An introduction to forcing

Wang Yunsong

School of Mathematical Science

1600010615@pku.edu.cn

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Overview

1. Generic extensions
2. Forcing

Our goal: $Con(ZFC) \to Con(ZFC + \neg CH)$

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Theorem(Godel)

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So if there is a $M \models ZFC$, we want to find a new model $N \models ZFC + \neg CH$

We will construct a model $M[G]$, s.t $G \notin M$, $M \subseteq M[G]$

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– Let $G \subseteq \mathbb{P}$, we call G is a filter on $\mathbb{P}$:

- $G \neq 0$
- $\forall p, q \in \mathbb{P}((p \in G \wedge q \in G) \rightarrow \exists r \in G(r \leq p \wedge r \leq q))$
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Proof: M is countable, so let $D_n(n \in \omega)$ enumerate all dense subsets of $\mathbb{P}$ which are in M.

Then choose a sequence $q_n(n \in \omega)$ so that $p = q_0 \geq q_1 \geq ...$ and $q_{n+1} \in D_n$. Let G be the filter generated by $\{q_n : n \in \omega\}$.

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Remark: Dense subset can be replaced by open dense subset or maximum anti-chain.

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If M is a transitive model of ZFC, $\mathbb{P} \in M$ is a p.o. such that $\forall p \in \mathbb{P}$ $\exists q, r \in \mathbb{P}\ (q \leq p \wedge r \leq p \wedge q \perp r)$, and G is $\mathbb{P}-$*generic* over M, then $G \notin M$.

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Remark: If condition fails for $\mathbb{P}$, then there is a filter G on $\mathbb{P}$ which intersects all dense subsets of $\mathbb{P}$, and if $\mathbb{P} \in M$, then $G \in M$.

Then forcing to such a $\mathbb{P}$ will be trivial. Thus, almost all p.o. we considered satisfy this condition, although it is never needed in the abstract treatment of forcing.

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$M[G]$ will be the least extension of M to a c.t.m for ZFC containing G. The fact that $G \in M[G]$ will imply, by Lemma, that in most cases $M \neq M[G]$.

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People living within M will be able to comprehend a name, τ, for an object in $M[G]$, but they will not in general be able to decide the object, τ_G, that τ names, since this will require a knowledge of G.

Definition

τ is a $\mathbb{P}$-name iff τ is a relation and

$$\forall < \sigma, p > \in \tau[\sigma \text{ is a } \mathbb{P}\text{-name } \wedge p \in \mathbb{P}].$$

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The collection of $\mathbb{P}$-name will be a proper class if $\mathbb{P} \neq 0$.

And this definition must be understood as a definition by transfinite recursion. Formally, one defines the characteristic function of the $\mathbb{P}$-name, $H(\mathbb{P}, \tau)$, by

$$H(\mathbb{P}, \tau) = 1 \text{ iff } \tau \text{ is a relation } \wedge \forall <\sigma, p> \in \tau[H(\mathbb{P}, \sigma) = 1 \wedge p \in \mathbb{P}].$$

$$H(\mathbb{P}, \tau) = 0 \text{ otherwise.}$$

Definition

$V^{\mathbb{P}}$ is the class of $\mathbb{P}$-names. If M is a transitive model of ZFC and $\mathbb{P} \in M$, $M^{\mathbb{P}} = V^{\mathbb{P}} \cap M$. Or, by absoluteness,

$$M^{\mathbb{P}} = \{\tau \in M : (\tau \text{ is a } \mathbb{P}\text{-name})^M\}.$$

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When forcing over M, use is made only of the $\mathbb{P}$-names in $M^{\mathbb{P}}$, which we may think of as being defined within M.

When $M \models ZFC$, τ is a $\mathbb{P}$-name $\Leftrightarrow$ (τ is a $\mathbb{P}$-name $)^M$.

Definition

$val(\tau, G) = \{val(\sigma, G) : \exists p \in G(<\sigma, p> \in \tau)\}$. We also write τ_G for $val(\tau, G)$. And 0 is also a $\mathbb{P}$-name, we define that $0_G = 0$ for any G.

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Lemma

If N is a transitive model of ZFC with $M \subset N$ and $G \in N$, then $M[G] \subset N$.

Thus, once we check that $M[G]$ is indeed a transitive extension of M containing G and satisfying ZFC, it will be the least such extension.

First, we want to check that $M \subseteq M[G]$:

Check Name

–$\check{x}$, $x \in M$ (check name)

$$\check{x} = \{< \check{y}, \mathbb{1}_{\mathbb{P}} > | y \in x\}$$

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Lemma

If M is a transitive model of ZFC, $\mathbb{P}$ is a p.o. in M, and G is a non-empty filter on $\mathbb{P}$, then:

(a)$\forall x \in M[\check{x} \in M^{\mathbb{P}} \wedge val(\check{x}, G) = x]$.

(b)$M \subset M[G]$.

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Proof:
$(G \notin M)$ If not, $\mathbb{P} \setminus G \subset \mathbb{P}$. For $\mathbb{P}$, we know that $\forall p \in \mathbb{P} \exists q, r \in \mathbb{P}, q \perp r$. So $\mathbb{P} \setminus G$ is dense. Contradiction!

$(G \in M[G])$ we need to find a name that represents it.
–$\Gamma = \{< \check{p}, p > | p \in \mathbb{P}\}$, and $\Gamma_G = G$.

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Remark: $M[G]$ satisfy Axiom of Extension, Union, Pairing, Foundation.

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Lemma

$Ord \cap M = Ord \cap M[G]$.

Proof: $M \subseteq M[G]$, and they are transitive, so $Ord \cap M \subseteq Ord \cap M[G]$.
For any set A, we define $rank(A) = min\{\alpha | A \in V_\alpha\}$.
$rk(A) = sup\{rk(B) + 1 | B \in A\}$, and $\forall \alpha, rk(\alpha) = \alpha$.
$Ord \cap M = \{rk(A) | A \in M\}$, $Ord \cap M[G] = \{rk(A) | A \in M[G]\}$.
We want to prove that $\forall \tau, rk(\tau_G) \leq rk(\tau) \in Ord \cap M$, then we will know $\forall A \in M[G]$, $rk(A) \in Ord \cap M$.

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First, we defind $\Vdash$:

$p \Vdash_{M,\mathbb{P}} \varphi$ iff for any $(M, \mathbb{P}) - \mathit{generic}$ G, if $p \in G$, then $M[G] \models \varphi$.

That is a semantic definition, later we need to find a syntax definition. And then we prove that they are equivalent.

Let us consider first a specific example.

- M is a c.t.m for ZFC.
- $\mathbb{P}$ is the set of finite partial functions from ω to 2 ordered by reverse inclusion.
- $\mathbb{1}_{\mathbb{P}}$ is the empty function.

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$G \in M[G]$ and $f_G = \cup G$, $f_G \in M[G]$ will follow immediately from the absoluteness of $\cup$ for transitive models of ZF.

However, we may check $f_G \in M[G]$ directly, Let

$$\Phi = \{< (< n, \check{m} >), p >: p \in \mathbb{P} \wedge n \in dom(p) \wedge p(n) = m\}$$

$$\Phi_G = \{< n, m >: \exists p \in G(n \in dom(p) \wedge p(n) = m)\} = f_G.$$

Thus, $f_G \in M[G]$.

If G is $\mathbb{P}-$ *generic* over M, then $G \notin M$.

For any $g : \omega \to 2$, if $g \in M$, $E = \{p : p \not\subset g\}$ is dense. So $G \cap E \neq 0$ means that $f_G \neq g$. So $f_G \notin M$.

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So in this example, we use forcing to find a new real number which is not in M.

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So in this example, we use forcing to find a new real number which is not in M.

Also, we know that $\{< 0, 0 >\} \Vdash \dot{f}_G(0) = 0$ and $\{< 0, 1 >\} \Vdash \dot{f}_G(0) = 1$.

Now we define $\Vdash^*$.

$p \Vdash^*_{M,\mathbb{P}} \varphi(\tau_1, \tau_2, ..., \tau_n)$, $\tau_1, \tau_2, ..., \tau_n$ are names in $M^{\mathbb{P}}$, $\varphi(x_1, ..., x_n)$ is a formula.

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(Induction of the complexity of names and formula)

(a) $(\tau_1 = \tau_2)$, $\tau_1 = \{< \pi_1, s_1 > | \dots\}$, $\tau_2 = \{< \pi_2, s_2 > | \dots\}$.

(α) for all $< \pi_1, s_1 > \in \tau_1$,
$\{q \leq p | q \leq s_1 \to \exists < \pi_2, s_2 > \in \tau_2 (q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$ is dense below p, and

(β) for all $< \pi_2, s_2 > \in \tau_2$,
$\{q \leq p | q \leq s_2 \to \exists < \pi_1, s_1 > \in \tau_1 (q \leq s_1 \wedge q \Vdash^* \pi_1 = \pi_2)\}$ is dense below p.

A is dense below p means that $\forall q \leq p \exists r \in A (r \leq q)$.

(b)$(\tau_1 \in \tau_2)$, $\tau_2 = \{< \pi_2, s_2 > |...\}$
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$p \Vdash^* \varphi \wedge \psi$ iff $p \Vdash^* \varphi$ and $p \Vdash^* \psi$.

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$p \Vdash^* \varphi \wedge \psi$ iff $p \Vdash^* \varphi$ and $p \Vdash^* \psi$.

(d)$(\neg\varphi)$

$p \Vdash^* \neg\varphi$ iff $\{q \leq p | q \nVdash^* \varphi\}$ is dense below p.

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(c)$(\varphi \wedge \psi)$
$p \Vdash^* \varphi \wedge \psi$ iff $p \Vdash^* \varphi$ and $p \Vdash^* \psi$.

(d)$(\neg\varphi)$
$p \Vdash^* \neg\varphi$ iff $\{q \leq p | q \nVdash^* \varphi\}$ is dense below p.

(e)$(\exists x \varphi)$
$p \Vdash^* \exists x \varphi(x, \dots)$ iff $\{r \leq p | \exists \sigma \in M^{\mathbb{P}} (r \Vdash^* \varphi(\sigma, \dots))\}$ is dense below p.

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Lemma

TFAE:

1. $p \Vdash^* \varphi(\vec{\tau})$.
2. $\{q \leq p | q \Vdash^* \varphi(\vec{\tau})\}$ is dense below p.
3. $r \leq p,\ r \Vdash^* \varphi(\vec{\tau})$.

Proof:

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$(1)_c$ $p \Vdash^* \varphi \wedge \psi$
$\leftrightarrow p \Vdash^* \varphi$ and $p \Vdash^* \psi$
$\leftrightarrow \forall r \leq p, r \Vdash^* \varphi$ and $r \Vdash^* \psi$ (By induction)

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$(1)_c$ $p \Vdash^* \varphi \wedge \psi$
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$\leftrightarrow \forall r \leq p, r \Vdash^* \varphi \wedge \psi$ $(3)_c$
$\rightarrow \{q \leq p | q \Vdash^* \varphi \wedge \psi\}$ is dense below p. $(2)_c$
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$\leftrightarrow (1)_c$

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$\leftrightarrow (1_d)$

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$\rightarrow (1)_e$

Theorem

(1) $p \in G,\ p \Vdash^* \varphi(\tau_1, \ldots, \tau_n)$

$\Rightarrow M[G] \models \varphi(\tau_{1_G}, \ldots, \tau_{n_G})$

(2) $M[G] \models \varphi(\tau_{1_G}, \ldots, \tau_{n_G})$

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Corollary

(1) $p \Vdash \varphi \Leftrightarrow p \Vdash^* \varphi$
(2) $M[G] \models \varphi(\tau_{1_G}, \ldots, \tau_{n_G}) \Leftrightarrow \exists p \in G(p \Vdash \varphi(\tau_1, \ldots, \tau_n))$

Proof of Corollary:
(1) ($\Leftarrow$) By the Theorem
($\Rightarrow$) $p \Vdash \varphi$, we need to prove that $\{r \leq p | r \Vdash^* \varphi\}$ is dense below p.
If not, there is a G, s.t. $G \cap \{r \leq p | r \Vdash^* \varphi\} = 0$. Then $M[G] \not\models \varphi$,
Contradiction!
(1) $\Rightarrow$ (2)

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(1) $(\tau_1 = \tau_2)$ We assume $p \Vdash^* \tau_1 = \tau_2$ and $p \in G$. We must show ${\tau_1}_G = {\tau_2}_G$.

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We shall show $\tau_{1_G} \subset \tau_{2_G}$
Every element of τ_{1_G} is of the form π_{1_G}, where $< \pi_1, s_1 > \in \tau_1$ for some $s_1 \in G$.

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Fix $r \in G$ with $r \leq p$ and $r \leq s_1$. Then $r \Vdash^* \tau_1 = \tau_2$, so there is a $q \in G$ such that $q \leq r$ and s.t. $q \leq s_1$ which implies

$$\exists < \pi_2, s_2 > \in \tau_2 (q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2).(*)$$

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So fix $< \pi_2, s_2 >$ as in (*), then $s_2 \in G$, so $\pi_{2_G} \in \tau_{2_G}$.
Also, by (1) for $\pi_1 = \pi_2$ (IH), $q \Vdash^* \pi_1 = \pi_2$ implies $\pi_{1_G} = \pi_{2_G}$, so $\pi_{1_G} \in \tau_{2_G}$.

The End