, X a set of worlds in the frame. Then

$$Choice(G/X) = \{K : K \in Choice(G) \text{ and } K \cap X \neq \emptyset\}$$

Intuitively, $Choice(G/X)$ is the collection of group G's choice which is consistent with condition X. We can further define conditional dominance relation over agent's choice. The intuition is, to compare whether the agent's choice K is dominated by K' under the condition X, we only need to concern other agents' choices which are consistent with the condition X and one of K and K'.

Definition 10 (conditional dominance; $\leq^F_{G/X}$). Let F, G be groups of agents from a consequentialist frame, X a set of worlds in the frame. Let K, $K' \in Choice(G/X)$. Then

$$K \leq^F_{G/X} K' \text{ iff for all } S \in Choice((A-G)/(X \cap (K \cup K'))),\ K \cap X \cap S \leq_F K' \cap X \cap S.$$

$K \leq^F_{G/X} K'$ can be read as "in the interest of group F, K' weakly dominates K under the condition of X". And we use $K <^F_{G/X} K'$ to express $K \leq^F_{G/X} K'$ and it is not that $K' \leq^F_{G/X} K$.

Proposition 11. Let F, G be groups of agents from a consequentialist frame, X a set of worlds in the frame. Let K, $K' \in Choice(G/X)$. Then $K <^F_{G/X} K'$ if and only if (1) $K \cap X \cap S \leq_F K' \cap X \cap S$ for each state $S \in Choice((A - G)/(X \cap (K \cup K')))$, and (2) $K \cap X \cap S <_F K' \cap X \cap S$ for some state $S \in Choice((A - G)/(X \cap (K \cup K')))$.

Proposition 12. Let F, G be groups of agents from a consequentialist frame, X a set of worlds in the frame. Let K, K', $K'' \in Choice(G/X)$. Then:

1. If $K \leq^F_{G/X} K'$ and $K' \leq^F_{G/X} K''$, then $K \leq^F_{G/X} K''$.
2. If $K \leq^F_{G/X} K'$ and $K' <^F_{G/X} K''$, then $K <^F_{G/X} K''$.
3. If $K <^F_{G/X} K'$ and $K' \leq^F_{G/X} K''$, then $K <^F_{G/X} K''$.
4. If $K <^F_{G/X} K'$ and $K' <^F_{G/X} K''$, then $K <^F_{G/X} K''$.

Proposition 12 corresponds to Proposition 5.4 in [5]. Notice that to make Proposition 12 true, the existential condition in Definition 2 is necessary. In [5] Definition 2 is replaced by Definition 2′, and $K \leq^F_{G/X} K'$ is defined as: for all $S \in Choice(A - G)$, $K \cap X \cap S \leq_F K' \cap X \cap S$. In that case we can construct the following counterexample to falsify the transitivity of $\leq^F_{G/X}$:

	S_1	S_2
K_1	w_1 (2)	w_2 (0) X
K_2	w_3 (3) X	w_4 (1)
K_3	w_5 (3)	w_6 (1) X

Fig. 3.

Here $W = \{w_1, ..., w_6\}$, $A = \{\alpha, \beta\}$ $Choice(\{\alpha\}) = \{K_1, K_2, K_3\}$, $Choice(\{\beta\}) = \{S_1, S_2\}$, $K_1 = \{w_1, w_2\}$, $K_2 = \{w_3, w_4\}$, $K_3 = \{w_5, w_6\}$, $S_1 = \{w_1, w_3, w_5\}$,

$S_2 = \{w_2, w_4, w_6\}$, $X = \{w_2, w_3, w_6\}$, and the number in the brackets represents the value of the world in the interest of group A. According to Horty's definition, we have $K_3 \leq^A_{\{\alpha\}/X} K_2$ and $K_2 \leq^A_{\{\alpha\}/X} K_1$ but we don't have $K_3 \leq^A_{\{\alpha\}/X} K_1$. Therefore in [5] the $\leq^F_{G/X}$ relation is not transitive and Proposition 5.4 is not totally true.

In [6], the definition of restricted choice set and conditional dominance is different from ours. See the following:

Definition 9′ Let F, G be groups of agents from a consequentialist frame, X a set of worlds in the frame. Then

$$Choice(G/X) = \{K \cap X : K \in Choice(G) \text{ and } K \cap X \neq \emptyset\}$$

Definition 10′ Let F, G be groups of agents from a consequentialist frame, X a set of worlds in the frame. Let k, $k' \in Choice(G/X)$. Then

$k \leq^F_{G/X} k'$ iff for all $S \in Choice(A - G)$ and for all w and $w' \in W$ it holds that if $w \in k \cap S$ and $w' \in k' \cap S$, then $Value_F(w) \leq Value_F(w')$

It's easy to verify that according to above definition, $Choice(G/X) = \{\{w_2\}, \{w_3\}, \{w_6\}\}$ and $\{w_6\} \leq^F_{G/X} \{w_3\}$, $\{w_3\} \leq^A_{\{\alpha\}/X} \{w_2\}$, but $\{w_6\} \leq^A_{\{\alpha\}/X} \{w_2\}$ does not hold. Hence this version of conditional dominance is not transitive.

2.3 Semantics

As in traditional modal logic, a model is a frame plus the valuation function.

Definition 13 (consequentialist model) A *consequentialist model* M is an ordered pair $\langle \mathfrak{F}, V \rangle$ where $\mathfrak{F}$ is a consequentialist frame and V a valuation function that assigns to each atomic proposition $p \in P$ a set of worlds $V(p) \subseteq W$.

In our semantics, we use the optimal choice and condition optimal choice to interpret our deontic operator. The definition of optimal (Definition 14) and conditional optimal (Definition 16) is rather simple.

Definition 14 ($Optimal^F_G$) Let F, G be groups of agents from a consequentialist frame,

$$Optimal^F_G = \{K \in Choice(G) : \text{there's no } K' \in Choice(G) \text{ such that } K <^F_G K'\}.$$

Proposition 15 Let F, G be groups of agents from a consequentialist frame, then for each $K \in Choice(G) - Optimal^F_G$, there exist $K' \in Optimal^F_G$ such that $K <^F_G K'$.

Definition 16. ($Optimal^F_{G/X}$) Let F, G be groups of agents from a consequentialist frame,

$$Optimal^F_{G/X} = \{K \in Choice(G/X) : \text{there's no } K' \in Choice(G/X) \text{ such that } K <^F_{G/X} K'\}.$$

Proposition 17. Let F, G be groups of agents from a consequentialist frame, for each $K \in Choice(G/X) - Optimal^F_{G/X}$, there exist $K' \in Optimal^F_{G/X}$ such that $K <^F_{G/X} K'$.

Here to ensure the truth of Proposition 17, the existential condition in Definition 2 is necessary. Otherwise we could have a counterexample as follows:

	S_1	S_2	S_3
K_1	w_1 (3) X	w_2 (0) X	w_3
K_2	w_4	w_5 (1) X	w_6 (4) X
K_3	w_7 (2) X	w_8	w_9 (5) X

Fig. 4.

In above case, $W = \{w_1, ..., w_9\}$, $A = \{\alpha, \beta\}$, $Choice(\{\alpha\}) = \{K_1, K_2, K_3\}$, $Choice(\{\beta\}) = \{S_1, S_2, S_3\}$, $K_1 = \{w_1, w_2, w_3\}$, $K_2 = \{w_4, w_5, w_6\}$, $K_3 = \{w_7, w_8, w_9\}$, $S_1 = \{w_1, w_4, w_7\}$, $S_2 = \{w_2, w_5, w_8\}$, $S_3 = \{w_3, w_6, w_9\}$ $X = \{w_1, w_2, w_5, w_6, w_7, w_9\}$, the numbers in the brackets represent the value of the world in the interest of A. If Definition 2 is replaced by Definition 2$'$, we would have $K_1 <^A_{\{\alpha\}/X} K_2$, $K_2 <^A_{\{\alpha\}/X} K_3$, and $K_3 <^A_{\{\alpha\}/X} K_1$. Hence $Optimal^A_{\{\alpha\}/X} = \emptyset$, contradict to Proposition 17.

Definition 18 (semantical rules). Let $M = \langle \mathfrak{F}, V \rangle$ be consequentialist model. Let $w \in W$ and let $\varphi, \psi \in \mathfrak{L}$. Then

(1) $M, w \models p$ iff $w \in \mathrm{V}(p)$;
(2) $M, w \models \neg\varphi$ iff it is not that $M, w \models \varphi$;
(3) $M, w \models \varphi \wedge \psi$ iff $M, w \models \varphi$ and $M, w \models \psi$;
(4) $M, w \models \Diamond\varphi$ iff there is a w' such that $M, w' \models \varphi$;
(5) $M, w \models [G]\varphi$ iff for all w' with $w \sim_G w'$ it holds that $M, w' \models \varphi$;
(6) $M, w \models \bigodot^F_G \varphi$ iff $K \subseteq ||\varphi||$ for each $K \in Optimal^F_G$;
(7) $M, w \models \bigodot^F_G(\varphi/\psi)$ iff $K \subseteq ||\varphi||$ for each $K \in Optimal^F_{G/\psi}$.

Here $\|\varphi\| = \{w \in W : M, w \models \varphi\}$. $Optimal^F_{G/\psi}$ is shorthand for $Optimal^F_{G/||\psi||}$.

We say φ is true in the world w of a consequentialist model M if $M, w \models \varphi$. Just like the standard modal logic in [4], we introduce the concept of validity as following: a formula φ is valid in a world w of a consequentialist frame $\mathfrak{F}$ (notation: $\mathfrak{F}, w \models \varphi$) if φ is true at w in every model $\langle \mathfrak{F}, V \rangle$ based on $\mathfrak{F}$; φ is valid in a consequentialist frame $\mathfrak{F}$ (notation: $\mathfrak{F} \models \varphi$) if it is valid at every world of $\mathfrak{F}$; φ is valid (notation: $\models \varphi$) if it is valid in the class of all consequentialist frames.

3 Revisit Principles of Conditional Ought

[1] suggests that the logic of commitment or conditional ought should satisfy six principles. In our language, they are as follows:

(1) $(\psi \wedge \odot^F_G(\varphi/\psi)) \rightarrow \odot^F_G \varphi$.

(2) $(\odot^F_G \psi \wedge \odot^F_G(\varphi/\psi)) \rightarrow \odot^F_G \varphi$.

(3) $(\mathbb{P}^F_G\psi \wedge \odot^F_G(\varphi/\psi)) \rightarrow \mathbb{P}^F_G\varphi$.

(4) $(\odot^F_G(\varphi/\psi) \wedge \odot^F_G(\chi/\varphi)) \rightarrow \odot^F_G(\chi/\psi)$

(5) $\odot^F_G(\varphi/\psi) \rightarrow \odot^F_G(\psi \rightarrow \varphi)$

(6) $\odot^F_G(\varphi/\neg\varphi) \rightarrow \odot^F_G \varphi$

The strongly normal system **G** in [2] excludes principle (1) and (4) and involves the others. In our logic, however, only principle (2) and (6) are valid. The invalidity of principle (1) and (4) and the validity of principle (6) are easy to prove, here we skip it. For the rest three principles, we have following propositions:

Proposition 19. The statement $(\odot^F_G \psi \wedge \odot^F_G(\varphi/\psi)) \rightarrow \odot^F_G \varphi$ is valid.

Proposition 20. The statement $(\mathbb{P}^F_G\psi \wedge \odot^F_G(\varphi/\psi)) \rightarrow \mathbb{P}^F_G\varphi$ is not valid.

Proposition 21. The statement $\odot^F_G(\varphi/\psi) \rightarrow \odot^F_G(\psi \rightarrow \varphi)$ is not valid.

The invalidity of principle (3) can be illustrated by a variation of the matching pennies game. In this new game, Lucy has three choices, showing head, showing tail and refraining from showing. If Lucy refrains from showing, then no matter how Lily acts, Lucy will receive 50 dollars. The detailed payoff of Lucy is indicated by numbers in brackets in Figure 5.

	Head	Tail
Head	w_1 (100)	w_2 (0) ψ
Tail	φ w_3 (20)	φ w_4 (30) ψ
Refrain	w_5 (50)	w_6 (50)

Fig. 5.

Denote the situation in which Lucy shows one side of her penny and Lily shows tail as situation ψ. Lucy is permitted to lead to situation ψ, since showing head is one of Lucy's optimal action and this action could lead to situation ψ. Apparently under situation ψ, Lucy ought to see to it that she shows tail. But Lucy is not permitted to show tail because this action is dominated by refraining.

4 Conclusion and Future Work

The main point of this paper is to introduce a new consequentianlist deontic logic in which the relation of preference over sets of worlds and the conditional dominance relation are both transitive. With transitivity a bunch of good properties could be proved. Our logic support the principle that absolute ought can be derived from conditional ought whenever the conditional statement is the agent's absolute ought. According to our semantics, conditional ought does not imply ought about conditionals.

One line of our future work is to create an axiomatic system for consequentianlist deontic logic and prove its soundness and completeness. Another line is to add epistemic modality to our system. For some related work in this area, see [10] and [8].

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Appendix

Proof of Lemma 3. Straightforward. ⊣

Proof of Proposition 4. Left to right. Assume $X <_F Y$, then we have $X \leq_F Y$ and it is not the case that $Y \leq_F X$. $X \leq_F Y$ plus Lemma 3 implies (1) and $X \neq \emptyset$, $Y \neq \emptyset$. By it is not the case that $Y \leq_F X$, we have either $X = \emptyset$ or $Y = \emptyset$ or for some $w' \in Y$, for some $w \in X$, $Value_F(w') > Value_F(w)$. But we already have $X \neq \emptyset$ and $Y \neq \emptyset$. Hence for some $w' \in Y$, for some $w \in X$ $Value_F(w') > Value_F(w)$, then (2) is true.

Right to left. By (2) we know $X \neq \emptyset$ and $Y \neq \emptyset$. This plus (1) implies $X \leq_F Y$. So it's sufficient to prove it is not the case that $Y \leq_F X$. Suppose $Y \leq_F X$, then for each $w' \in Y$ and for each $w \in X$, $Value_F(w') \leq Value_F(w)$. But according to (2), for some $w' \in Y$ and for some $w \in X$, $Value_F(w') > Value_F(w)$. Contradiction. ⊣

Proof of Proposition 5. Here we just prove Clause 1. Assume $X \leq_F Y$ and $Y \leq_F Z$, then $X \neq \emptyset$, $Y \neq \emptyset$ and $Z \neq \emptyset$. Let w be arbitrary history in X, w'' be arbitrary history in Z. By $Y \neq \emptyset$ we know there exist some $w' \in Y$. By $X \leq_F Y$ and $Y \leq_F Z$ we know $Value_F(w) \leq Value_F(w')$, $Value_F(w') \leq Value_F(w'')$, hence $Value_F(w) \leq Value_F(w'')$. Therefore $X \leq_F Z$. ⊣

Proof of Proposition 7. Left to right. Assuming $K <^F_G K'$, then $K \leq^F_G K'$ and it is not the case that $K' \leq^F_G K$. (1) directly follows from $K \leq^F_G K'$. By it is not the case that $K' \leq^F_G K$, we have for some state $S \in Choice(A - G)$, it is not the case that $K' \cap S \leq_F K \cap S$. But according to (1) we already have $K \cap S \leq_F K' \cap S$. Hence $K \cap S <_F K' \cap S$. Which means (2) is true.

Right to left. By (1) we know $K \leq^F_G K'$. So we only need to prove it is not that $K' \leq^F_G K$. That is, for some state $S \in Choice(A - G)$, it is not that $K' \cap S \leq_F K \cap S$. This is implied by (2), because (2) means for some $S \in Choice(A-G)$, $K \cap S \leq_F K' \cap S$ and it is not the case that $K' \cap S \leq_F K \cap S$. ⊣

Proof of Proposition 8. See Proposition 4.7 of [5]. ⊣

Proof of Proposition 11. Similar to the proof of Proposition 7. ⊣

To prove Proposition 12, we need following Lemmas:
Lemma A. Let F, G be groups of agents from a consequentialist frame, X a set of worlds in the frame. Let $K, K' \in Choice(G/X)$. If $K \leq^F_{G/X} K'$, then $Choice((A - G)/(X \cap K))$=$Choice((A - G)/(X \cap K'))$.

Proof of Lemma A. We are going to prove $Choice((A - G)/(X \cap K)) \subseteq Choice((A-G)/(X \cap K'))$ and $Choice((A-G)/(X \cap K)) \supseteq Choice((A-G)/(X \cap K'))$.

For $Choice((A - G)/(X \cap K)) \subseteq Choice((A - G)/(X \cap K'))$, assume there exist some $S \in Choice(A - G)$ such that $S \in Choice((A - G)/(X \cap K))$ but

$S \notin Choice((A-G)/(X \cap K'))$. Then $S \cap X \cap K \neq \emptyset$ and $S \cap X \cap K' = \emptyset$. Therefore $S \cap X \cap (K \cup K') \neq \emptyset$ and $S \in Choice((A-G)/(X \cap (K \cup K')))$. Now by $K \leq^F_{G/X} K'$ we have $K \cap X \cap S \leq_F K' \cap X \cap S$. This plus Lemma 3 implies $K' \cap X \cap S \neq \emptyset$. Contradiction. Hence $Choice((A-G)/(X \cap K)) \subseteq Choice((A-G)/(X \cap K'))$.

The case for $Choice((A-G)/(X \cap K)) \supseteq Choice((A-G)/(X \cap K'))$ is similar. $\dashv$

Lemma B. Let G be a group of agents, X and Y be sets of worlds from a consequentialist frame. If $Choice((A-G)/X) = Choice((A-G)/Y)$, then $Choice((A-G)/X) = Choice((A-G)/(X \cup Y))$.

Proof of Lemma B. For $Choice((A-G)/X) \subseteq Choice((A-G)/(X \cup Y))$. If $S \in Choice((A-G)/X)$, then $S \in Choice((A-G)/Y)$ and $S \cap X \neq \emptyset$, $S \cap Y \neq \emptyset$. Hence $(S \cap X) \cup (S \cap Y) \neq \emptyset$, $S \cap (X \cup Y) \neq \emptyset$. So we have $S \in Choice((A-G)/(X \cup Y))$.

For $Choice((A-G)/X) \supseteq Choice((A-G)/(X \cup Y))$. If $S \in Choice((A-G)/(X \cup Y))$, then $S \cap (X \cup Y) \neq \emptyset$, $(S \cap X) \cup (S \cap Y) \neq \emptyset$. Now assume $S \notin Choice((A-G)/X)$, then $S \notin Choice((A-G)/Y)$. Hence $S \cap X = \emptyset$ and $S \cap Y = \emptyset$. So we have $(S \cap X) \cup (S \cap Y) = \emptyset$. Contradiction. $\dashv$

Proof of Proposition 12. Here we just prove clause 1. Other clauses are similar.

Assume $K \leq^F_{G/X} K'$ and $K' \leq^F_{G/X} K''$. By Lemma A,

$Choice((A-G)/(X \cap K)) = Choice((A-G)/(X \cap K')) = Choice((A-G)/(X \cap K''))$.

By Lemma B, we now have

$Choice((A-G)/(X \cap K)) = Choice((A-G)/((X \cap K) \cup (X \cap K'))) = Choice((A-G)/(X \cap (K \cup K')))$,

$Choice((A-G)/(X \cap K)) = Choice((A-G)/((X \cap K) \cup (X \cap K''))) = Choice((A-G)/(X \cap (K \cup K'')))$,

$Choice((A-G)/(X \cap K')) = Choice((A-G)/((X \cap K') \cup (X \cap K''))) = Choice((A-G)/(X \cap (K' \cup K'')))$.

Hence for each $S \in Choice((A-G)/(X \cap (K \cup K'')))$, we have $S \in Choice((A-G)/(X \cap (K \cup K')))$ and $S \in Choice((A-G)/(X \cap (K' \cup K'')))$. Therefore by $K \leq^F_{G/X} K'$ we have $K \cap X \cap S \leq_F K' \cap X \cap S$, by $K' \leq^F_{G/X} K''$ we have $K' \cap X \cap S \leq_F K'' \cap X \cap S$. Since the relation $\leq_F$ is transitive, we have $K \cap X \cap S \leq_F K'' \cap X \cap S$. Therefore $K \leq^F_{G/X} K''$. $\dashv$

Proof of Proposition 15. Similar to Proposition 4.11 of [5]. $\dashv$

Proof of Proposition 17. Similar to Proposition 5.7 of [5]. $\dashv$

Proof of Proposition 19. Assume this formula is not valid, then there is a consequentialist frame $\mathfrak{F}$ and a world w in $\mathfrak{F}$ such that for some model M

based on $\mathfrak{F}$, $M, w \models \bigodot_G^F \psi \wedge \bigodot_G^F(\varphi/\psi)$ but $M, w \nvDash \bigodot_G^F \varphi$. Hence there must be some $K \in Optimal_G^F$ such that $K \nsubseteq ||\varphi||$, $K \subseteq ||\psi||$.

As $K \subseteq ||\psi||$, for arbitrary $S \in Choice(A-G)$, for arbitrary $K' \in Choice(G)$, we must have $S \cap (||\psi|| \cap (K \cup K')) \neq \emptyset$ because $S \cap K \subseteq S \cap (||\psi|| \cap (K \cup K'))$ and $S \cap K \neq \emptyset$ by the definition of choice function. Hence $S \in Choice((A-G)/(||\psi|| \cap (K \cup K')))$ and $Choice(A-G) = Choice((A-G)/(||\psi|| \cap (K \cup K')))$.

Obviously either $K \in Optimal_{G/\psi}^F$ or $K \notin Optimal_{G/\psi}^F$. If $K \in Optimal_{G/\psi}^F$, then by $M, w \models \bigodot_G^F(\varphi/\psi)$, $K \subseteq ||\varphi||$. Contradiction. Therefore $K \notin Optimal_{G/\psi}^F$. Then by Proposition 17 there exist some $K' \in Choice(G/\psi)$ with $K <_{G/\psi}^F K'$. It must be either $K' \in Optimal_G^F$ or $K' \notin Optimal_G^F$. We can show both these two cases imply contradictions.

For the first case, assume $K' \in Optimal_G^F$. Then by $M, w \models \bigodot_G^F \psi$, $K' \subseteq ||\psi||$. According to $K <_{G/\psi}^F K'$, for each $S \in Choice((A-G)/(||\psi|| \cap (K \cup K')))$, $K \cap S \cap ||\psi|| \leq_F K' \cap S \cap ||\psi||$. Since $K \subseteq ||\psi||$ and $K' \subseteq ||\psi||$, we know $K \cap S \cap ||\psi|| = K \cap S$, $K' \cap S \cap ||\psi|| = K' \cap S$. Therefore $K \cap S \leq_F K' \cap S$. Note we have already proved $Choice(A-G) = Choice((A-G)/(||\psi|| \cap (K \cup K')))$, hence for each $S \in Choice(A-G)$, $K \cap S \leq_F K' \cap S$. That is, $K \leq_G^F K'$. By proposition 11, $K <_{G/\psi}^F K'$ also implies that for some $S \in Choice((A-G)/(||\psi|| \cap (K \cup K'))) = Choice(A-G)$, $K \cap S \cap ||\psi|| <_F K' \cap S \cap ||\psi||$. Use $K \cap S \cap ||\psi|| = K \cap S$, $K' \cap S \cap ||\psi|| = K' \cap S$ one more time we have the conclusion that for some $S \in Choice(A-G)$, $K \cap S <_F K' \cap S$. Now by Proposition 7 we know $K <_G^F K'$, contradict to $K \in Optimal_G^F$.

For the second case, assume $K' \notin Optimal_G^F$. Then there exist some $K'' \in Choice(G)$ with $K' <_G^F K''$. So we have for each $S \in Choice(A-G) = Choice((A-G)/(||\psi|| \cap (K \cup K'')))$, $K' \cap S \leq_F K'' \cap S$. By $K <_{G/\psi}^F K'$, $K \cap S \cap ||\psi|| \leq_F K' \cap S \cap ||\psi||$. It then follows from Lemma 3 that $K' \cap S \cap ||\psi|| \neq \emptyset$. This plus $K' \cap S \leq_F K'' \cap S$ implies $K' \cap S \cap ||\psi|| \leq_F K'' \cap S$. Note that $K \cap S = K \cap S \cap ||\psi||$ since $K \subseteq ||\psi||$, therefore $K \cap S \leq_F K' \cap S \cap ||\psi||$. Now by Proposition 5 we have $K \cap S \leq_F K'' \cap S$. Hence $K \leq_G^F K''$. By Proposition 11, $K <_{G/\psi}^F K'$ also implies for some $S \in Choice((A-G)/(||\psi|| \cap (K \cup K'))) = Choice(A-G)$, $K \cap S \cap ||\psi|| <_F K' \cap S \cap ||\psi||$. Use $K \cap S = K \cap S \cap ||\psi||$ again we have $K \cap S <_F K' \cap S \cap ||\psi||$. Since $K' \cap S \leq_F K'' \cap S$ and $K' \cap S \cap ||\psi|| \neq \emptyset$, we have $K' \cap S \cap ||\psi|| \leq_F K'' \cap S$. By Proposition 5 we now have $K \cap S <_F K'' \cap S$. It then follow from proposition 7 that $K <_G^F K''$, contradict to $K \in Optimal_G^F$. $\dashv$

Proof of Proposition 20. It's sufficient to construct a model M such that for some world w in M, $M, w \models \mathbb{P}_G^F \psi \wedge \bigodot_G^F(\varphi/\psi)$ but $M, w \nvDash \mathbb{P}_G^F \varphi$.

As illustrated by Figure 5. Let $M = \langle W, A, Choice, \{Value_F\}_{F \subseteq A}, V \rangle$, $W = \{w_1, ..., w_6\}$, $A = \{\alpha, \beta\}$, $Choice(\{\alpha\}) = \{\{w_1, w_2\}, \{w_3, w_4\}, \{w_5, w_6\}\}$, $Choice(\{\beta\}) = \{\{w_1, w_3, w_5\}, \{w_2, w_4, w_6\}\}$, $Value_{\{\alpha\}}(w_1) = 100$, $Value_{\{\alpha\}}(w_2) = 0$, $Value_{\{\alpha\}}(w_3) = 20$, $Value_{\{\alpha\}}(w_4) = 30$, $Value_{\{\alpha\}}(w_5) = 50$, $Value_{\{\alpha\}}(w_6) = 50$. Let $F = \{\alpha\}$, $G = \{\alpha\}$, $||\varphi|| = \{w_3, w_4\}$, $||\psi|| = \{w_2, w_4\}$. Then $Optimal_G^F = \{\{w_1, w_2\}, \{w_5, w_6\}\}$, $Optimal_{G/\psi}^F = \{\{w_3, w_4\}\}$. Note that $M, w \models \mathbb{P}_G^F \varphi$ if and only if for some $K \in Optimal_G^F$, $K \cap ||\varphi|| \neq \emptyset$. Hence by the semantics we have $M, w_1 \models \mathbb{P}^F_G\psi$ and $M, w_1 \models \bigodot^F_G(\varphi/\psi)$. But as $\{w_1, w_2\} \cap ||\varphi|| = \emptyset$ and $\{w_3, w_4\} \cap ||\varphi|| = \emptyset$, we have $M, w_1 \not\models \mathbb{P}^F_G\varphi$. $\dashv$

Proof of Proposition 21. It's sufficient to construct a model M such that for some world w in M, $M, w \models \bigodot^F_G(\varphi/\psi)$ but $M, w \not\models \bigodot^F_G(\psi \to \varphi)$.

Revisit Figure 1. Let $M = \langle W, A, Choice, \{Value_F\}_{F\subseteq A}, V\rangle$, $W = \{w_1, ..., w_4\}$, $A = \{\alpha, \beta\}$, $Choice\ (\{\alpha\}) = \{\{w_1, w_2\}, \{w_3, w_4\}\}$, $Choice(\{\beta\}) = \{\{w_1, w_3\}, \{w_2, w_4\}\}$, $Value_{\{\alpha\}}(w_1) = 1$, $Value_{\{\alpha\}}\ (w_2) = 0$, $Value_{\{\alpha\}}(w_3) = 0$, $Value_{\{\alpha\}}(w_4) = 1$. Let $F = \{\alpha\}$, $G = \{\alpha\}$, $||\varphi|| = \{w_1, w_2\}$, $||\psi|| = \{w_1, w_3\}$. In this situation, $Optimal^F_{G/\psi} = \{\{w_1, w_2\}\}$, hence $M, w_1 \models \bigodot^F_G(\varphi/\psi)$. As $Optimal^F_G = \{\{w_1, w_2\}, \{w_3, w_4\}\}$ and $\{w_3, w_4\} \not\subseteq ||\psi \to \varphi||$, we have $M, w_1 \not\models \bigodot^F_G(\psi \to \varphi)$.$\dashv$