Logic and Social Choice Theory

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Introduction

Why do we need social choice theory?

When a group needs to make a decision, we are faced with the problem of aggregating the views of the individual members of that group into a single collective view that adequately reflects the "will of the people". And logic has played an important role in the development of social choice theory from the very beginning.

A typical (but not the only) problem studied in social choice theory is preference aggregation.

The Majority Rule

Why are social choice theorists and logicians interested in preference aggregation?

Example

Donald Trump winned the 2016 election due to Electoral College although Hillary Clinton winned 2,864,974 more votes than him.

What preference order would best represent the collective view of the group? An approach that suggests itself is to use the **majority rule**: rank x above y if and only if a majority of the individuals do, and similarly for all other pairs of alternatives.

But...

Condorcet Paradox

Example

(Condorcet Paradox) Suppose five individuals each declare their preferences by providing a ranking of the elements of a set of alternatives X = x, y, z, as follows:

Individual 1:	$x \succ y \succ z$
Individual 2:	$x \succ y \succ z$
Individual 3:	$y \succ z \succ x$
Individual 4:	$z \succ y \succ x$
Individual 5:	$z \succ x \succ y$

What should the collective preference order be by the mojority rule?

Condorcet Paradox

Example

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Individual 1:	$\mathbf{x} \succ \mathbf{y} \succ z$
Individual 2:	$\mathbf{x} \succ \mathbf{y} \succ z$
Individual 3:	$\mathbf{y} \succ z \succ \mathbf{x}$
Individual 4:	$z \succ \mathbf{y} \succ \mathbf{x}$
Individual 5:	$z \succ \mathbf{x} \succ \mathbf{y}$

$x \succ y$

Condxorcet Paradox

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Individual 2:	$x \succ y \succ z$
Individual 3:	$y \succ z \succ x$
Individual 4:	$z \succ y \succ x$
Individual 5:	$z \succ x \succ y$

$x \succ y, z \succ x$

Condorcet Paradox

Example

(Condorcet Paradox) Suppose five individuals each declare their preferences by providing a ranking of the elements of a set of alternatives X = x, y, z, as follows:

Individual 1:	$x \succ \mathbf{y} \succ \mathbf{z}$
Individual 2:	$x \succ \mathbf{y} \succ \mathbf{z}$
Individual 3:	$\mathbf{y} \succ \mathbf{z} \succ x$
Individual 4:	$\mathbf{z} \succ \mathbf{y} \succ x$
Individual 5:	$\mathbf{z} \succ x \succ \mathbf{y}$

$x \succ y$, $z \succ x$, $y \succ z$

It turns out $\ldots x \succ y \succ z \succ x \ldots$.

Axiomatic Method

The question now arises whether there are better methods of aggregation than the majority rule. Social choice theorists have approached this question using the so-called **axiomatic method**. This method amounts to formulating normatively desirable properties of aggregation rules as “axioms” in a mathematically rigorous manner, to then obtain precise characterisations of the aggregation rules that satisfy these properties.

What properties do we want?
Condorce Candidate, **Independence**, **Monoticity** etc.
Social choice theorists care about the relations between those properties.

Axiomatic Method

The best known example is the Impossibility Theorem from Kenneth J. Arrow's seminal work "Social Choice and Individual Values", originally published in 1951 (Arrow, 1963).

Arrow argued that any acceptable method of aggregation should satisfy at least the following two axioms:

1. **Pareto:** If every individual ranks x above y, then so should society.
2. **Independence:** It should be possible to determine the relative social ranking of x and y by considering only the relative ranking of x and y supplied by each of the individuals.

Axiomatic Method

Arrow then proved a truly astonishing result: If there are at least three alternatives, then the only kind of mechanism that will respect both of our axioms and that will return a collective preference that is a linear order for any combination of individual preferences is a **dictatorship**, i.e., a function that simply copies the preferences of a fixed individual and returns it as the collective preference! In other words, satisfying both axioms and the requirement of being nondictatorial is **impossible**.

Question

Dose the majortity rule satisfy Arrow's axioms? If so, who is the dictator in previous example ?

Axiomatic Method

The Majority Rule

Rank x above y if and only if a majority of the individuals do, and similarly for all other pairs of alternatives.

Arrow's Axioms

1. If every individual ranks x above y, then so should society.
2. It should be possible to determine the relative social ranking of x and y by considering only the relative ranking of x and y supplied by each of the individuals.

The majority rule does satisfy both of these requirements.

Axiomatic Method

Example

(Condorcet Paradox) Suppose five individuals each declare their preferences by providing a ranking of the elements of a set of alternatives X = x, y, z, as follows:

Individual 1:	$x \succ y \succ z$
Individual 2:	$x \succ y \succ z$
Individual 3:	$y \succ z \succ x$
Individual 4:	$z \succ y \succ x$
Individual 5:	$z \succ x \succ y$

Given $\ldots x \succ y \succ z \succ x \ldots$, who is the dictator? It needs more clarifications of Arrow's Theorem in a formal way.

Axiomatic Method

The axiomatic method often makes reference to notions from logic, albeit only in an informal manner. For instance, the notion of axiom used here is inspired by, although different from, the use of the term in mathematical logic, and most of the results we will discuss establish the "logical inconsistency" of certain requirements. And that preferences are modelled as binary relations also provides a bridge to formal logic.

1. Introduction
   - Condorcet Paradox
   - Axiomatic Method
2. The Axiomatic Method in Social Choice Theory
   - Basics
   - Social Welfare Function: Arrow's Theorem
   - Social Choice Functions: Sen, Muller and Satterthwaite
3. A Logic for Social Choice Theory
   - A Logic for SWFs
   - Judgement Aggregation Logic

Basics

- Let $\mathcal{N} = \{i_1, \ldots, i_n\}$ be a finite set of (at least two) **individuals** (or voters, or agents).
- Let $\mathcal{X} = \{x_1, x_2, x_3, \ldots\}$ be a (not necessarily finite) nonempty set of **alternatives** (or candidates).
- Each voter in $\mathcal{N}$ is endowed with, and will be asked to express, a preference over the alternatives in $\mathcal{X}$.

Remark

Preferences are either linear or weak orders on $\mathcal{X}$. A linear order is a binary relation that is irreflexive, transitive, and complete, while a weak order is reflexive, transitive, and complete. We assume preferences are linear orders, but all definitions given and results proven can be adapted to the case of weak orders.

Basics

- Let $\mathcal{N} = \{i_1, \ldots, i_n\}$ be a finite set of (at least two) **individuals** (or voters, or agents).
- Let $\mathcal{X} = \{x_1, x_2, x_3, \ldots\}$ be a (not necessarily finite) nonempty set of **alternatives** (or candidates).
- Each voter in $\mathcal{N}$ is endowed with, and will be asked to express, a **linear oder** over the alternatives in $\mathcal{X}$.
- Let $\mathcal{L}(\mathcal{X})$ denote the set of all linear orders on $\mathcal{X}$.
- A **profile R** $= (R_1, \ldots, R_n) \in \mathcal{L}(\mathcal{X})^{\mathcal{N}}$.
- $N^{\mathbf{R}}_{x \succ y}$ denotes the set of individuals that rank x above y under **R**.

Social Welfare Function

The first type of preference aggregation mechanism we consider are functions that map a profile of preference orders to a single (collective) preference order. Such a function is called a **social welfare function** (SWF). Formally, a SWF is a function $F : \mathcal{L}(\mathcal{X})^{\mathcal{N}} \to \mathcal{L}(\mathcal{X})$. We assume F is total.

Let us now give a precise account of the two axioms mentioned in the introduction above, which Arrow (1963) argued to be basic requirements for any acceptable SWF.

Pareto Condition

The first is a fundamental principle in economic theory, due to the Italian economist Vilfredo Pareto (1848–1923), that states that if x is at least as good as y for all and strictly better for some members of a society, then x should be socially preferred to y. Given that we assume that preferences are strict, i.e., no individual will be indifferent between two distinct alternatives, this simplifies to asking that x should be socially preferred to y if everybody prefers x to y.

Pareto Condition

A SWF F satisfies the **Pareto condition** if, for all profile $\mathbf{R}$ and pair (x, y), $N^{\mathbf{R}}_{x \succ y} = N$ implies $(x, y) \in F(\mathbf{R})$.

Independence of Irredlevant Alternatives

In addition to the Pareto condition, a widely accepted standard requirement, Arrow proposed an independence axiom that states that the relative social ranking of two alternatives should not change when an individual updates her preferences regarding a third alternative. That is, social choices should be independent of irrelevant alternatives.

Independence of Irrelevant Alternatives (IIA)

A SWF F satisfies **IIA** if the relative social ranking of two alternatives only depends on their relative individual rankings: for all $\mathbf{R}, \mathbf{R}'$ and (x, y), $N^{\mathbf{R}}_{x \succ y} = N^{\mathbf{R}'}_{x \succ y}$ implies $(x, y) \in F(\mathbf{R}) \iff (x, y) \in F(\mathbf{R}')$.

Arrow's Theorem

Dictatorship

Any dictatorship $i \in N$ will map any profile $\mathbf{R}$ to the dictator's reported ranking $\mathbf{R}_i$.

Theorem (Arrow, 1951)

Any SWF for three or more alternatives that satisfies the Pareto condition and IIA must be a dictatorship.

Now we can answer why we cannot talk about the dictator in previous example.

Proof of Arrow's Theorem

Proof Strategy

Our proof broadly follows Sen (1986) and is based on the idea of "decisive coalitions". The main idea of the proof is to show that, whenever some coalition G (with $|G| \geq 2$) is decisive, then there exists a nonempty $G' \subset G$ that is decisive as well. Given the finiteness of $\mathcal{N}$, this means that F dictatorial. Actually, $\mathcal{N}$ is decisive due to **Pareto Condition**.

Decisive Coalition

Let us call a coalition $G \subseteq \mathcal{N}$ decisive on alternatives x, y if for any $\mathbf{R}$, $G \subseteq N^{\mathbf{R}}_{x \succ y}$ entails $(x, y) \in F(\mathbf{R})$. When G is decisive on all pairs of alternatives, then we simply say that G is decisive.

Contraction Lemma

Lemma (Contraction Lemma)

Let $G \subseteq \mathcal{N}$ with $|G| \geq 2$. For any G_1 and G_2 s.t. $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \emptyset$, if G is decisive, then either G_1 or G_2 is decisive as well.

This concludes the proof of the theorem: repeated application of the Contraction Lemma will produce a dictator.

Proof of Contraction Lemma

Consider a profile where all individuals in G_1 rank $x \succ y \succ z$, all individuals in G_2 rank $y \succ z \succ x$, and all others rank $z \succ x \succ y$. As G is decisive, we have $y \succ z$ in the social ranking. We distinguish two cases: society ranks $x \succ z$ and society ranks $z \succ x$.

Contraction Lemma

Proof Cont.

1. Society ranks x above z. Note that it is exactly the individuals in G_1 that rank x above z. Thus, by **IIA**, in any profile **R** where exactly the individuals in G_1 rank x above z, society will do the same. Does this mean that G_1 is decisive? Yes, but we need to prove it later.
2. Society ranks z above x. Society ranks z above x, and thus y above x. As exactly the individuals in G_2 rank y above x, by the same kind of argument as above, G_2 must be decisive.

□

Decisiveness Lemma

Lemma (Decisiveness Lemma)

If there is a pair (x, y) s.t. for every $\mathbf{R}$, $N^{\mathbf{R}}_{x \succ y} = G$ implies $(x, y) \in F(\mathbf{R})$, then G is decisive on any given pair (x^*, y^*).

Note that the antecedent of Decisiveness Lemma is stronger than decisive condition.

We prove the case where x, y, x^*, y^* are all distinct (the other cases are similar).

Decisiveness Lemma

Proof.

Suppose individuals in G rank $x^* \succ y^*$ under **R**. We consider a special $\mathbf{R}'$ under which every individual has the same preference over x^* and y^* with respect to **R**.

R	G	$\overline{G}$
	x^*	
	y^*	

$\mathbf{R}'$	G	$\overline{G}$		
	x^*			
	x	x^*	y	y
	y	x	y^*	x
	y^*			

□

Decisiveness Lemma

Proof.

Suppose individuals in G rank $x^* \succ y^*$ under $\mathbf{R}$. We consider a special $\mathbf{R}'$ under which every individual has the same preference over x^* and y^* with respect to $\mathbf{R}$.

$\mathbf{R}$	G	$\overline{G}$
	x^*	
	y^*	

$\mathbf{R}'$	G	$\overline{G}$		
	x^*			
	$\mathbf{x}$	x^*	y	$\mathbf{y}$
	$\mathbf{y}$	x	y^*	$\mathbf{x}$
	y^*			

Society rank $x \succ y$ under $\mathbf{R}'$ by antecedent of Decisiveness Lemma. □

Decisiveness Lemma

Proof.

Suppose individuals in G rank $x^* \succ y^*$ under **R**. We a special **R**′ under which every individual has the same preference over x^* and y^* with respect to **R**.

R	G	$\overline{G}$
	x^*	
	y^*	

R′	G	$\overline{G}$		
	x^*			
	x	x^*	y	y
	y	x	y^*	x
	y^*			

$x \succ y$. Society rank $x^* \succ x$ and $y \succ y^*$ under **R**′ by **Pareto**. □

Decisiveness Lemma

Proof.

Suppose individuals in G rank $x^* \succ y^*$ under $\mathbf{R}$. We consider a special $\mathbf{R}'$ under which every individual has the same preference over x^* and y^* with respect to $\mathbf{R}$.

$\mathbf{R}$	G	$\overline{G}$
	x^*	
	y^*	

$\mathbf{R}'$	G	$\overline{G}$		
	x^*			
	x	x^*	y	y
	y	x	y^*	x
	y^*			

By transitivity, We have $x^* \succ x \succ y \succ y^*$. Thus $x^* \succ y^*$ under $\mathbf{R}'$. □

Decisiveness Lemma

Proof.

Suppose individuals in G rank $x^* \succ y^*$ under **R**. We consider a special **R**$'$ under which every individual has the same preference over x^* and y^* with respect to **R**.

R	G	$\overline{G}$
	x^*	
	y^*	

R$'$	G	$\overline{G}$		
	x^*			
	x	x^*	y	y
	y	x	y^*	x
	y^*			

Society rank $x^* \succ y^*$ under **R**$'$. By **IIA**, society rank $x^* \succ y^*$ under **R**. □

Another Proof via Ultrafilter

Several alternative proofs for Arrow's Theorem may be found in the literature (Geanakoplos, 2005). We want to briefly mention one such proof here, due to Kirman and Sondermann (1972), which reduces Arrow's Theorem to a well- known fact in the theory of ultrafilters. Given the importance of ultrafilters in model theory and set theory, this proof provides additional evidence for the close connections between logic and social choice theory.

Another Proof via Ultrafilter

Definition

An **ultrafilter** $\mathcal{G}$ for a set $\mathcal{N}$ is a set of subsets of $\mathcal{N}$ satisfying the following conditions:

1. The empty set is not included: $\emptyset \notin \mathcal{G}$.
2. If $G_1 \subseteq G_2$ and $G_1 \in \mathcal{G}$, then $G_2 \in \mathcal{G}$
3. $\mathcal{G}$ is closed under intersection: if $G_1 \in \mathcal{G}$ and $G_2 \in \mathcal{G}$, then $G_1 \cap G_2 \in \mathcal{G}$.
4. $\mathcal{G}$ is maximal: for all $G \subseteq \mathcal{N}$, either $G \in \mathcal{G}$ or $(\mathcal{N} \setminus G) \in \mathcal{G}$.

Let us now interpret $\mathcal{N}$ as a set of individuals and $\mathcal{G}$ as the set of decisive coalitions for a given SWF satisfying the Pareto condition and IIA. It turns out that $\mathcal{G}$ satisfies the four conditions above, i.e., it is an ultrafilter(principle).

The Way to Find the Dictator (Geanakoplos, 2005)

Find an arbitrary alternative x and a profile in which all individuals rank x at the very bottom. Then move x to the very top from individual one to the last individual. If i is the first individual whose change causes the social ranking of x to change, then i is the dictator under such SWF.

It is also another method to prove Arrow's Theorem.

Question

What makes the dictator be a dictator?

More on Arrow's Theorem

Arrow's Theorem may be read either as a **characterisation** of dictatorships in terms of the axioms of **Pareto** and **IIA**, or as an **impossibility theorem**: it is impossible to devise a SWF for three or more alternatives that is Pareto efficient, independent, and nondictatorial.

More on Arrow's Theorem

Observe that we have made explicit use of both the assumption that there are at least three alternatives and the assumption that the set of individuals is finite. Indeed, if either assumption is dropped, then Arrow's Theorem ceases to hold: First, for two alternatives, the majority rule, which returns the ranking made by the majority of individuals (with ties broken in favour of, say, the first alternative), satisfies both the Pareto condition and IIA and clearly is not dictatorial. Second, for an infinite number of individuals, Fishburn (1970) has shown how to design a nondictatorial SWF that is Pareto efficient and independent. Whether or not the set of alternatives is infinite is uncritical.

Social Choice Function

A **social choice function** SCF is a function $F : \mathcal{L}(\mathcal{X})^{\mathcal{N}} \to 2^{\mathcal{X}} \setminus \{\emptyset\}$ mapping profiles of linear orders on alternatives to nonempty sets of alternatives. Intuitively, for a given profile of declared preferences, F will choose the "best" alternatives. If F always returns a singleton, then F is called resolute. We can think of a SCF as a voting rule, mapping profiles of ballots cast by the voters to winning candidates.

Liberalism

The first result we shall review is Sen's Theorem on the Impossibility of a Paretian Liberal (Sen, 1970b). Sen introduced a new type of axiom, liberalism, which requires that for each individual there should be at least one pair of alternatives for which she can determine the relative social ranking (i.e., she should be able to ensure that at least one of them does not win).

Liberalism

A SCF F satisfies the axiom of **liberalism** if, for every individual $i \in \mathcal{N}$, there exist two distinct alternatives $x, y \in \mathcal{X}$ such that i is **two-way decisive** on x and y in the sense that whichever of the two i ranks lower cannot win: $i \in N^{\mathbf{R}}_{x \succ y}$ implies $y \notin F(\mathbf{R})$ and $i \in N^{\mathbf{R}}_{y \succ x}$ implies $x \notin F(\mathbf{R})$.

Pareto Condition

The second axiom required to state the theorem is again the **Pareto condition**, which takes the following form in the context of SCFs:

Pareto

A SCF F satisfies the **Pareto condition** if, whenever all individuals rank x above y then y cannot win: $N^{\mathbf{R}}_{x \succ y} = \mathcal{N}$ implies $y \notin F(\mathbf{R})$.

Sen's Theorem

Theorem (Sen, 1970)

No SCF satisfies both liberalism and the Pareto condition.

Proof Strategy

For the sake of contradiction, suppose there exists a SCF F satisfying both liberalism and the Pareto condition. Let i_1 and i_2 be two distinguished individuals, let x_1 and y_1 be the alternatives on which i_1 is two-way decisive, and let x_2 and y_2 be the alternatives on which i_2 is two-way decisive. We shall derive a contradiction for the case where x_1, y_1, x_2, y_2 are pairwise distinct.

Proof of Sen's Theorem

Proof.

Consider a profile with the following properties:

1. Individual i_1 ranks x_1 above y_1.
2. Individual i_2 ranks x_2 above y_2.
3. All individuals rank y_1 above x_2 and also y_2 above x_1 .
4. All individuals rank x_1, x_2, y_1, y_2 at the top.

Due to liberalism, (1) rules out y_1 as a winner and (2) rules out y_2 as a winner. Due to the Pareto condition, (3) rules out x_1 and x_2 as winners and (4) rules out all other alternatives as winners. As a SCF must return a nonempty set of winners, we have thus derived a contradiction and are done. □

Monotonicity

We now turn to yet another type of axiom: **monotonicity**. Intuitively, a SCF is monotonic if any additional support for a winning alternative will benefit that alternative. Somewhat surprisingly, not all commonly used voting rules do satisfy this property.

Example (Failure of monotonicity under plurality with runoff)

6 voters:	$x \succ z \succ y$
5 voters:	$y \succ x \succ z$
6 voters:	$z \succ y \succ x$

x and z will make it into the second round where x will beat z.

Monotonicity

We now turn to yet another type of axiom: **monotonicity**. Intuitively, a SCF is monotonic if any additional support for a winning alternative will benefit that alternative. Somewhat surprisingly, not all commonly used voting rules do satisfy this property, including some SWFs.

Example (Failure of monotonicity under plurality with runoff)

6 voters:	$x \succ z \succ y$
5 voters:	$y \succ x \succ z$
6 voters:	$z \succ y \succ x$

8 voters:	$x \succ z \succ y$
5 voters:	$y \succ x \succ z$
4 voters:	$z \succ y \succ x$

Even though x received additional support, x and y will make it into the second round where y will beat x.

Weak Monotonicity

Intuitively, monotonicity requires that whenever an alternative x is amongst the winners and some individuals raise x in their linear orders without affecting the relative rankings of any other pairs of alternatives, then x should also be a winner.

Weak Monotonicity

A SCF F satisfies **weak monotonicity** if $x \in F(\mathbf{R})$ implies $x \in F(\mathbf{R}')$ for any alternative x and distinct profiles $\mathbf{R}$ and $\mathbf{R}'$ with $N^{\mathbf{R}}_{x \succ y} \subseteq N^{\mathbf{R}'}_{x \succ y}$ and $N^{\mathbf{R}}_{y \succ z} \subseteq N^{\mathbf{R}'}_{y \succ z}$ for all $y, z \in \mathcal{X} \setminus \{x\}$.

As Example above has demonstrated, plurality with runoff violates weak monotonicity.

Strong Monotonicity

Muller-Satterthwaite Theorem (Muller and Satterthwaite, 1977) shows how a stronger form of monotonicity can lead to an impossibility similar to Arrow's Theorem. This result applies to **resolute** SCFs. If F is resolute, we shall simply write $x = F(\mathbf{R})$ rather than $x \in F(\mathbf{R})$ to indicate that x is the winner under profile $\mathbf{R}$.

Strong Monotonicity

A resolute SCF F satisfies **strong monotonicity** if $x = F(\mathbf{R})$ implies $x = F(\mathbf{R}')$ for any alternative x and distinct profiles $\mathbf{R}$ and $\mathbf{R}'$ with $N^{\mathbf{R}}_{x \succ y} \subseteq N^{\mathbf{R}'}_{x \succ y}$ for all $y \in \mathcal{X} \setminus \{x\}$.

Muller and Satterthwaite's Theorem

One further axiom we require is **surjectivity**. F is surjective if it does not rule out certain alternatives as a possible winners from the outset: for every $x \in \mathcal{X}$ there exists a profile $\mathbf{R}'$ such that $F(\mathbf{R}') = x$. Finally, a resolute SCF F is **dictatorial** if there exists an individual $i \in \mathcal{N}$ such that the winner under F is always the top-ranked alternative of i.

Theorem (Muller and Satterthwaite, 1977)

Any resolute SCF for three or more alternatives that is surjective and strongly monotonic must be a dictatorship.

Proof of Muller and Satterthwaite's Theorem

Proof Strategy

We will show that any resolute SCF that is surjective and strongly monotonic must also satisfy the Pareto condition and an independence property similar to IIA, thereby reducing the claim to a variant of Arrow's Theorem for resolute SCFs.

Independence

If $F(\mathbf{R}) = x$, $x \neq y$, and $N^{\mathbf{R}}_{x \succ y} = N^{\mathbf{R}'}_{x \succ y}$ then $F(\mathbf{R}') \neq y$.

Pareto

A SCF F satisfies the **Pareto condition** if, whenever all individuals rank x above y then y cannot win: $N^{\mathbf{R}}_{x \succ y} = \mathcal{N}$ implies $y \notin F(\mathbf{R})$.

Summary

	Arrow	Sen	Muller
F	SWF	SCF	resolute SCF
$\mathcal{N}$	finite	≥ 2	≥ 2
$\mathcal{X}$	≥ 3	≥ 3	≥ 3
conditions	Pareto, IIA	Pareto, Liberalism	Pareto, Independence
result	dictatorial	non-exist	dictatorial

So far we do not distinguish between the preferences an individual declares when reporting to an aggregation mechanism and the true preferences of that individual. Some researchers do care about this distinction and they have found some results on **strategic manipulation** (Gibbard and Satterthwaite, 1973/1975).

What Do the Impossibility Results Say?

Marc Pauly views the impossibility results as definability results of corresponding classes of models (Pauly, 2008).

- Given a **semantic domain** $\mathcal{D}$ and a target class $\mathcal{T} \subseteq \mathcal{D}$
- Fix a language $\mathcal{L}$ and a satisfaction relation $\models \subseteq \mathcal{D} \times \mathcal{L}$
- $\Delta \subseteq \mathcal{L}$ be a set of axioms

Δ axiomatizes $\mathcal{T}$ iff for all $\mathcal{M} \in \mathcal{D}$, $\mathcal{M} \in \mathcal{D}$ iff $\mathcal{M} \models \Delta$.

Arrow's Theorem

Δ is the set of SWFs w.r.t. 3 or more candidates, $\mathcal{T}$ is the class of dictatorships, $\mathcal{L}$ is the given language. Δ is the properties of Arrow's theorem, then Δ axiomatizes $\mathcal{T}$.

We now give an example of given language $\mathcal{L}$ that can express Arrow's framework.

A Logic for SWFs

Ågotnes et al. (2011) define a modal logic for reasoning about SWFs. The language of this logic is parametric in $\mathcal{N}$(individuals) and $\mathcal{X}$ (alternatives).

$$\Pi = \{p_i \mid i \in \mathcal{N}\} \cup \{q_{(x,y)} \mid x, y \in \mathcal{X}\} \cup \{\sigma\}$$
$$\phi := \alpha \mid [\mathrm{PROF}]\phi \mid [\mathrm{PAIR}]\phi \mid \phi \wedge \phi \mid \neg\phi,\ \alpha \in \Pi$$

A model is a triplet $(\mathbf{R}, (x, y), F)$.

$$\begin{array}{lcl}
\mathbf{R}, (x, y), F \models p_i & \Leftrightarrow & \text{i ranks } x \succ y \text{ under } \mathbf{R} \\
\mathbf{R}, (x, y), F \models q_{(x',y')} & \Leftrightarrow & (x, y) = (x', y') \\
\mathbf{R}, (x, y), F \models \sigma & \Leftrightarrow & (x, y) \in F(\mathbf{R}) \\
\mathbf{R}, (x, y), F \models [\mathrm{PROF}]\phi & \Leftrightarrow & \forall \mathbf{R}'\ \mathbf{R}', (x, y), F \models \phi \\
\mathbf{R}, (x, y), F \models [\mathrm{PAIR}]\phi & \Leftrightarrow & \forall (x', y')\ \mathbf{R}, (x', y'), F \models \phi
\end{array}$$

bollean connectives as usual

A Logic for SWFs

We are now able to express properties of the SWF F.

$$\text{Pareto} := \ [\text{PROF}][\text{PAIR}](p_1 \wedge \cdots \wedge p_n \rightarrow \sigma)$$

That is, in every state $\mathbf{R}, (x, y)$ it must be the case that, whenever all individuals rank $x \succ y$ (i.e., all p_i are true), then also the collective preference will rank $x \succ y$(i.e., σ is true).

A Logic for SWFs

We are now able to express properties of the SWF F.

$$\begin{aligned}\text{Pareto} &:= [\text{PROF}][\text{PAIR}](p_1 \wedge \cdots \wedge p_n \to \sigma)\\ \text{IIA} &:= [\text{PROF}] \bigwedge_{o \in O} [\text{PAIR}]((o \wedge \sigma) \to [\text{PROF}](o \to \sigma))\end{aligned}$$

Let an outcome o be a maximal conjunction of literals $k_1 \wedge \ldots k_n$, where each k_i is either p_i or $\neg p_i$. The set O is the set of all possible outcomes. IIA says for any profile $\mathbf{R}$, any pair (x, y) and any outcome o considering x and y under $\mathbf{R}$, when $(x, y) \in F(\mathbf{R})$, for any other profile $\mathbf{R}'$ with same outcome o w.r.t. x and y (i.e. the voters' preferences over x and y do not change), $(x, y) \in F(\mathbf{R}')$ holds (i.e. σ is ture).

A Logic for SWFs

We are now able to express properties of the SWF F.

$$\begin{aligned} \text{Pareto} &:= [\text{PROF}][\text{PAIR}](p_1 \wedge \cdots \wedge p_n \rightarrow \sigma) \\ \text{IIA} &:= [\text{PROF}] \bigwedge_{o \in O} [\text{PAIR}]((o \wedge \sigma) \rightarrow [\text{PROF}](o \rightarrow \sigma)) \\ \text{Dict} &:= \bigvee_{i \in \mathcal{N}} [\text{PROF}][\text{PAIR}](p_i \leftrightarrow \sigma) \end{aligned}$$

The formula expresses that there exists an individual i (the dictator) such that, to whichever state $\mathbf{R}, (x, y)$ we move in terms of the profile (by application of [PROF]) and the pair of alternatives under consideration (by application of [PAIR]), it will be the case that the collective preference will rank $x \succ y$ (i.e., σ will be true) if and only if individual i ranks $x \succ y$ (i.e., p_i is true).

A Logic for SWFs

We are now able to express properties of the SWF F.

$$\text{Pareto} := [\text{PROF}][\text{PAIR}](p_1 \wedge \cdots \wedge p_n \to \sigma)$$

$$\text{IIA} := [\text{PROF}] \bigwedge_{o \in O} [\text{PAIR}]((o \wedge \sigma) \to [\text{PROF}](o \to \sigma))$$

$$\text{Dict} := \bigvee_{i \in \mathcal{N}} [\text{PROF}][\text{PAIR}](p_i \leftrightarrow \sigma)$$

$$\text{MT2} := \neg[\text{PROF}]\neg(\neg[\text{PAIR}]\neg(p_1 \wedge p_2) \wedge \neg[\text{PAIR}]\neg(p_1 \wedge \neg p_2))$$

Intuitively, MT2 ('more than 2 (alternatives)') says that there is a profile and two pairs such that both the p_1 and p_2 rank $x \succ y$, and p_1 ranks $x' \succ y'$ but p_2 ranks $y' \succ x'$.

A Logic for SWFs

We are now able to express properties of the SWF F.

$$\begin{aligned}
\text{Pareto} &:= [\text{PROF}][\text{PAIR}](p_1 \wedge \cdots \wedge p_n \rightarrow \sigma) \\
\text{IIA} &:= [\text{PROF}] \bigwedge_{o \in O} [\text{PAIR}]((o \wedge \sigma) \rightarrow [\text{PROF}](o \rightarrow \sigma)) \\
\text{Dict} &:= \bigvee_{i \in \mathcal{N}} [\text{PROF}][\text{PAIR}](p_i \leftrightarrow \sigma) \\
\text{MT2} &:= \neg[\text{PROF}]\neg(\neg[\text{PAIR}]\neg(p_1 \wedge p_2) \wedge \neg[\text{PAIR}]\neg(p_1 \wedge \neg p_2)) \\
\text{Arrow} &:= (\text{MT2} \wedge \text{Pareto} \wedge \text{IIA}) \rightarrow \text{Dictatorship}
\end{aligned}$$

Ågotnes et al. give a sound and complete axiomatization of this logic. It follows that the theorem is derivable in the logic. We now present the original and general logic of judgement aggregation given by them.

Basics Again

- Let $\mathcal{N} = \{i_1, \ldots, i_n\}$ be a set of (at least two) **individuals** (or voters, or agents).
- Let $\mathcal{A}$ be a nonempty set (not necessarily finite) of **agendas**.
- Each individual in $\mathcal{N}$ is endowed with, and will be asked to express a **judgment set** R_i that is complete and consistent over $\mathcal{A}$.
- Let $\mathcal{L}(\mathcal{A})$ denote the set of all complete and consistent individual judgment set over $\mathcal{A}$.
- A **judgement profile** $\mathbf{R} = (R_1, \ldots, R_n) \in \mathcal{L}(\mathcal{A})^{\mathcal{N}}$.
- A **judgment aggregation rule** (JAR) is a function F that maps each judgment profile $\mathbf{R} = (R_1, \ldots, R_n)$ to a complete and consistent collective judgment set $F(\mathbf{R}) \in \mathcal{L}(\mathcal{A})$.

Judgement Aggregation Logic (JAL)

The language of Judgment Aggregation Logic (JAL) is parameterised by a set of agents $\mathcal{N} = \{i_1, i_2, \ldots, i_n\}$ (we will assume that there are at least two agents) and an agenda $\mathcal{A}$.

$$\Pi = \{p_i \mid i \in \mathcal{N}\} \cup \{\mathbf{h}_q \mid q \in \mathcal{A}\} \cup \{\sigma\}$$
$$\phi := \alpha \mid \Box\phi \mid \blacksquare\phi \mid \phi \wedge \phi \mid \neg\phi,\ \alpha \in \Pi$$

A model is a triplet $(\mathbf{R}, q, F)$.

$$\begin{array}{lcl}
\mathbf{R}, q, F \models p_i & \Leftrightarrow & q \in R_i \\
\mathbf{R}, q, F \models \mathbf{h}_{q'} & \Leftrightarrow & q = q' \\
\mathbf{R}, q, F \models \sigma & \Leftrightarrow & q \in F(\mathbf{R}) \\
\mathbf{R}, q, F \models \Box\phi & \Leftrightarrow & \forall \mathbf{R}' \;\; \mathbf{R}', q, F \models \phi \\
\mathbf{R}, q, F \models \blacksquare\phi & \Leftrightarrow & \forall q' \;\; \mathbf{R}, q, F \models \phi
\end{array}$$

bollean connectives as usual

Axiomatization

We define $\Diamond\phi := \neg\Box\neg\phi$, $\blacklozenge := \neg\blacksquare\neg\phi$ and $x \in \{\sigma\} \cup \{p_i \mid i \in \mathcal{N}\}$.

$\neg(\mathbf{h}_q \wedge \mathbf{h}'_q)$	$q \neq q'$	Atmost
$\bigvee_{q\in\mathcal{A}} \mathbf{h}_q$		Atleast
$\blacklozenge\mathbf{h}_q$		Agenda
$\blacklozenge(\mathbf{h}_q \wedge \phi) \rightarrow \blacksquare(\mathbf{h}_q \rightarrow \phi)$		Once
$\blacklozenge(\mathbf{h}_q \wedge x) \vee \blacklozenge(\mathbf{h}'_q \wedge x)$		CpJS
all instantiations of propositional tautologies		
K, T, 4, 5 for two modalities		
$\vdots$		
MP, Nec for two modalities		
if $q_1, \dots q_n \vdash q$, then		
$\vdash (\blacklozenge(\mathbf{h}_{q_1} \wedge x) \wedge \cdots \wedge \blacklozenge(\mathbf{h}_{q_n} \wedge x))$		
$\rightarrow (\blacksquare(\mathbf{h}_q \rightarrow x) \wedge \blacksquare(\mathbf{h}'_q \rightarrow \neg x))$		Closure

Axiomatization

Completeness

If the agenda is finite, we have $\vdash \phi$ iff $\vDash \phi$.

- JAL is simple, intuitive and general.
- A limitation of this logic is that we need to fix $\mathcal{N}$ before we can start writing down formulas.
- Arrow's theorem that has been formalized is actually weaker than the original theorem.
- The fact that Arrow's Theorem ceases to hold when we move to an infinite electorate cannot be modelled in this logic.

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