Basic Forcing Theory

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Outline

The Continuum Hypothesis

Forcing is a technique, invented by Cohen in the early 1960s, for proving the independence, or at least the consistency of, of certain statements relative to ZFC. The most famous statement independent of ZFC may be

Convention 1.1 (Continuum Hypothesis, or CH)

$2^\omega = \omega_1$.

In fact,

- Cohen constructed in 1962 from models of ZFC a model of ZFC in which CH fails as well as a model of ZF in which AC fails;
- Gödel constructed in 1938 a model (L, the constructible universe) of ZFC in which CH holds.

By combining these results we get that the AC is independent of ZF and that CH is independent of ZFC.

Two Ways to Look at Forcing

Before we discuss Cohen's forcing technique, let's briefly recall what it means for a sentence φ to be independent of ZFC:

Definition 1.2

For any set Γ of formulas and a formula φ, φ is independent of Γ iff $\Gamma \nvdash \varphi$ and $\Gamma \nvdash \neg\varphi$.

And since we also have the lemma:

Lemma 1.3

For any set Γ of formulas and formula φ, we have $\Gamma \nvdash \varphi$ iff $\Gamma \cup \{\neg\varphi\}$ is satisfiable, and $\Gamma \nvdash \neg\varphi$ iff $\Gamma \cup \{\varphi\}$ is satisfiable. □

then to prove a sentence φ is independent of ZFC, it suffices for us to produce models for both $\mathsf{ZFC} \cup \{\varphi\}$ and $\mathsf{ZFC} \cup \{\neg\varphi\}$. There are two main ways for this: for any sentence φ,

- starting from a model of ZFC, one could construct directly a model of $\mathsf{ZFC} \cup \{\varphi\}$;
- one could apply compactness and show that whenever $\mathsf{ZFC}^* \subseteq \mathsf{ZFC}$ is a finite set of axioms $\mathsf{ZFC}^* \cup \{\varphi\}$ has a model.

These two approaches correspond to two different ways to look at forcing.

Example 1.4 (An Example from Group Theory)

Consider the group $\mathcal{G} = (\mathbb{Q}^+, \cdot)$ where $\mathbb{Q}^+ = \{x \in \mathbb{Q} | x > 0\}$ and the group theory GT and the sentence $\varphi = \exists x(x \cdot x = 2)$. It's clear that $\mathcal{G} \models \mathsf{GT}$ and $\mathcal{G} \not\models \varphi$. Extend the domain of $\mathcal{G}$ by adding elements of the form qX where $q \in \mathbb{Q}^+$, and for all $p, q \in \mathbb{Q}^+$ we define:

- $p * q = p \cdot q$;
- $p * qX = (p \cdot q)X$;
- $pX * q = (p \cdot q)X$;
- $pX * qX = 2 \cdot p \cdot q$;
- $(pX)^{-1} = (\frac{1}{2} \cdot p^{-1})X$.

And set $\mathbb{Q}^+[X] = \mathbb{Q}^+ \cup \{qX | q \in \mathbb{Q}^+\}$and $\mathcal{G}[X] = (\mathbb{Q}^+[X], *)$. It's easy to check that $\mathcal{G}[X] \models \mathsf{GT}$ and $\mathcal{G}[X] \models \varphi$ (where $x = 1 \cdot X$ is the witness).

Let's consider the other way. First let $\mathsf{ZFC}^* \subseteq \mathsf{ZFC}$ be any finite set of axioms. As known, $V \models \mathsf{ZFC}$, then by the Reflection theorem there is a set model $M \in V$ such that $M \models \mathsf{ZFC}^*$. The goal is now to show that for any finite subset $\mathsf{ZFC}^* \subseteq \mathsf{ZFC}$ it's possible to extend M to a set model $M[X]$ of $\mathsf{ZFC}^* \cup \{\varphi\}$. Then since ZFC^* is arbitrary we would get the consistency of $\mathsf{ZFC} \cup \{\varphi\}$. To illustrate this approach we give an example:

Example 1.5

Let's work with the group $\overline{\mathcal{G}} = (\mathbb{R}^+, \cdot)$ where $\mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$. Now the group $\mathcal{G} = (\mathbb{Q}^+, \cdot)$ is just a subgroup of $\overline{\mathcal{G}}$, and in $\overline{\mathcal{G}}$ we can extend $\mathcal{G}$ to the group $\mathcal{G}[\sqrt{2}]$ with domain $\mathbb{Q}^+ \cup \{q \cdot \sqrt{2} | q \in \mathbb{Q}^+\}$, which is still a subgroup of $\overline{\mathcal{G}}$.

- in the first way we just need to extend the original model once; while in the second one many many times maybe needed;
- in the second one the entire forcing construction can be carried out in the model V: because M is a set in V, we can extend M within the model V to the desired model $M[X]$, such that $M[X]$ is still a model in V. So all these take place within the model V; while in the first one it's usually not that case;
- in Example 1.4, we look $\mathcal{G}$ from $\mathcal{G}$ and extend $\mathcal{G}$ also from $\mathcal{G}$; while in Example 1.5, we look $\mathcal{G}$ in the model $\overline{\mathcal{G}}$ and extend $\mathcal{G}$ within $\overline{\mathcal{G}}$;
- in Example 1.5, $\sqrt{2}$ at least for people living in $\overline{\mathcal{G}}$ is a proper real than any other symbol, although it's just a symbol for people in $\mathcal{G}$; while in Example 1.5 the symbol X at least for people living in $\mathcal{G}$ is just a symbol with some specified properties. On the other hand, in the latter example, people living in $\overline{\mathcal{G}}$ already know that $\sqrt{2}$ exists, whereas in the former example there are no such people since our universe is just $\mathcal{G}$.

In this proposal, we mainly introduce the basic theory of forcing used to extend models of ZFC. The main ingredients to construct such an extension are a transitive model V of ZFC ($V = L$), a partially ordered set $\mathbb{P} = (P, \leq)$ contained in V, as well as a special subset G of P which will not belong to V. The extended model $V[G]$ will then consist of all sets which can be "described" or "named" in V, where the naming depends on the set G. The main task will be to prove that $V[G]$ is a model of ZFC as well as to decide within V whether a given statement is true or false in a certain extension $V[G]$.

To get an idea how this is done, think for a moment that there are people living in V. For these people, V is the unique set-theoretic universe which contains all the sets. Now, the key point is that, for any statement, these people are able to compute whether the statement is true or false in a particular extension $V[G]$, even though they have almost no information about the set G (in fact they would actually deny the existence of such a set).

The Forcing Notion

Definition 2.1

- $\mathbb{P} = (P, \leq)$ is a partial order iff $\leq$ is a relation on P which is transitive and reflexive. And elements of P are called conditions. If $p \leq q$, we read that as "p is weaker than q";
- $C \subseteq P$ is a chain in P iff for all $p, q \in P$ either $p \leq q$ or $q \leq p$;
- $p, q \in P$ are compatible iff there is some $r \in P$ such that $p \leq r$ and $q \leq r$; otherwise, p, q are incompatible, in symbols, $p \perp q$;
- $A \subseteq P$ is an antichain in P iff for all $p, q \in A$, p, q are incompatible;
- A partial order $\mathbb{P} = (P, \leq)$ has the countable chain condition iff every antichain in P is countable;
- $D \subseteq P$ is directed iff for all $p, q \in D$ there is some $r \in D$ such that $p \leq r$ and $q \leq r$;

Definition 2.1 (Continued)

- $D \subseteq P$ is open iff for all $p \in D$ and $q \in P$, if $p \leq q$ then $q \in D$;
- $D \subseteq P$ is dclosed iff for all $p \in D$ and $q \in P$, if $q \leq p$ then $q \in D$;
- $D \subseteq P$ is dense in P iff for all $p \in P$, there is some $q \in D$ such that $p \leq q$;
- $\varnothing \neq G \subseteq P$ is a filter in P iff it's directed and dclosed.

Definition 2.2

A forcing notion is just a partially ordered set $\mathbb{P} = (P, \leq)$ satisfying

- P has a smallest element, i.e., $\exists p \in P \forall q \in P (p \leq q)$;
- P has the property that there are incompatible elements above each $p \in P$, i.e.,
  $\forall p \in P \exists q_0 \in P \exists q_1 \in P (p \leq q_0 \wedge p \leq q_1 \wedge q_0 \perp q_1)$.

Example 2.3

(1) Recall that $\text{Fin}(I, J)$ is the set of all finite partial functions from I to J. Now for any cardinal $\kappa > 0$, define the partially ordered set $\mathbb{C}_\kappa = (\text{Fin}(\kappa \times \omega, 2), \sqsubseteq)$, i.e., $p, q \in \mathbb{C}_\kappa$, p is stronger than q iff the function p extends q. Now we check that $\mathbb{C}_\kappa$ has the two properties in Definition 2.2:

- $\mathbb{P}$ is a partially ordered set;
- P has a smallest element $\varnothing$, i.e., the empty function;
- P has the property that there are incompatible elements above each $p \in P$. Fix some $p \in \text{Fin}(\kappa \times \omega, 2)$, there is an ordered pair $(\alpha, n) \in \kappa \times \omega$ which doesn't belong to $\text{dom}(p)$. Now let $q_0 = q \cup \{((\alpha, n), 0)\}$ and $q_1 = q \cup \{((\alpha, n), 1)\}$, then it's clear that $p \sqsubseteq q_0, p \sqsubseteq q_1$ and $q_0 \perp q_1$.

So $\mathbb{C}_\kappa$ is a forcing notion. In particular, the forcing notion $\mathbb{C}_1$, denoted $\mathbb{C}$, is called Cohen forcing.

Example 2.3 (Continued)

(2) Define an equivalence relation on $[\omega]^\omega$ by stipulating $x \sim y \Leftrightarrow x \triangle y$ is finite. And let $[\omega]^\omega / \text{Fin} = \{[x] | x \in [\omega]^\omega\}$ and $[x] \leq [y] \Leftrightarrow y - x$ is finite, and let $\mathbb{U} = ([\omega]^\omega / \text{Fin}, \leq)$.

- $\mathbb{U}$ is a partially ordered set;
- $[\omega]^\omega / \text{Fin}$ has a smallest element $[\omega]$;
- $[\omega]^\omega / \text{Fin}$ has the property that there are incompatible elements above each $[x] \in [\omega]^\omega / \text{Fin}$. Fix some $x \in [\omega]^\omega$, we can easily find disjoint y_0 and y_1 in $[x]^\omega$. It's clearly that $y_0 - x = y_1 - x = \varnothing$, and it suffices to show that there is no $y \in [\omega]^\omega$ such that both $y - y_0$ and $y - y_1$ are finite, i.e., $[y_0] \perp [y_1]$. Suppose there is such y, then we have $(y - y_0) \cup (y - y_1)$ is finite. But $(y - y_0) \cup (y - y_1) = y$ is infinite since $y_0 \cap y_1 = \varnothing$, a contradiction.

So $\mathbb{U}$ is a forcing notion, and we call it ultrafilter forcing.

On Naming Sets

Let V be a model of ZFC and $\mathbb{P} = (P, \leq)$ be a forcing notion which belongs to V, i.e., the set P as well as the relation "$\leq$" which is a subset of $P \times P$ belongs to the transitive model V. The goal is to extend the so-called ground transitive model V, by adding a certain subset $G \subseteq P$ to V, and then construct a model $V[G]$ of ZFC which contains V. In order to get a proper extension, the set G which even though a subset of P must not belong to V.

Roughly speaking, $V[G]$ consists of all sets which can be constructed from G by applying the set-theoretic processes definable in V. In fact each set in the extension will have a name in V, which tells how it has been constructed from G. We use symbols such as $\hat{x}, \hat{y}, \hat{f}, \hat{X}$ and so on for ordinary names, but also $\check{x}, \check{y}, \check{c}, \check{G}$ and so on for some special names (for example, names for sets in V). Strictly speaking, we call such names $\mathbb{P}$-names and we use $V^{\mathbb{P}}$ denote the class of all $\mathbb{P}$-names.

Definition 2.4

By transfinite recursion, we have

$$\begin{array}{lcll} V_\alpha^{\mathbb{P}} & = & \varnothing & \alpha = 0; \\ V_\alpha^{\mathbb{P}} & = & \mathscr{P}(V_\beta^{\mathbb{P}} \times P) & \alpha = \beta + 1; \\ V_\alpha^{\mathbb{P}} & = & \bigcup_{\delta \in \alpha} V_\delta^{\mathbb{P}}, & \alpha \in \mathsf{Lim}. \end{array}$$

And we set $V^{\mathbb{P}} = \bigcup_{\alpha \in \mathsf{ON}} V_\alpha^{\mathbb{P}}$.

Lemma 2.5

Let V be a transitive model of ZFC *and $\mathbb{P} = (P, \leq)$ be a forcing notion which belongs to V. Then for any ordinal α the following hold*

1. $0 = \varnothing \in V^{\mathbb{P}}$;
2. *if* $\alpha \neq 0$, *then* $\alpha \notin V^{\mathbb{P}}$;
3. $V^{\mathbb{P}}_{\alpha} \in V$, *and hence* $V^{\mathbb{P}}_{\alpha} \subseteq V$;
4. $V^{\mathbb{P}} \subseteq V$;
5. $V^{\mathbb{P}}$ *is a proper class.* □

Definition 2.6

For any forcing notion $\mathbb{P} = (P, \leq)$ and $\mathbb{P}$-names $\hat{x} \in V^{\mathbb{P}}$, we define the rank-function ρ by setting

$$\rho(\hat{x}) = \bigcup \{\rho(\hat{y}) + 1 | \exists p \in P((\hat{y}, p) \in \hat{x})\}.$$

On Translating Names

Names are objects in V intended to designate sets in the extension $V[G]$ ($G \subseteq P$). In other words, names are special sets in V which stand for sets in the extension. So the next step in the construction of $V[G]$ is to translate the names into the sets they stand for:

Definition 2.7

Let V be a model of ZFC, $\mathbb{P} = (P, \leq)$ be a forcing notion and $G \subseteq P$ (later, G will always be a generic filter). Then by transfinite recursion on $\mathbb{P}$-names $\hat{x}$ we define

$$\hat{x}[G] = \{\hat{y}[G] | \exists q \in G((\hat{y}, q) \in \hat{x})\},$$

and in general we set

$$V[G] = \{\hat{x}[G] | x \in V^{\mathbb{P}}\}.$$

Remark 2.8

1. Since $0 \in V^{\mathbb{P}}$, then $0[G] = 0$;
2. If $G = \varnothing$, then $V[G] = \varnothing$;
3. For example, let's consider again the three conditions of $\mathbb{U}$: $u_0 = [\omega], u_1 = [\{2n | n \in \omega\}]$ and $u_2 = [\{3n | n \in \omega\}]$, and three $\mathbb{U}$-names: $\hat{x} = \{(\varnothing, u_0)\}, \hat{y} = \{(\hat{x}, u_1), (\varnothing, u_2)\}$ and $\hat{z} = \{(\hat{y}, u_0), (\hat{x}, u_1), (\varnothing, u_2), (\hat{y}, u_2)\}$, and $G_0 = \{u_0\}$, and $G_1 = \{u_0, u_1\}$. Then $\hat{x}[G_0] = 1, \hat{x}[G_1] = 1, \hat{y}[G_0] = 0$, $\hat{y}[G_1] = \{1\}, \hat{z}[G_0] = 1$ and $\hat{z}[G_1] = \{\{1\}, 1\}$.

Since $V[G]$ supposed to be an extension of V, we have to show that V is in general a subclass of $V[G]$. Furthermore, G should belong to $V[G]$, no matter whether G belongs to V or not.

Let $\mathbb{O}$ be the smallest element of P and $\mathbb{O} \in G$. For any $x \in V$, there is a canonical name $\check{x} \in V^{\mathbb{P}}$ such that $\check{x}[G] = x$ if we have the following definition:

Definition 2.9

By transfinite recursion, for any $x \in V$ we define $\check{x} = \{(\check{y}, \mathbb{O}) | y \in x\}$.

Remark 2.10

1. Clearly $\check{0} = \check{\varnothing} = \varnothing = 0$, $\check{1} = \{(\check{0}, \mathbb{O})\}$, $\check{2} = \{(\check{0}, \mathbb{O}), (\check{1}, \mathbb{O})\}$ and so on;
2. Since $\mathbb{O} \in G$, then for all $x \in V$ we have $\check{x}[G] = \{\check{y}[G] | y \in x\}$.

Lemma 2.11

If $G \subseteq P$ with $\mathbb{O} \in G$, then $V \subseteq V[G]$.

Proof.

It suffices to prove that, for any $x \in V$, $\check{x}[G] = x$. We proceed by transfinite induction on $\rho(\check{x})$.

- Basic step. If $\rho(\check{x}) = 0$, then $x = \varnothing$, and $\check{x}[G] = \check{\varnothing}[G] = \varnothing = x$;
- Inductive step. Now suppose $\rho(\check{x}) = \alpha$ and assume that $\check{y}[G] = y$ for all $\rho(\check{y}) \in \alpha$. Then

$$\check{x}[G] = \{\check{y}[G] | y \in x\} = \{y | y \in x\} = x.$$

This completes the proof. □

In order to make sure that G belongs to $V[G]$, we need a $\mathbb{P}$-name $\hat{G}$ such that $\hat{G}[G] = G$.

Definition 2.12

Let V be a model of ZFC, $\mathbb{P} = (P, \leq)$ be a forcing notion and $G \subseteq P$, we define $\hat{G} = \{(\check{p}, p) | p \in P\}$.

For V, we usually let it be a transitive model. And since $P \in V$, then $\check{p} \in V^{\mathbb{P}}$. Therefore $\hat{G} \in V^{\mathbb{P}}$. As a corollary we have

Corollary 2.13

If $G \subseteq P$ with $\mathbb{O} \in G$, then $\hat{G}[G] = G$. And so $G \in V[G]$.

Proof.

$\hat{G}[G] = \{\check{p}[G] | \exists q \in G((\check{p}, q) \in \hat{G})\} = \{\check{p}[G] | p \in G\} = \{p | p \in G\} = G.$ □

We can also define names for unordered and ordered pairs of sets:

Definition 2.14

Let V be a model of ZFC, $\mathbb{P} = (P, \leq)$ be a forcing notion and $G \subseteq P$. For any $\hat{x}, \hat{y} \in V^{\mathbb{P}}$, we define

$$\mathrm{up}(\hat{x}, \hat{y}) = \{(\hat{x}, \mathbb{O}), (\hat{y}, \mathbb{O})\},$$

and

$$\mathrm{op}(\hat{x}, \hat{y}) = \{(\{(\hat{x}, \mathbb{O})\}, \mathbb{O}), (\{(\hat{x}, \mathbb{O}), (\hat{y}, \mathbb{O})\}, \mathbb{O})\}.$$

Lemma 2.15

If $G \subseteq P$ with $\mathbb{O} \in G$, then we have $\mathrm{up}(\hat{x}, \hat{y})[G] = \{\hat{x}[G], \hat{y}[G]\}$ *and* $\mathrm{op}(\hat{x}, \hat{y}) = (\hat{x}[G], \hat{y}[G])$. □

The Forcing Language

A sentence φ of the forcing language is like a first-order sentence, except that the parameters appearing in φ are some $\mathbb{P}$-names in $V^{\mathbb{P}}$. Sentences of the forcing language use the $\mathbb{P}$-names in $V^{\mathbb{P}}$ to assert something about $V[G]$ (for certain $G \subseteq P$).

Definition 2.16

Let $\mathbb{P}$ be a forcing notion. Then the forcing language $\mathcal{L}_{\mathbb{P}}$ consists of logical formulas using $\in$ and all the $\mathbb{P}$-names as constant symbols.

The people living in the ground model V may not know whether a given sentence is true in $V[G]$. The truth or falsity of φ in $V[G]$ will generally depend on the $G \subseteq P$. To illustrate this, we give an example:

Example 2.17

Recall our definition for ultrafilter forcing $\mathbb{U}$ in Example 2.3. Give a condition of $\mathbb{U}$: $u_0 = [\omega]$ and a $\mathbb{U}$-name: $\hat{x} = \{(\varnothing, u_0)\}$. Consider the sentence $\varphi = \exists y(y \in \hat{x})$. Now

$$\begin{aligned}
& \varphi \text{ is true in } V[G] \\
\Leftrightarrow\quad & V[G] \models \varphi, \\
\Leftrightarrow\quad & V[G] \models \exists y(y \in \hat{x}), \\
\Leftrightarrow\quad & \hat{x}[G] \neq \varnothing, && \hat{x} \text{ is interpreted as } \hat{x}[G] \text{ in } V[G], \\
\Leftrightarrow\quad & u_0 \in G, && \text{the definition of } \hat{x}[G] \text{ and } \hat{x}.
\end{aligned}$$

Even though people living in V don't know whether $V[G] \models \varphi$, they know that $V[G] \models \varphi \Leftrightarrow u_0 \in G$.

Moreover, it will turn out that people living in V are able to verify that in certain models $V[G]$ all axioms of ZFC remain true.

Generic Filters

Let $\mathbb{P} = (P, \leq)$ be an arbitrary forcing notion which belongs to a transitive model V of ZFC. First let's recall some definitions:

- $D \subseteq P$ is open dense iff $\forall p \in D \forall q \in P(p \leq q \to q \in D)$ (open) and $\forall p \in P \exists q \in D(p \leq q)$ (dense);
- $A \subseteq P$ is an antichain in P iff $\forall p, q \in P(p \perp q)$;
- $\varnothing \neq G \subseteq P$ is a filter in P iff $\forall p, q \in G \exists r \in G(p \leq r \wedge q \leq r)$ (directed) and $\forall p \in D \forall q \in P(q \leq p \to q \in D)$ (dclosed).

Generic Filters

Now we define generic filters:

Definition 3.1

Let $\mathbb{P} = (P, \leq)$ be an arbitrary forcing notion which belongs to a transitive model V of ZFC. A filter $G \subseteq P$ is said to be $\mathbb{P}$-generic over V if $G \cap D \neq \varnothing$ for all open dense sets $D \subseteq P$ which belong to V.

In other words, a filter $G \subseteq P$ is said to be $\mathbb{P}$-generic over V iff it meets every open dense subsets belong to V. The restriction that open dense subsets have to belong to V which at first glance seems to be superficial is in fact crucial.

Alternations of Generic Filters

It's sometimes useful to have a few alternative definitions of $\mathbb{P}$-generic filters at hand.

Lemma 3.2

Let $\mathbb{P} = (P, \leq)$ be a forcing notion which belongs to a transitive model V of ZFC. *Then, for a filter G on P, the following are equivalent:*

1. *G is $\mathbb{P}$-generic over V;*
2. *G meets every maximal antichain in P which belongs to V;*
3. *G meets every dense subset of P which belongs to V.*

Proof.

(2) $\Rightarrow$ (3) needs Kurepa's Principle: Each partially ordered set has a maximal subset of pairwise incomparable elements. Notably that it's equivalent to AC. □

Definition 3.3

Let $p \in P$, then a set $D \subseteq P$ is dense above p if, for any $q \in P$ with $p \leq q$, there is an $r \in D$ such that $q \leq r$.

Remark 3.4

Notice that if $D \subseteq P$ is dense above p and $p \leq q$, then D is also dense above q.

Lemma 3.5

Let $\mathbb{P} = (P, \leq)$ be a forcing notion which belongs to a model V of ZFC. *Then, for a filter G on P which contains the condition p. Then the following are equivalent.*

1. *G is $\mathbb{P}$-generic over V;*
2. *G meets every set $D \subseteq P$ which is dense above p.* □

Generic Models

First we give the notion of generic models.

Definition 4.1

Let $\mathbb{P} = (P, \leq)$ be an arbitrary forcing notion which belongs to a transitive transitive model V of ZFC. If a filter $G \subseteq P$ is $\mathbb{P}$-generic over V, then the class $V[G]$ is a generic extension of V, or just a generic model.

Now we shall define a relation, denoted $\Vdash_{\mathbb{P}}$ or $\Vdash$ (if $\mathbb{P}$ is clear in the context), between conditions $p \in P$ and sentences φ of the forcing language. Further, we define it recursively:

Definition 4.2

Let $\mathbb{P} = (P, \leq)$ be a forcing notion and V be a transitive model of ZFC. For any $p \in P$, we define $p \Vdash \varphi$ by recursion on $\varphi \in \mathcal{L}_{\mathbb{P}}$:

- $p \Vdash \hat{x}_0 = \hat{x}_1$ iff for all $(\hat{y}_0, s_0) \in \hat{x}_0$, the set

  $$\{q \geq p | s_0 \leq q \to \exists(\hat{y}_1, s_1) \in \hat{x}_1(s_1 \leq q \wedge q \Vdash \hat{y}_0 = \hat{y}_1)\}$$

  is dense above p and for all $(\hat{y}_1, s_1) \in \hat{x}_1$, the set

  $$\{q \geq p | s_1 \leq q \to \exists(\hat{y}_0, s_0) \in \hat{x}_0(s_0 \leq q \wedge q \Vdash \hat{y}_0 = \hat{y}_1)\}$$

  is dense above p;

Definition 4.2 (Continued)

- $p \Vdash \hat{x}_0 \in \hat{x}_1$ iff the set

$$\{q \geq p | \exists(\hat{y}_1, s_1) \in \hat{x}_1(s_1 \leq q \wedge q \Vdash \hat{x}_1 = \hat{y}_1)\}$$

  is dense above p;
- $p \Vdash \neg\varphi(\hat{x})$ iff for all $q \geq p$ we have $q \nVdash \varphi(\hat{x})$;
- $p \Vdash \varphi(\hat{x}) \wedge \psi(\hat{x})$ iff we have $q \Vdash \varphi(\hat{x})$ and $q \Vdash \psi(\hat{x})$;
- $p \Vdash \exists y \varphi(y, \hat{x})$ iff the set

$$\{q \geq p | \exists \hat{y} \in V^{\mathbb{P}}(q \Vdash \varphi(\hat{y}, \hat{x}))\}$$

  is dense above p.

Lemma 4.3

Let $\mathbb{P} = (P, \leq)$ be a forcing notion, then for any sentence $\varphi \in \mathcal{L}_{\mathbb{P}}$,

1. *if $p \Vdash \varphi$ and $p \leq q$, then $q \Vdash \varphi$;*
2. *the set $\Delta_\varphi = \{p \in P | (p \Vdash \varphi) \vee (p \Vdash \neg\varphi)\}$ is open dense in P.*

Proof.

(1) By induction on φ.
(2) We have to show two points:

- Δ_φ is open. Fix some $p \in \Delta_\varphi$, $q \in P$ with $p \leq q$. Then by (1) we have $q \in \Delta_\varphi$;
- Δ_φ is dense. For any $p \in P$, either there is a $q \geq p$ such that $q \Vdash \varphi$, or for all $q \geq p$ we have $q \nVdash p$. In the former case, $q \in \Delta_\varphi$, and in the latter case we have $p \Vdash \neg\varphi$ and $p \in \Delta_\varphi$. And since $\mathbb{P}$ is a forcing notion, then there is a $r \geq p$, and then by (1) we have $r \in \Delta_\varphi$.

□

Forcing Theorem and Generic Model Theorem

Until now we didn't prove the forcing relation is doing what we want, for example, $p \Vdash \varphi$ implies $p \nVdash \neg\varphi$. While the following theorem which is the core result of forcing tells us that we are in the right way.

Theorem 4.4 (Forcing)

Let $\mathbb{P} = (P, \leq)$ be a forcing notion which belongs to the transitive model V of ZFC*, and let $G \subseteq P$ be $\mathbb{P}$-generic over V, and let $\varphi(\hat{x_0}, \cdots, \hat{x_{n-1}})$ be a formula of the forcing language such that for the first order formula $\varphi(x_0, \cdots, x_{n-1})$ we have $\mathrm{Fr}(\varphi) \subseteq \{x_0, \cdots, x_{n-1}\}$. Then*

$$V[G] \models \varphi(\hat{x_0}, \cdots, \hat{x_{n-1}}) \Leftrightarrow \exists p \in G(p \Vdash \varphi(\hat{x_0}, \cdots, \hat{x_{n-1}})).$$

Corollary 4.5

Let $\mathbb{P} = (P, \leq)$ be a forcing notion, G be generic over V and $p \in G$. Then

1. *if $p \Vdash \hat{z} \in \hat{y}$, then there is a $\mathbb{P}$-name $\hat{x}$ with $\rho(\hat{x}) < \rho(\hat{y})$ and a $\mathbb{P}$-condition $q \geq p$ in G such that $q \Vdash \hat{z} = \hat{x}$;*
2. *if $p \Vdash \hat{f} \in {}^{\hat{A}}\hat{B} \wedge \hat{x} \in \hat{A}$, then there is a $\mathbb{P}$-name $(\hat{y}, r) \in \hat{B}$ with $r \in G$ and there is some $q \geq p$ such that $q \Vdash \hat{f}(\hat{x}) = \hat{y}$.*

Lemma 4.6

Let $\mathbb{P} = (P, <)$ be a forcing notion and $\varphi \in \mathcal{L}_{\mathbb{P}}$, then the following are equivalent:

1. *$p \Vdash \varphi$;*
2. *$q \Vdash \varphi$ for all $q \in P$ with $p \leq q$;*
3. *$\{q \in P | q \Vdash \varphi\}$ is dense above p.*

With forcing theorem in hands, we can show another core result of forcing:

Theorem 4.7 (Generic Model)

Let $\mathbb{P} = (P, \leq)$ be a forcing notion which belongs to the transitive model V of ZFC *and G be generic over V. Then $V[G] \models$*ZFC*. Moreover, $V \subseteq V[G]$, $G \in V[G]$ and $V[G]$ is the smallest standard model of* ZFC *which contains V as a subclass and G as an element. Furthermore, $\mathsf{ON}^V = \mathsf{ON}^{V[G]}$.*

Forcing Notions not Adding Reals

Definition 4.8

A forcing notion $\mathbb{P} = (P, \leq)$ is said to be κ-closed if whenever $\alpha < \kappa$ and $\{p_\xi | \xi < \alpha\}$ is an increasing sequence of elements of P $(\xi_0 < \xi_1 \rightarrow p_{\xi_0} < p_{\xi_1})$, then there is some $q \in P$ such that $p_\xi \leq q$ for all $\xi < \alpha$.

In particular, we also call "ω-closed" as "σ-closed".

The following result shows that forcing with κ-closed forcing notion doesn't add new reals to the ground model.

Lemma 4.9

Let $\mathbb{P} = (P, \leq)$ be a κ-closed forcing notion, $\lambda < \kappa$, G a $\mathbb{P}$-generic filter over V, X a set in V, and $f : \lambda \rightarrow X$ a function in $V[G]$, i.e., $V[G] \models f \in {}^{\lambda}X$, then f belongs to V.

Forcing Notions Preserving Cardinalities

Definition 4.10

Suppose α is a limit ordinal, and $A \subseteq \alpha$. A is unbounded in α if for all $\gamma < \alpha$ there is some $\xi \in A$ such that $\gamma < \xi$.

The cofinality of α, i.e., $\mathrm{cf}(\alpha)$, is the least limit ordinal θ such that there is an increasing θ-sequence $\langle \gamma_\nu | \nu < \theta \rangle$ whose range is unbounded in α.

Definition 4.11

Let $\mathbb{P} = (P, \leq)$ be a forcing notion in the transitive model V of ZFC. We say $\mathbb{P}$ preserves κ if $\kappa^V = \kappa^{V[G]}$ for all $\mathbb{P}$-generic G over V; otherwise we say $\mathbb{P}$ collapses κ; and we say $\mathbb{P}$ preserve cardinalities if $\mathbb{P}$ preserves all cardinals κ.

We say $\mathbb{P}$ preserve cofinalities if $\mathrm{cf}(\kappa)^V = \mathrm{cf}(\kappa)^{V[G]}$ for all $\mathbb{P}$-generic G over V.

Lemma 4.12

For any $\kappa \leq \omega$ and any forcing notion $\mathbb{P}$, $\mathbb{P}$ preserves κ.

Lemma 4.13

Let $\mathbb{P} = (P, \leq)$ be a forcing notion in the transitive model V of ZFC. *If $\mathbb{P}$ preserve cofinalities, then $\mathbb{P}$ preserves cardinalities.*

More generally, we have

Theorem 4.14

Let $\mathbb{P} = (P, \leq)$ be a forcing notion in the transitive model V of ZFC. *Then if $\mathbb{P}$ satisfies the countable chain condition, then $\mathbb{P}$ preserves cofinalities, and hence preserves cardinalities.*

Consistency of CH

Definition 5.1

For any ordinal α, we define a forcing notion $\mathbb{K}_\alpha = (K_\alpha, \sqsubseteq)$ as follows:

$$K_\alpha = \{p | p : A \to \mathscr{P}(\aleph_\alpha) \wedge A \subseteq \aleph_{\alpha+1} \wedge |A| < \aleph_{\alpha+1}\};$$
$$p \sqsubseteq q \Leftrightarrow p = q \upharpoonright \mathrm{dom}(p) \text{ and } q(x) = p(x) \text{ for all } x \in \mathrm{dom}(p).$$

Lemma 5.2

For any ordinal α, $\mathbb{K}_\alpha = (K_\alpha, \sqsubseteq)$ is $\aleph_{\alpha+1}$-closed.

Lemma 5.3

For any ordinal α, $\mathbb{K}_\alpha$ preserves cardinals $\leq \aleph_{\alpha+1}$ and doesn't add subsets of $\aleph_\alpha$ into $V[G_\alpha]$.

Theorem 5.4

Let $\mathbb{K}_\alpha = (K_\alpha, \sqsubseteq)$ be defined as above, and V be a transitive model of ZFC, *and G_α be $\mathbb{K}_\alpha$-generic over V, then $V[G_\alpha] \models 2^{\aleph_\alpha} = \aleph_{\alpha+1}$. In particular, $V[G_0] \models$*CH.

Proof.

We show that $\bigcup G_\alpha$ is a surjective function from $\aleph_{\alpha+1}$ to $\mathscr{P}(\aleph_\alpha)$. We work in $V[G_\alpha]$. For any $\xi \in \aleph_{\alpha+1}$ and $x \in \mathscr{P}(\aleph_\alpha)$ define:

$$D_{\xi,x} = \{p \in K_\alpha | \xi \in \mathrm{dom}(p) \wedge x \in \mathrm{ran}(p)\}.$$

It's easy to show that $D_{\xi,x}$ is open dense. Then $G_\alpha \cap D_{\xi,x} \neq \varnothing$. Thus for any $\xi \in \aleph_{\alpha+1}$ and $x \in \mathscr{P}(\aleph_\alpha)$, there is always some $p \in G_\alpha$ such that $\xi \in \mathrm{dom}(p) \wedge x \in \mathrm{ran}(p)$. And since G_α is directed, then this implies that $\bigcup G_\alpha$ in $V[G_\alpha]$ is indeed a surjective function from $\aleph_{\alpha+1}$ to $\mathscr{P}(\aleph_\alpha)$. And hence $V[G_\alpha] \models |\mathscr{P}(\aleph_\alpha)| \leq \aleph_{\alpha+1}$. □

Consistency of ¬CH

Firstly, lets's recall definition for the forcing notion $\mathbb{C}_\kappa = (\mathrm{Fin}(\kappa \times \omega, 2), \sqsubseteq)$:

$$\mathrm{Fin}(\kappa \times \omega, 2) = \{p | p : A \to 2 \wedge A \subseteq I \wedge |A| < \omega\};$$
$$p \sqsubseteq q \Leftrightarrow p = q \restriction \mathrm{dom}(p) \text{ and } q(x) = p(x) \text{ for all } x \in \mathrm{dom}(p).$$

Lemma 5.5

$\mathbb{C}_\kappa$ satisfies the countable chain condition.

Theorem 5.6

Let $\mathbb{C}_\kappa = (\mathrm{Fin}(\kappa \times \omega, 2), \sqsubseteq)$ be as above, and V be a transitive model of ZFC*, and G be $\mathbb{C}_\kappa$-generic over V, then $V[G] \models 2^{\aleph_0} \geq \kappa$. In particular, if $\kappa > \aleph_1$, then $V[G] \models \neg$*CH.

Proof.

To keep notations short, let $C_\kappa = \mathrm{Fin}(\kappa \times \omega, 2)$. We work in $V[G]$.

Firstly we show that $\bigcup G$ is a function from $\kappa \times \omega$ to 2. For $\alpha \in \kappa$ and $n \in \omega$ define

$$D_{\alpha,n} = \{p \in C_\kappa | (\alpha, n) \in \mathrm{dom}(p)\}.$$

Then for any $\alpha \in \kappa$ and $n \in \omega$, it's easy to check that $D_{\alpha,n}$ is open dense in C_κ. And hence $G \cap D_{\alpha,n} \neq \varnothing$. So for any $\alpha \in \kappa$ and $n \in \omega$ there is a $p \in G$ such that $(\alpha, n) \in \mathrm{dom}(p)$, and since G is directed $\bigcup G$ is a function with $\mathrm{dom}(\bigcup G) = \kappa \times \omega$.

Secondly we construct κ different real numbers from G. For any $\alpha \in \kappa$ define $r_\alpha \in {}^\omega 2$ by setting $r_\alpha(n) = \bigcup G((\alpha, n))$. Now for any α, β define

$$D_{\alpha,\beta} = \{p \in C_\kappa | \exists n \in \omega((\alpha, n), (\beta, n) \in \mathrm{dom}(p) \wedge p((\alpha, n)) \neq p((\beta, n)))\}.$$

Then for any $\alpha, \beta \in \kappa$ it's easy to check that $D_{\alpha,\beta}$ is open dense in C_κ, and so $G \cap D_{\alpha,\beta} \neq \varnothing$. Thus for any $\alpha \neq \beta \in \kappa$ there is some $n \in \omega$ and some $p \in G$ such that $p((\alpha, n)) \neq p((\beta, n))$, and so $r_\alpha(n) \neq r_\beta(n)$. This gives $2^{\aleph_0} \geq \kappa$ in $V[G]$. □

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