Great Expectations: A Surreal Decision Theory

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December 22, 2015
Logic Group, Peking University

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Overview

1. Standard Expected-Utility (EU) Theory

A gamble G:

Let X be a set of possible prizes / consequences / states:

$$\{x_1, x_2, x_3, ... x_n\}$$

Let U be a utility function over X and its values are:

$$\{u(x_1), u(x_2), u(x_3), ... u(x_n)\}$$

Let Cr be a credence distribution over X:

$$\{cr_1, cr_2, cr_3, ... cr_n\}$$

The expected utility of the gamble:

$$EU(G) = \sum_{i=1}^{n} cr_i u(x_i)$$

Example: Lottery_1 and Lottery_2

$$X_1 = \{\text{Winning ticket, Losing ticket}\}$$
$$U_1 = \{100, -10\}$$
$$Cr_1 = \{.1, .9\}$$
$$EU(Lottery_1) = \sum_{i=1}^{2} cr_i u(x_i) = 0.1 \times 100 + 0.9 \times (-10) = 1$$
$$X_2 = \{\text{Winning ticket, Losing ticket}\}$$
$$U_2 = \{100, -15\}$$
$$Cr_2 = \{.2, .8\}$$
$$EU(Lottery_2) = \sum_{i=1}^{2} cr_i u(x_i) = 0.2 \times 100 + 0.8 \times (-15) = 8$$

EU theory: the agent should prefer Lottery_2 over Lottery_1, because the agent should maximize expected utility, and

$$EU(Lottery_2) = 8 > 1 = EU(Lottery_1)$$

2. Problems with Infinities

Standard EU Theory:

- (with reasonable modifications) successful in the finite cases.
- (perhaps hopelessly) problematic in the infinite cases.

$$EU(G) = \sum_{i=1}^{n} cr_i u(x_i) \Longrightarrow EU(G) = \sum_{i=1}^{\infty} cr_i u(x_i)$$

Example 1: Failure of Dominance Reasoning

Infinity or Nothing: you are offered a coin flip that yields infinite utility if heads, and nothing if it lands tails. The gamble is thus $G_1 = \{.5, \infty; .5, 0\}$.

Infinity or Something: you are offered a coin flip that yields infinite utility if heads, and utility 10,000 if it lands tails. The gamble is thus $G_2 = \{.5, \infty; .5, 10,000\}$.

Infinity or Bust: you are offered a coin flip that yields infinite utility if heads, and -10,000 utility if it lands tails. The gamble is thus $G_3 = \{.5, \infty; .5, -10,000\}$.

Example 1: Failure of Dominance Reasoning

- Dominance reasoning: G_2 weakly dominates both G_1 and G_3, while G_1 weakly dominates G_3.
- Rational preferences (by Dominance): $G_2 \succ G_1 \succ G_3$.
- Standard EU Theory: $EU(G_1) = EU(G_2) = EU(G_3) = \infty$.
- Rational preferences (by EU Theory): $G_2 \sim G_1 \sim G_3$.

Example 2: Non-Well-Defined Expected Utility

Fair Infinity: you are offered a coin flip that yields infinite utility if heads, and infinite disutility if it lands tails. The gamble is thus G_4 $= \{.5, \infty; .5, -\infty\}$.

Biased Positive Infinity: you are offered a coin flip that yields infinite utility if heads, and infinite disutility if it lands tails. the coin is biased 9:1 in favor of heads. The gamble is thus $G_5 = \{.9, \infty; .1, -\infty\}$.

Biased Negative Infinity: you are offered a coin flip that yields infinite utility if heads, and infinite disutility if it lands tails. The coin is biased 9:1 against heads. The gamble is thus $G_6 = \{.1, \infty; .9, -\infty\}$.

Example 2: Non-Well-Defined Expected Utility

- Rational preferences: $G_5 \succ G_4 \succ G_6$.

- Standard EU Theory: $\infty - \infty$ is not well-defined.
- Rational preferences (by EU Theory): *silent.*

Diagnosis

∞ is (infinitely) vague.

Cantor's Paradise?

Assign cardinal numbers as utilities and uses cardinal arithmetic to calculate the expected utilities. We would get:

$$
\begin{aligned}
G_1 &= .5\aleph_0 \\
G_2 &= .5\aleph_0 + 5,000 \\
G_3 &= .5\aleph_0 - 5,000 \\
G_4 &= .5\aleph_0 - .5\aleph_0 \\
G_5 &= .9\aleph_0 - .1\aleph_0 \\
G_6 &= .1\aleph_0 - .9\aleph_0
\end{aligned}
$$

Problems with Cardinal Arithmetic

1. Cardinal arithmetic also has the absorption property. Assuming the Axiom of Choice, if either κ or μ is infinite, then $\kappa + \mu = max\{\kappa, \mu\}$, and $\kappa \times \mu = max\{\kappa, \mu\}$.
2. $\aleph_0 - \aleph_0$ is still not well-defined.

Ordinal Arithmetic?

1. Ordinal arithmetic lacks the absorption property, but is non-commutative.
2. $\omega - \omega$ is still not well-defined.

Searching for a Solution

Summarizing our previous observations about the problems with infinities, we need a solution which includes:

1. an ordered-field including all reals and ordinals;
2. addition in that field that is commutative, non-absorptive, and such that each element has an additive inverse;
3. multiplication in that field that is commutative, non-absorptive, and such that each non-zero element has a multiplicative inverse.

In short: we require a number system and accompanying operations that allow us to treat finite and transfinite numbers in similar and familiar ways.

A Surreal Solution

- John Conway discovered (or invented, depending on your philosophy of mathematics) such a field, and began its exploration in his *On Games and Numbers* (1976). [Norman L. Alling (1962) discovered a similar number field.]
- Conway called the objects he discovered *surreal* numbers.
- Construction: similiar to Dedekind's construction of the reals out of the rationals, Conway's construction uses the ordinals. We can think of surreal numbers as being something analogous to performing "Dedekind cuts" on ordinals.

A Surreal Solution

Surreal Numbers:

1. an ordered-field including all reals and ordinals (in the sense that their ordered fields can be realizes as subfields of the surreals);
2. addition in that field that is commutative, non-absorptive, and such that each element has an additive inverse;
3. multiplication in that field that is commutative, non-absorptive, and such that each non-zero element has a multiplicative inverse.

Surreals: Definitions and Constructions

DEFINITION 1: If L and R are sets of numbers, and no $x \in L \geq$ any $y \in R$, then $\{L|R\}$ is a number.

DEFINITION 2: $x \geq y$ iff no $x^R \leq y$ and no $y^L \geq x$

Surreals: Definitions and Constructions

DEFINITION 1 looks circular. Fortunately, the null set is trivially a set of numbers, and so our first surreal number is $\{\varnothing|\varnothing\} = 0$.

From 0, we gain two new numbers: $\{0|\varnothing\} = 1$ and $\{\varnothing|0\} = -1$.

From these numbers, we can find yet more numbers. We use **No** to denote the class of numbers created by repeated application of definition 1, and the iteration of definition 1 on which n is found its 'birthday.'

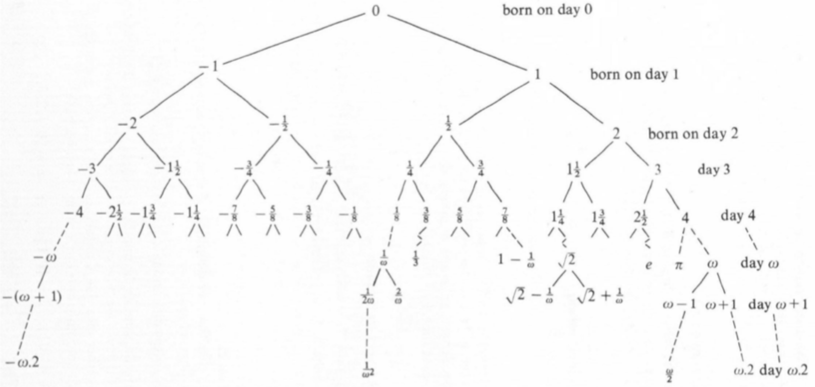


Figure : The Surreal Tree

DEFINITION 3: $x + y = \{x^L + y, x + y^L | x^R + y, y^R + x\}$

DEFINITION 4: $-x = \{-x^L | -x^R\}$

DEFINITION 5:
$x \times y = \{x^L \times y + y^L \times x - x^L \times y^L, x^R \times y + y^R \times x - x^R \times y^R | x^L \times y + y^R \times x - x^L \times y^R, x^R \times y + y^L \times x - x^R \times y^L\}$

These definitions make **No** an ordered field including all reals and all ordinals (in fact, Conway proves that it is a universally embedding field). We refer the interested reader to Conway for the proofs and further details. (Conway (1974), pp.15-44)

Question: Can we use surreal numbers in decision theory?

Answer: Yes, we shall state a surreal von Neumann-Morgenstern Representation Theorem.

THEOREM 3 (SURREAL VON NEUMANN-MORGENSTERN THEOREM): Let X be a space of lotteries, and let $\preceq$ be a binary relation $\subseteq X \times X$. There exists an affine function $U : X \to \mathbf{No}$ such that $\forall x, y \in X$
$U(x) \leq U(y) \Leftrightarrow x \preceq y$
if and only if $\preceq$ satisfies all of the following:

1. Completeness: $\forall x, y \in X$, either $x \preceq y$ or $y \preceq x$.
2. Transitivity: $\forall x, y, z \in X$, if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.
3. Continuity$\star$: $\forall x, y, z \in X$, if $x \prec y \prec z$, then there exist surreals $p, q \in \star(0, 1)$ such that $px + (1 - p)z \prec y \prec qx + (1 - q)z$.
4. Independence$\star$: $\forall x, y, z \in X, \forall p \in \star(0,\ 1]$,x $\preceq y$ if and only if $px + (1 - p)z \preceq py + (1 - p)z$.

Proof: See bonus slides if interested.

A Simple Application

With Surreal arithmetic thus defined, we now return to the games that seem problematic for the standard EU theory. Given the representation theorem, we can use surreal numbers to represent the values of the various gambles.

Recall the calculations we needed to make (with ∞ precisified as the ordinal ω):

$$
\begin{aligned}
G_1 &= .5\omega \\
G_2 &= .5\omega + 5,000 \\
G_3 &= .5\omega - 5,000 \\
G_4 &= .5\omega - .5\omega \\
G_5 &= .9\omega - .1\omega \\
G_6 &= .1\omega - .9\omega
\end{aligned}
$$

A Simple Application

With surreal arithmetic operations, we get the intuitive results.
$G_2 > G_1 > G_3$, and $G_5 > G_4 > G_5$.
To illustrate how the calculations can be carried out, we will show (with the help of some theorems in surreal analysis) that $G_2 > G_1$.
We can use the definitions to show:

$$.5\omega = .5\omega$$

$$0 < 5,000$$

$$.5\omega + 0 = .5\omega$$

Moreover, it is a theorem that $x < x' \wedge y < y' \Rightarrow x + y < x' + y'$.
Therefore, $.5\omega < .5\omega + 5,000$.
Note: although it plays no part here, that $\omega - \omega$ is defined, and is 0 (because -ω is the additive inverse of ω).

Summary

Surreal numbers are well-suited for modeling decision problems with infinite utilities.

Infinite State Spaces—Different Problems

Recall Standard EU Theory extending from finite cases to cases involving infinities:

$$EU(G) = \sum_{i=1}^{n} cr_i u(x_i) \Longrightarrow EU(G) = \sum_{i=1}^{\infty} cr_i u(x_i)$$

Two places where infinities can occur:

1. $u(x_i) = \infty$ for some i is infinite
2. $n = \infty$, i.e. infinite number of terms in the summation

Infinite State Spaces—Different Problems

$$EU(G) = \sum_{i=1}^{\infty} cr_i u(x_i)$$

To allow infinite sums in the calculation of EU is to consider infinite state spaces.

Two problems associated with infinite sums:

1. Dominance
2. Order Dependence

Review: General Problems with Infinities in Decision Theory

1. Failure of Dominance
   - St Petersburgh: $1 + 1 + 1 + \ldots\ldots$
     St Petersburgh$^+$: $2 + 2 + 2 + \ldots\ldots$
2. Paralysis from Order Dependence
   - Pasadena: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \ldots\ldots$
   - A game with an infinite state space does not come with privileged orderings, unless the mathematical machinery privileges some orderings.

Review: Infinite Series

- Sequence: $\{a_1, a_2, ...a_n\}$
- Series: $a_1 + a_2 + ... + a_n$
- Infinite sequence: $\{a_n\}_{n \in \mathbb{N}}$
- Infinite series: $\sum_{i=1}^{\infty} a_i$

In standard mathematical analysis, the sum of an infinite sequence is calculated as the limit of the associated sequence of partial sums.

$$L = \sum_{i=1}^{\infty} a_i \Leftrightarrow L = \lim_{k \to \infty} S_k$$

where

$$S_k = \sum_{i=1}^{k} a_k$$

Modes of convergence for the real-valued infinite sum $\sum_{i=1}^{\infty} a_i$:

- It is convergent $\Leftrightarrow$ it has a limit in the real numbers.
- It is divergent $\Leftrightarrow$ it doesn't have a limit in the real numbers.
- It is absolutely convergent $\Leftrightarrow$ both it and its companion series $\sum_{i=1}^{\infty} |a_i|$ are convergent.
- It is conditionally convergent $\Leftrightarrow$ it is convergent but its companion series is divergent.

- Absolutely convergent series, such as the series $\sum_{n=1}^{\infty}(-1)^n 2/n^2$, behave nicely. Infinite state space games with absolutely convergent payoff series work perfectly in standard EU theory.
- Divergent series, such as the series $1+1+1...$, introduce infinite expected utilities, even though no single state is assigned infinite value. These games seem counterintuitive, but they have well-defined values. The St. Petersburg game is the classic case.
- However, the sweetened St Petersburgh game has the same expected utility with the original St Petersburgh game: $2+2+2....$ EU theory says we should be indifferent between the two games.

Order Dependence ⇒ Paralysis

A much more challenging problem lies in the conditionally convergent series.

The classic case is Hájek and Nover's Pasadena game, which, like the St. Petersburg game, is played by flipping a coin until a coin lands heads, paying on the nth flip $\$(-1)^{n-1}2^n/n$.

Heads on *n*th Flip	1	2	3	4	...
Probability	1/2	1/4	1/8	1/16	...
Payoff	\$2	\$-2	\$8/3	\$-4	...
Expected Payoff	\$1	\$-1/2	\$1/3	\$-1/4	...

Table : Pasadena Game

Assuming utility linear in dollars, the expected payoffs of this game match the alternating harmonic series:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... = \ln(2)$$

However, its companion series is divergent:

$$\sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$$

$$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}..$$

$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ... = \infty$$

Riemann Rearrangement Theorem

THEOREM: If an infinite series is conditionally convergent, then its terms can be rearranged so that the new series converges to any given value or diverges to positive or negative infinities.

In short: Infinite summation is highly order-dependent.

Riemann Rearrangement Theorem

1. Infinite summation is highly order-dependent.
2. However we add up the expected payoffs, for every real number (or positive/negative infinity) there is a rearrangement of the series which sums to it.
3. But many infinite games do not come with privileged orderings.
4. The Riemann Rearrangement Theorem, therefore, leaves us with no way to consistently assign an expected value to many games such as the Pasadena Game.

Riemann Rearrangement Theorem

Example: The Pasadena Game Payoff Series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... = \ln(2)$$

A rearrangement that gives us ∞:

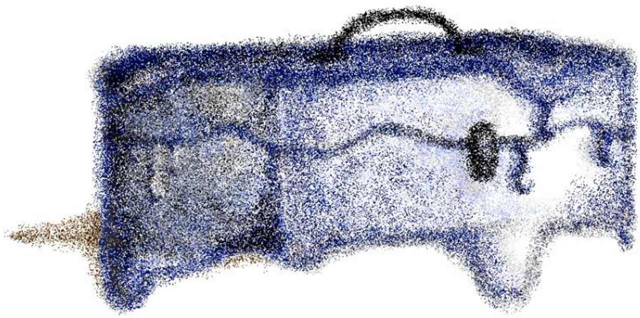
$$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{6} + ...$$

$$> \frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{6} + \frac{1}{4} - \frac{1}{6} + \frac{1}{4} - \frac{1}{6}.... = \infty$$

Riemann Rearrangement Theorem

Shall we explain away the Pasadena Game as somehow flawed, incoherent, or ill-stated?

No.



The Surreal Toolbox

Decision theory with infinite state spaces has 2 major problems: (A) Failure of Dominance and (B) Paralysis. We shall once again look at the "surreal toolbox" and (briefly) explore 4 ways to address these problems:

1. Order Restriction.
2. Selective Representation. [Solves (B)]
3. Surrealization of RET. [Solves (A)]
4. Infinite Sums of Series of Exact Lengths. [Solves (A) and (B)]

1. Order Restriction

The general strategy: check whether the mathematical machinery imposes independently-motivated constraints on the order of summation and see if those constraints rule out the problematic cases.

For example, to avoid absurdities such as an infinite sequence of negative numbers summing to a positive number, Conway's proposal of infinite sum imposes additional constraints on the order of terms.

We can think of stronger constraints on the order of terms for infinite sums.

- A definition of infinite sum **order-restricting** if it bans certain ordering of terms as inadmissible. For example, an infinite sum that requires the absolute values of the terms be non-increasing is order-restricting. So: $(1-\omega)+(\omega-\omega^2)+(\omega^2-\omega^3)$ is inadmissible. No rearrangement of the Pasadena payoff series is admissible.
- A definition of infinite sum is **extremely order-restricting** if it allows at most one admissible ordering for every sequence. For example, an infinite sum that requires the terms be arranged in a non-increasing order is extremely order-restricting.

The latter kind of sum can trivially avoid paralysis: the value of the game is given by infinite sum in the privileged ordering and no others. The former kind of sum can avoid at least some paralysis.

2. Selective Representation

The general strategy: provide reasons for privileging certain representations over the others in a mathematical equivalence class.

We have assumed that utility is linear in dollars. But this *ipso facto* does not imply that a state paying \$2 is worth 2 utils.

Utility functions are only unique up to positive affine transformation:

$u(x)$ represents an agent's preferences $\Leftrightarrow$
some $au(x) + b$ represents an agent's preferences.

So there is a large class of mathematically equivalent representations for one's rational preferences.

Meacham and Weisberg (2011) argue that even if a preference structure satisfies the axioms of EU theory, there's always an available representation that uses a non-probabilistic credence function. Even the most respectable preference structures can be represented by functions that we take to be problematic or somehow suboptimal.

They think that this result undermines any normativity representation theorems might claim.

We disagree.

Rather, it shows that there are better and worse ways to represent a preference structure quantitatively, even inside a mathematical equivalence class.

Mathematical equivalence does not preserve everything that we care about.

We have good reasons to break the symmetry in the mathematical equivalence.
E.g. from philosophy of physics.

1. A N-particle Bohmian system in $\mathbb{R}^3 \sim$ a one-particle Bohmian system in $\mathbb{R}^{3N}$
2. A wave function in the position representation $\sim$ A wave function in the momentum representation

- Modeling scientific phenomena by mathematical models, therefore, seems to rely on considerations more fine-grained than mathematical equivalence.
- Since here we are providing a model of rational agents, we can play favorites among mathematically equivalent models, on the basis of pragmatic and epistemological concerns.
- Here, we a strategy for solving the Pasadena Game and indeed the whole class of games involving conditional convergence. These games show another way in which a representation can be deficient: it can allow for gambles with conditionally convergent expected payoffs series.

- CC-VULNERABILITY: Let u be a utility function and p the probability function which together with u represents some agent's preferences. We will say that u is cc-vulnerable with respect to p iff there is some $X \subseteq \{x_i : x$ is the product of p_i and $u_i \}$ such that the members of X can be all and only the members of a conditionally convergent series.
- CC-INVULNERABILITY: Let u be a utility function and p be a probability function. We will say that u is cc-invulnerable with respect to p iff u is not cc-vulnerable with respect to p.
- CC-INVULNERABLE TRANSFORMATION: Let p be a probability function, u be a utility function and u^+ be a positive affine transformation of u. We say that u^+ is a cc-invulerable transformation of u with respect to p iff u is cc-vulnerable with respect to p and u^+ is cc-invulnerable with respect to p.

By Corollary 6.1, every utility function with codomain **No** (which is to say, surreal-valued) has a cc-invulnerable transformation (in fact, infinitely many).
Proof: Bonus.

We note that this is not true of utility functions with codomain $\mathbb{R}$, and thus surreal utilities are vital to our proposed resolution.

Any representable preference structure can be adequately represented with a cc-invulnerable utility function. So we contend that cc-vulnerability in the utility function is inessential to adequately representing a preference structure.

CC-vulnerable utility functions give rise to problematic gambles like the Pasadena Game, we contend that cc-vulnerability is a representational defect.

We propose that cc-vulnerable representations be rejected in favor of their cc-invulnerable transformations.

This proposal does not allow the problematic payoff series to arise; but because we have a representation theorem, it ensures that any preference structure satisfying the VNM axioms—and, *a fortiori*, any VNM preference structure which can be represented by a utility function that is linear in dollars—can still be represented.

The Question of Dominance

Dominance principle(s): perhaps our most firmly held principle of decision-making in an information-poor situation (covering both risk and uncertainty; Newcombe's Problem notwithstanding).

The Question of Dominance

WEAK DOMINANCE: Act 1 *weakly dominates* Act 2 iff Act 1 and 2 contain the same states, every state in Act 1 pays at least as well as it does in Act 2, and one state pays more in Act 1 than it does in Act 2.

STRICT DOMINANCE: Act 1 *strictly dominates* Act 2 iff Act 1 and 2 contain the same states, and every state in Act 1 pays better than it does in Act 2.

It is known that in the finite case, standard EU maximization respects dominance.(See Easwaran [Forthcoming].) But it is also known that in the infinite case, standard EU maximization need not respect dominance.

The Question of Dominance

Are WEAK DOMINANCE and STRICT DOMINANCE true in the surreal decision theory?

The Question of Dominance

CONJECTURE 4 (Surreal Strict Dominance Theorem): Let $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ be two surreal-valued expected payoff series. If $\{a_i\}_{i\in\mathbb{N}}$ strictly point-wise dominates $\{b_i\}_{i\in\mathbb{N}}$, then $\sum_{i=1}^{\infty} a_i > \sum_{i=1}^{\infty} b_i$.

CONJECTURE 5 (Surreal Weak Dominance Theorem): Let $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ be two surreal-valued expected payoff series. If $\{a_i\}_{i\in\mathbb{N}}$ strictly dominates $\{b_i\}_{i\in\mathbb{N}}$ at least in one term and weakly dominates $\{b_i\}_{i\in\mathbb{N}}$ in all other terms, then $\sum_{i=1}^{\infty} a_i > \sum_{i=1}^{\infty} b_i$.

The Question of Dominance

If the following sum rule holds, then we can easily prove Conjectures 4 and 5:

$$\sum_{i=1}^{\infty}(b_i + k_i) = \sum_{i=1}^{\infty} b_i + \sum_{i=1}^{\infty} k_i$$

But this principle is in tension with the CC-transformation maneuver. [Thanks to Paul Bartha for discussing with us and confirming this worry.] Other proofs?

In any case: Solution 2 addresses the Paralysis Problem.

3. Surrealization of RET

- Goal: to show that we can outperform our competitors by "surrealizing" their proposals.
- Adopting the surreal framework is good for almost all proposals.
- Case Study: Relative Expectation Theory (Colyvan).
- Surrealized RET respects Dominance in the infinite and finite cases; it outperforms RET in finite cases with single shot infinities.

Relative Expectation Theory (RET) Version 1

Finite state space:

$$REU(A_k, A_l) = \sum_{i=1}^{n} p_i(u_{ki} - u_{li})$$

Infinite state space:

$$REU(A_k, A_l) = \sum_{i=1}^{\infty} p_i(u_{ki} - u_{li})$$

RET-1 Decision Rules: Choose act A_k over A_l iff $REU(A_k, A_l) > 0$. If $REU(A_k, A_l) = 0$ an agent should be indifferent between the two acts in question.

Strengths of RET-1

- Enforce the dominance principle to many infinite cases.

Problems of RET-1

- Identification of states with probability profiles.
- Unable to handle comparisons such as Fair Infinity vs. Biased Infinity.
- Unable to handle single-state infinities.

Relative Expectation Theory (RET) Version 2

Finite state space:

$$REU(A_k, A_l) = \sum_{i=1}^{n}(p_{ki}u_{ki} - p_{li}u_{li})$$

Infinite state space:

$$REU(A_k, A_l) = \sum_{i=1}^{\infty}(p_{ki}u_{ki} - p_{li}u_{li})$$

RET-2 Decision Rules: Choose act A_k over A_l iff $REU(A_k, A_l) > 0$. If $REU(A_k, A_l) = 0$ an agent should be indifferent between the two acts in question.

Strengths of RET-2

- Enforce the dominance principle in more cases involving the infinite state spaces.

Problems of RET-2

- Still unable to handle single-state infinities.

Surrealized RET-2

Finite state space:

$$REU(A_k, A_l) = \sum_{i=1}^{n} (p_{ki}u_{ki} - p_{li}u_{li})$$

Infinite state space:

$$REU(A_k, A_l) = \sum_{i=1}^{\infty} (p_{ki}u_{ki} - p_{li}u_{li})$$

S-RET-2 Decision Rules: Choose act A_k over A_l iff $REU(A_k, A_l) > 0$. If $REU(A_k, A_l) = 0$ an agent should be indifferent between the two acts in question.

Strengths of S-RET-2

- Enforce dominance in finite and infinite state spaces.
- More flexible than RET-1 with different probability profiles.
- Better than RET-2: no problem with finitely many single-state infinities.

Remaining Problems of S-RET-2

- Infinite state space with infinitely many infinities.
- Joyce-style examples: silent about some cases with infinite state space.

Key Differences

- Finite sum: surreal pairwise addition. No problem with single-state infinities.
- Infinite sum: agree with classical infinite sum. If finitely many single-state infinities occur in the series, first separate them from the infinite sum, do surreal addition on them, and sum up the remaining terms in the classical way.

4. Exact Lengths

Strategy: formulate a definition of infinite sum that requires exact length of infinite sequence. This may block the Riemann Rearrangement Theorem (RR) from the get-go.

- RR holds for real numbers and real countable sum.
- Its proof requires a crucial property in real analysis (with extended reals): vagueness of infinity.
- The surreal field has much more structure than the real field – thus we can provide more precision in the infinity in countable sum.

Riemann Rearrangement Theorem

Example: The Pasadena Game Payoff Series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... = \ln(2)$$

A construction of Riemann Rearrangement—-a rearrangement that gives us ∞:

$$1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{5} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{6} + ...$$

$$> \frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{6} + \frac{1}{4} - \frac{1}{6} + \frac{1}{4} - \frac{1}{6}.... = \infty$$

Precise Countable Sum in **No**:

$$\sum_{n=1}^{\infty} a_n \Longrightarrow \sum_{n=1}^{X} a_n,$$

where X can be finite or infinite. Interpret the surreal countable sum as a formal symbol with four inputs: state, utility function, probability function, and the **exact length** of the sequence. This introduces precision in the sum, even if the exact length of the series is infinite.

E.g. if $X = \omega_0$, then $X + 1 \neq X, 2X \neq X,$ and $X/2 \neq X$.

Precise Countable Sum in **No**: $\sum_{n=1}^{X}(-1)^{n+1}1/n$

1. Let $X = \omega_0$. If ω_0 is even, then there will be $\frac{X}{2}$ positive terms and $\frac{X}{2}$ negative terms. A rearrangement of $\sum_{n=1}^{X}(-1)^{n+1}1/n$ should also contain X terms.
2. In the rearrangement of the Pasadena series, there are more negative terms than positive terms. If there are $\frac{X}{2}$ negative terms, there will be more than $\frac{X}{2}$ positive terms. The entire series has more terms than X.
3. Therefore, it is not a rearrangement of $\sum_{n=1}^{X}(-1)^{n+1}1/n$.

- Therefore, no order-dependency results from conditionally convergent series such as the Pasadena payoff series.
- Conjecture: its value is infinitesimally close to $ln2$.
- If the Sum Rule holds, then Dominance Theorems also hold.

Conclusion

1. Infinities (infinite utilities and infinite state spaces) present difficult problems for the standard EU theory.
2. We develop a surreal decision theory—a conservative extension from standard EU theory of finite cases.
3. We prove a surreal representation theorem and two dominance theorems.
4. We apply our theory to games with finite state spaces (Pascal's Wager and Many Gods in the bonus slides) and show how to assign consistent values that respect dominance reasoning.
5. We apply our theory to games with infinite state spaces (St Petersburgh's Game, the Pasadena Game, and their sweetened cousins) and explore four approaches that can be fruitful for further investigation.

In closing, we note future potential applications for surreal numbers.

We see our project as one of the first steps in a program of bringing cutting-edge mathematical tools from non-standard analysis to bear on old philosophical problems. We expect the use of surreals to be particularly helpful in solving problems in transfinite axiology and in dissolving many of the traditional paradoxes of infinity, which rely on absorption by cardinal operations.

Acknowledgement (unordered):

- Branden Fitelson
- Adam Elga
- John Conway
- Andy Egan
- Alan Hájek
- Paul Bartha
- Kenny Easwaran
- Philip Ehrlich
- David Black
- Eric Carlen
- Rutgers Formal Epistemology Reading Group

Thanks for your attention!

Pascal's Wager

	God	No God
Christian	∞	10
Non-Christian	5	10
Expected Payoff	∞	10

Table : Pascal's Wager, Classical Presentation

Pascal reasoned, best to lead a Christian life as long as one's credence that there is a god is non-zero. The rule of expected utility maximization confirms this.

Objection 1: Mixed Strategies

Whenever we have gambles, we can adopt mixtures of those gambles. We can think of mixtures heuristically as using coin flips

to decide which gamble to take. So someone presented with Pascal's Wager might make her choice by flipping a fair coin. In that case, the expected utility of the flip strategy = the expected utility of simply picking "Christian."

This is counterintuitive because gambles with arbitrary biases will have the same expected utility and the agent (according to the standard EU theory) ought to be indifferent among the different gambles.

Objection 1: Mixed Strategies

By standard lights, a chance at ∞ is as good as the genuine artifact. But not so in surreal arithmetic.

The surreal ω is strictly greater than the surreal $.5\omega$. Thus, our proposal correctly predicts that the pure "Christian" strategy beats all mixed strategies.

Objection 2: Many Gods

Another common objection to Pascal is that his decision problem is too simple, and as a result, the use of infinite utilities looks less problematic than it is.

For there are a great many purported gods, many of which treat their followers well, and their doubters cruelly. Moreover, there are any number of other potential eschatological situations.

The objection goes that once we see all these situations, and their accompanying infinite utilities and disutilities in the decision problem, we conclude that there's nothing interesting to say, and so problems of this sort aren't sensibly posed.

Objection 2: Many Gods

Our proposal allows us to formulate and analyze this objection precisely. Let $E_1 \ldots E_n \ldots$ be a (potentially infinite) partition over states in an expanded Pascalian decision problem. With each E_i, we associate some surreal number n, corresponding to $u(E_i)$ in the agent's utility function. Let $cr(E_i)$ be value of the agent's credence function over E_i. We can then give the EU of each of the E_i's.

Objection 2: Many Gods

	Zeus	Athena	Apollo	Atheism
Zeusian	ω	ω	-ω	100
Athenian	-ω	ω	-ω	100
Apollinist	-ω	ω	-ω	100
Atheist	-ω	ω	ω	100

Table : Pascal's Wager With Three Gods

Objection 2: Many Gods

- Zeus is the best option if: $Cr(Zeus) = .5, Cr(Athena) = .3, Cr(Apollo) = .1, Cr(Atheism) = .1,$
- $EU(Zeusian) = .7\omega - 10 > EU(Atheist) = -.1\omega + 10 > EU(Athenian) = EU(Apollinist) = -.3\omega + 10.$

- Atheist is the best option, if $Cr(Zeus) = .1, Cr(Athena) = .2, Cr(Apollo) = .2, Cr(Atheism) = .5$
- $EU(Atheist) = .3\omega + 50 > EU(Zeusian) = .1\omega + 50 > EU(Athenian) = EU(Apollinist) = -.1\omega + 50.$

Which one to choose? It depends on the gambler's credences.

Key Theorems and Proofs

THEOREM 3 (SURREAL VON NEUMANN-MORGENSTERN THEOREM): Let X be a space of lotteries, and let $\preceq$ be a binary relation $\subseteq X \times X$. There exists an affine function $U : X \to \mathbf{No}$ such that $\forall x, y \in X,\ U(x) \leq U(y) \Leftrightarrow x \preceq y$
if and only if $\preceq$ satisfies all of the following:

1. Completeness: $\forall x, y \in X$, either $x \preceq y$ or $y \preceq x$.
2. Transitivity: $\forall x, y, z \in X$, if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.
3. Continuity$\star$: $\forall x, y, z \in X$, if $x \prec y \prec z$, then there exist surreals $p, q \in \star(0, 1)$ such that $px + (1-p)z \prec y \prec qx + (1-q)z$.
4. Independence$\star$: $\forall x, y, z \in X, \forall p \in\star(0,\ 1]$,x $\preceq y$ if and only if $px + (1-p)z \preceq px + (1-p)z$.

Proof: We adopt the usual constructive proof strategy for the von Neumann-Morgenstern representation theorem. We will use the proof to illustrate the content of Continuity$\star$ and Independence$\star$ as well as some properties of surreal numbers.

($\Rightarrow$) This is, as usual, the easier direction. Suppose the existence of a $\star$-affine function $U : X \to \mathbf{No}$ such that $\forall x, y \in X$, $U(x) \leq U(y) \Leftrightarrow x \preceq y$. We want to show that $\preceq$ satisfies Completeness, Transitivity, Continuity$\star$ and Independence$\star$.

(Completeness) Take any $x, y \in X$, suppose that it is not the case that $x \preceq y$. Then it is not the case that $U(x) \leq U(y)$. Since $U(x), U(y) \in \mathbf{No}$, $U(x) \geq U(y)$. (Application of a theorem about $\leq$ as a linear ordering of the surreal field.) Thus, $y \preceq x$.

(Transitivity) Take any $x, y, z \in X$, suppose that $x \preceq y$ and $y \preceq z$. Then $U(x) \leq U(y)$ and $U(y) \leq U(z)$. Since $U(x), U(y), U(z) \in \mathbf{No}$, $U(x) \geq U(z)$. (Application of a theorem about $\leq$ as a linear ordering of the surreal field.) Thus, $x \preceq z$.

(Continuity$\star$) Take any $x, y, z \in X$, suppose that $x \preceq y \preceq z$. Then $U(x) < U(y) < U(z)$. Now, $U(x), U(y), U(z) \in$ **No**. Since in **No** infinitesimals are well-defined, take $p = \frac{U(y)-U(z)}{2U(x)-2U(z)}$. Then, $pU(x) + (1-p)U(z) < U(y)$. Similarly, $q = \frac{U(x)-U(y)}{2U(x)-2U(z)}$. Thus, $U(y) < qU(x) + (1-q)U(z)$. Thus, $px + (1-p)z \prec y \prec qx + (1-q)z$.

(Independence$\star$) Take any $x, y, z \in X, p \in \star(0,1]$. We have:

$$\begin{aligned} x \preceq y &\Leftrightarrow U(x) \leq U(y) \\ &\Leftrightarrow pU(x) \leq pU(y) \\ &\Leftrightarrow pU(x) + (1-p)U(z) \leq pU(y) + (1-p)U(z) \\ &\Leftrightarrow px + (1-p)z \preceq px + (1-p)z \end{aligned}$$

($\Leftarrow$) Suppose that $\preceq$ satisfies Completeness, Transitivity, Continuity$\star$ and Independence$\star$. We want to construct a $\star$-affine function $U : X \to \mathbf{No}$ such that $\forall x, y \in X, U(x) \leq U(y) \Leftrightarrow x \preceq y$. As usual[1], let $\overline{p}$ and $\underline{p}$ denote the $\preceq$-top and $\preceq$-bottom elements in X. If $\preceq$ admits several maximals and several minimals, then let $\overline{p}$ and $\underline{p}$ denote some representatives of the equivalence classes of maximals / minimals. If $\overline{p} = \underline{p}$, then choose any constant surreal function and we are done. Suppose $\overline{p} > \underline{p}$. By Continuity$\star$ and Independence$\star$, suppose that $1 > b > a > 0$, we have:

[1] The following proof follows closely Jonathan Levin's online notes at: http://web.stanford.edu/ jdlevin/Econ%20202/Uncertainty.pdf. Accessed on March 7, 2015.

$$
\begin{aligned}
\overline{p} &= b\overline{p} + (1-b)\overline{p} \\
&> b\overline{p} + (1-b)\underline{p} \\
&> (b-a)\overline{p} + a\overline{p} + (1-b)\underline{p} \\
&> (b-a)\underline{p} + a\overline{p} + (1-b)\underline{p} \\
&= a\overline{p} + (1-a)\underline{p} \\
&> \underline{p}
\end{aligned}
$$

Thus,

$$
\overline{p} > b\overline{p} + (1-b)\underline{p} > a\overline{p} + (1-a)\underline{p} > \underline{p} \qquad (1)
$$

(Lemma) $\forall p \in X, \exists! \lambda_p$ such that $\lambda_p \overline{p} + (1 - \lambda_p) \underline{p} \sim p$.

The existence of such a λ_p is guaranteed by Continuity$\star$, for $\overline{p} \geq p \geq \underline{p}$. The uniqueness of λ_p is guaranteed by Inequality (1). Thus, (Lemma) is true.

Now, because of (Lemma), we can construct the desired utility function as $U(p) = \lambda_p$. We know:

$$p \succeq q \Leftrightarrow \lambda_p \overline{p} + (1 - \lambda_p)\underline{p} \geq \lambda_q \overline{q} + (1 - \lambda_q)\underline{q} \Leftrightarrow \lambda_p \geq \lambda_q$$

.

In the final step of the proof, we show that U is affine, i.e.

$$\forall a \in \star[0,1], \forall p, p' \in X, U(ap + (1-a)p') = aU(p) + (1-a)U(p')$$

.

Take $a \in \star[0,1], p, p' \in X$. By the construction of U(p), we have: $p \sim U(p)\overline{p} + (1 - U(p))\underline{p}$ and $p' \sim U(p')\overline{p'} + (1 - U(p'))\underline{p'}$. Thus,

$$ap + (1-a)p' \sim (aU(p) + (1-a)U(p'))\overline{p} + (1 - (aU(p) + (1-a)U(p')))\underline{p}$$

By the construction of U(p), we thus have:
$U(ap + (1-a)p') = aU(p) + (1-a)U(p')$.
So U is indeed an affine utility function. This completes *Proof 2*. □

THEOREM 4 (Surreal Strict Dominance Theorem): Let $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ be two surreal-valued expected payoff series. If $\{a_i\}_{i\in\mathbb{N}}$ strictly point-wise dominates $\{b_i\}_{i\in\mathbb{N}}$, then $\sum_{i=1}^{\infty} a_i > \sum_{i=1}^{\infty} b_i$.

Proof: Suppose $\{a_i\}_{i\in\mathbb{N}}$ strictly point-wise dominates $\{b_i\}_{i\in\mathbb{N}}$. Since they are both expected payoff series, they are CC-invulnerable by stipulation, i.e. invulnerable to conditional convergence. Moreover,

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty}(a_i - b_i + b_i) = \sum_{i=1}^{\infty}(k_i + b_i),$$

for some positive k_i. Since surreal addition is commutative and irreflexive (non-absorptive), and k_i's are positive,

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} b_i + \sum_{i=1}^{\infty} k_i > \sum_{i=1}^{\infty} b_i . \square$$

THEOREM 5 (Surreal Weak Dominance Theorem): Let $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ be two surreal-valued expected payoff series. If $\{a_i\}_{i\in\mathbb{N}}$ strictly dominates $\{b_i\}_{i\in\mathbb{N}}$ at least in one term and weakly dominates $\{b_i\}_{i\in\mathbb{N}}$ in all other terms, then $\sum_{i=1}^{\infty} a_i > \sum_{i=1}^{\infty} b_i$.

THEOREM 5 (Surreal Weak Dominance Theorem): Let $\{a_i\}_{i\in\mathbb{N}}$ and $\{b_i\}_{i\in\mathbb{N}}$ be two surreal-valued expected payoff series. If $\{a_i\}_{i\in\mathbb{N}}$ strictly dominates $\{b_i\}_{i\in\mathbb{N}}$ at least in one term and weakly dominates $\{b_i\}_{i\in\mathbb{N}}$ in all other terms, then $\sum_{i=1}^{\infty} a_i > \sum_{i=1}^{\infty} b_i$.

Proof: We leave this as an exercise for the reader. □

THEOREM 8: Suppose X is a set of surreal numbers. Then there exists some $\mathfrak{n}$ s.t. $\mathfrak{n} > X$.

Proof: By theorem 7, for any surreal number n, the class of ordinals $\ngeq n$ is set-sized. By theorem 8, for any set of ordinals, there is a greater one. Let X_o by the set $\{x : x \text{ is an ordinal in } X\} \cup \{x : x \text{ is an ordinal less than } X\}$. By construction, X_o is a set of ordinals. So there is an ordinal α greater than it. But since X_o includes every ordinal that is a member of, or less than a member of, X, $\alpha > X$. Let $\mathfrak{n} = \alpha$. $\square$

COROLLARY 8.1 Let u be a surreal-valued utility function. There exists a utility function u' that is a cc-invulnerable transformation of u.

Proof: Let X be the set of negative numbers in the image of u. Let $\mathfrak{n}$ be greater than $\{|X|\}$. Then $u'(x) = u(x) + \mathfrak{n}$. $\square$