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Logic of primary-conditionals and secondary-conditionals

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Abstract Firstly, the authors analyzed the properties of primary-conditionals and secondary-conditionals, establish the minimum system $C2L_m$ of primary-conditionals and secondary-conditionals, and then prove some of the formal theorems of the system which have important intuitive meanings. Secondly, the authors constructed the neighborhood semantics, prove the soundness of $C2L_m$, introduce a general concept of canonical model by the neighborhood semantics, and then prove the completeness of $C2L_m$ by the canonical model. Finally, according to the technical results of the minimum system $C2L_m$, the authors discuss some of the important problems concerning primary-conditionals and secondary-conditionals.

Keywords logic of conditionals, primary-conditionals and secondary-conditionals, neighborhood semantics, canonical model, monotonicity

Conditionals are characterized by the relationship between conditions and results. An ordinary logic of conditionals only characterizes results under a single condition, and the conjunction is applied in the cases of more than one conditions. However, sometimes this is not proper, because different condi-

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tions may be of different functions to achieve results, and the way to conjoin them ignores the differences. Therefore, we need to directly deal with conditionals with multi-conditions.

We will discuss conditionals with two different conditions—the primary and the secondary. Therefore, we have our logic named as the logic of primary-conditionals and secondary-conditionals (hereafter, LPSC).

1 Basic ideas

To consider two conditions and a single result at a time, we need a ternary connective \geq . The meaning of $\alpha\beta \geq \gamma$ is:

- (i) conditions α and β lead to the result γ ,
- (ii) α is the primary condition, whereas, β is the secondary condition.

Note that (ii) did not exclude the case that α is the secondary condition and β is the primary one, and we can have them at the same time. So \geq characterizes a more general two-conditional.

However, we can use the two-conditional to define the conditional in which the two conditions are strictly distinguished. $(\alpha\beta \geq \gamma) \wedge \neg((\beta\alpha \geq \gamma))$ serves as an example, which confirms that α is only the primary condition and β only the secondary condition.

With the two-conditional, we can also define the conditional with only a single condition: $\alpha > \gamma$ is defined by $\alpha\alpha \geq \gamma$.

Item (i) says that, α and β lead to γ , which does not mean that γ cannot be achieved from α or β alone. The difference between primary and secondary conditions can be seen in that case.

If two conditions lead to one result, and one of the conditions can lead to a result alone, then the other condition which cannot singly lead to a result cannot be the primary condition.

We conclude from the hypothesis above that, if $\alpha\beta \geq \gamma$ holds, $\neg(\alpha > \gamma)$ and $\beta > \gamma$ cannot both hold. That is to say, if both $\alpha\beta \geq \gamma$ and $\beta > \gamma$ hold, $\alpha > \gamma$ also holds. Therefore, we have the following axiom:

$$(\alpha\beta \geq \gamma) \wedge (\beta > \gamma) \rightarrow (\alpha > \gamma).$$

Nevertheless, in the case that $\alpha\beta \geq \gamma$ and $\alpha > \gamma$ both hold, $\beta > \gamma$ does not necessarily hold. Thus, the formula

$$(\alpha\beta \geq \gamma) \wedge (\alpha > \gamma) \rightarrow (\beta > \gamma)$$

is not an axiom.

LPSC with the axiom above is called a *minimal* LPSC. Certainly, primary and secondary conditionals are still conditionals, so they have the properties of conditionals.

In our opinion (Liu 1999, pp. 58–64), to take $>$ as implication we de-

mand at least:

- (i) **Truth preserving:** If both α and $\alpha > \gamma$ are true, then γ is also true. This can be formalized as the axiom: $\alpha \wedge (\alpha > \gamma) \rightarrow \gamma$ (or $(\alpha > \gamma) \rightarrow (\alpha \rightarrow \gamma)$).
- (ii) **Completeness:** If in every case, α implies γ , then $\alpha > \gamma$ holds. α implying γ for all cases is just to say that $\alpha \rightarrow \gamma$ holds, hence completeness tells this deduction rule: $\alpha > \gamma$ follows from $\alpha \rightarrow \gamma$.

Conditionals can be regarded as necessary sentences with respect to the premises. $\alpha > \gamma$ can be taken as $\Box_{\alpha} \gamma$. (Mao 1995, pp. 255–273) In our opinion, a necessity operator demands at least:

- Monotonicity:** $\gamma \rightarrow \delta$ implies $\Box_{\alpha} \gamma \rightarrow \Box_{\alpha} \delta$,
- Conjunctivity:** $\Box_{\alpha} (\gamma \wedge \delta) \rightarrow \Box_{\alpha} \gamma \wedge \Box_{\alpha} \delta$ is an axiom. (Liu, 2002, pp. 72–76) Therefore, the condition implication demands at least:
- (iii) **Monotonicity of results:** $\gamma \rightarrow \delta$ implies $(\alpha > \gamma) \rightarrow (\alpha > \delta)$.
- (iv) **Conjunctivity of results:** $(\alpha > \gamma) \wedge (\alpha > \delta) \rightarrow (\alpha > \gamma \wedge \delta)$ is an axiom.
- (ii) implies $\alpha > \alpha$, and from $\alpha > \alpha$ as well as (iii) we obtain completeness. Hence, in the logic of conditionals, we often use the axiom $\alpha > \alpha$ instead of using the deduction rule: $\alpha > \gamma$ follows from $\alpha \rightarrow \gamma$.

Although the logic of conditionals does not necessarily satisfy the principle of compositionality (PC), we confine ourselves here to the logic satisfying PC. PC appears as the basic replacement theorem in systems. Classical connectives satisfy PC, and by monotonicity of results, so do the results. Hence, we just need to add the replacement principle of conditions.

- (v) **Replacement principle of conditions:** $\alpha_1 \leftrightarrow \alpha_2$ implies $(\alpha_1 > \gamma) \leftrightarrow (\alpha_2 > \gamma)$.
- We need to generalize $>$ to \geq . That is to generalize (i), (iii), (iv), and (v) to the following respectively:

- (i') $\alpha \wedge \beta \wedge (\alpha \beta \geq \gamma) \rightarrow \gamma$ is an axiom,
- (iii') $\gamma \rightarrow \delta$ implies $(\alpha \beta \geq \gamma) \rightarrow (\alpha \beta \geq \delta)$,
- (iv') $(\alpha \beta \geq \gamma) \wedge (\alpha \beta \geq \delta) \rightarrow (\alpha \beta \geq \gamma \wedge \delta)$ is an axiom,
- (v') $\alpha_1 \leftrightarrow \alpha_2$ and $\beta_1 \leftrightarrow \beta_2$ imply $(\alpha_1 \beta_1 \geq \gamma) \leftrightarrow (\alpha_2 \beta_2 \geq \gamma)$.

The generalization of “ $\alpha > \alpha$ is an axiom” is “ $\alpha \beta \geq \alpha$ is an axiom”. Note that $\alpha \beta \geq \beta$ is not an axiom, which shows a difference between primary and secondary conditions.

2 A minimal system of LPSC

The formal language of LPSC is the language of classical propositional logic (only \neg and \wedge as the connectives) together with the ternary connective \geq . We add to the formation rules of formulas:

If α, β, γ are formulas, then $\alpha \beta \geq \gamma$ is also a formula.

The set of all formulas is denoted by **Form**.

We use replacement $\alpha[\gamma/\beta]$ and substitution $\alpha(\beta_1/p_1, \dots, \beta_n/p_n)$ of a formula α in their usual meaning.

We define \vee , \rightarrow and \leftrightarrow as in the classical propositional logic. We define $>$, the *single condition implication*, as follows:

$$\alpha > \gamma =_{df} \alpha \alpha \geq \gamma.$$

The minimal system of LPSC, denoted by **C2L_m**, is stated as follows:

Axioms

- (i) All instances of tautologies;
- (ii) Axiom of truth preserving: $\alpha \wedge \beta \wedge (\alpha \beta \geq \gamma) \rightarrow \gamma$;
- (iii) Axiom of conjunctivity of results: $(\alpha \beta \geq \gamma) \wedge (\alpha \beta \geq \delta) \rightarrow (\alpha \beta \geq \gamma \wedge \delta)$;
- (iv) Axiom of self-implication of the primary condition: $\alpha \beta \geq \alpha$;
- (v) Basic axiom of primary and secondary conditions:
 $(\alpha \beta \geq \gamma) \wedge (\beta > \gamma) \rightarrow (\alpha > \gamma)$.

Deduction rules

- (i) Modus ponens (MP): β follows from α and $\alpha \rightarrow \beta$;
- (ii) Monotonicity of results: $(\alpha \beta \geq \gamma) \rightarrow (\alpha \beta \geq \delta)$ follows from $\gamma \rightarrow \delta$;
- (iii) Replacement of conditions: $(\alpha_1 \beta_1 \geq \gamma) \leftrightarrow (\alpha_2 \beta_2 \geq \gamma)$ follows from $\alpha_1 \leftrightarrow \alpha_2$ and $\beta_1 \leftrightarrow \beta_2$.

(In the notation **C2L_m**, **C2** stands for “two-conditional”, and the subscript **m** means “minimal”).

We use $\vdash \alpha$ to mean “ α is a theorem of **C2L_m**”, and the set of all **C2L_m** theorems is denoted by **Th(C2L_m)**.

The single condition implication defined in **C2L_m** meets the minimal demands of condition implication.

Theorem 2.1 (Properties of single condition implication)

- (i) Truth preserving: $\vdash (\alpha > \gamma) \rightarrow (\alpha \rightarrow \gamma)$;
- (ii) Self-implication: $\vdash \alpha > \alpha$;
- (iii) Conjunctivity of results: $\vdash (\alpha > \gamma) \wedge (\alpha > \delta) \rightarrow (\alpha > \gamma \wedge \delta)$;
- (iv) Monotonicity of results: if $\vdash \gamma \rightarrow \delta$, then $\vdash (\alpha > \gamma) \rightarrow (\alpha > \delta)$;
- (v) Completeness: if $\vdash \alpha \rightarrow \gamma$, then $\vdash \alpha > \gamma$;
- (vi) Replacement of conditions: $\vdash \alpha_1 \leftrightarrow \alpha_2$, then $\vdash (\alpha_1 > \gamma) \leftrightarrow (\alpha_2 > \gamma)$.

Proof. It can be easily proved by the definition, and we omit the details here. ■

There are some important conclusions about the primary and secondary conditions in **C2L_m**.

Theorem 2.2 (The completeness of the primary condition) If $\vdash \alpha \rightarrow \gamma$,

then $\vdash \alpha\beta \geq \gamma$.

Proof. The precondition and the rule of monotonicity of results imply $(\alpha\beta \geq \alpha) \rightarrow (\alpha\beta \geq \gamma)$, and then with the axiom of self-implication of the primary condition, $\alpha\beta \geq \alpha$, we obtain $\alpha\beta \geq \gamma$ by MP. ■

Theorem 2.3 (A single condition is a primary condition) If $\vdash \alpha > \gamma$, then $\vdash \alpha\beta \geq \gamma$.

Proof. From Theorem 2.1(i) and MP, we get $\alpha \rightarrow \gamma$, and then $\alpha\beta \geq \gamma$ is obtained by Theorem 2.2. ■

Two conditions can lead to a result, while any single condition cannot. This is usually called *non-degenerate*, and is the situation we mainly talk about.

Theorem 2.4 (Principle of non-degenerate) $\vdash \neg(\alpha\beta \geq \gamma) \wedge \neg(\alpha > \gamma) \rightarrow \neg(\beta > \gamma)$.

Proof. It can be easily obtained by the basic axiom of primary and secondary conditions. ■

This principle tells us that whenever the primary condition cannot lead to any result alone, the two-conditional is non-degenerate.

The primary condition can be a result (Axiom iv), but when can a secondary condition be a result? That is related to whether the secondary condition is “unnecessary”.

Let α and β be two conditions. If α can lead to β , then β is called *unnecessary*.

Theorem 2.5 $\vdash (\alpha\beta \geq \beta) \rightarrow (\alpha > \beta)$

Proof. The axiom $(\alpha\beta \geq \beta) \wedge (\beta > \beta) \rightarrow (\alpha > \beta)$ implies

$$(\beta > \beta) \rightarrow ((\alpha\beta \geq \beta) \rightarrow (\alpha > \beta)).$$

Hence by Theorem 2.1(ii) and MP, $(\alpha\beta \geq \beta) \rightarrow (\alpha > \beta)$ is obtained. ■

From Theorem 2.3 (β for γ) and 2.5 we get

Theorem 2.6 (The bi-condition of the secondary condition being a result) $\vdash \alpha > \beta$ iff $\vdash \alpha\beta \geq \beta$. ■

C2L_m satisfies PC (viz. the principle of compositionality):

Theorem 2.7 (Replacement principle of results) If $\vdash \gamma_1 \leftrightarrow \gamma_2$, then

$$\vdash (\alpha\beta \geq \gamma_1) \leftrightarrow (\alpha\beta \geq \gamma_2).$$

Proof. By the rule of monotonicity of results. ■

Theorem 2.8 (Basic replacement theorem) If $\vdash \beta \leftrightarrow \gamma$, then $\vdash \alpha \leftrightarrow \alpha[\gamma/\beta]$

Proof. It is easy to prove by induction on formulas, hence we omit the details here. ■

3 Neighborhood semantics

We are going to construct the neighborhood semantics for LPSC. Note that the semantics are only an instance of the authors' general neighborhood semantics (Liu, 1995, pp. 52–56) in LPSC, and are of course not the same as the general neighborhood semantics which can be used in the study of the logic of conditionals.

For any set A , we denote the power set of A by $\mathcal{P}(A)$, which is to say

$$\mathcal{P}(A) = \{X \mid X \subseteq A\}.$$

Definition 3.1 (Neighborhood functions) Let W be a non-empty set. A function from W to $\mathcal{P}(\mathcal{P}(W))^n$ is called an n -ary *neighborhood function* on W .

Definition 3.2 (Frames) Assume $K = \langle W, N \rangle$. If K satisfies:

(i) W is a non-empty set with its elements called *possible worlds* and itself *the set of possible worlds*,

(ii) N is a ternary neighborhood function on W ,
then we call K a *frame*.

Definition 3.3 (Evaluations and models) Let $K = \langle W, N \rangle$ be a frame, and V be a function from **Form** to $\mathcal{P}(W)$. If V satisfies:

- (i) $x \in V(\neg\alpha)$ iff $x \notin V(\alpha)$,
 - (ii) $x \in V(\alpha \wedge \beta)$ iff $x \in V(\alpha)$ and $x \in V(\beta)$,
 - (iii) $x \in V(\alpha \beta \geq \gamma)$ iff $\langle V(\alpha), V(\beta), V(\gamma) \rangle \in N(x)$,
- then we call V an *evaluation* on K , and $\langle K, V \rangle$ a *model*.

We can learn from the definition that the classical connectives have the following properties:

Lemma 3.4

- (i) $V(\neg\alpha) = W \setminus V(\alpha)$,
 $V(\alpha \wedge \beta) = V(\alpha) \cap V(\beta)$,
 $V(\alpha \vee \beta) = V(\alpha) \cup V(\beta)$;
- (ii) If $x \in V(\alpha)$ and $x \in V(\alpha \rightarrow \beta)$, then $x \in V(\beta)$;
- (iii) $V(\alpha \rightarrow \beta) = W$ iff $V(\alpha) \subseteq V(\beta)$;
- (iv) $V(\alpha \leftrightarrow \beta) = W$ iff $V(\alpha) = V(\beta)$.

Definition 3.5 (Satisfaction) Let $K = \langle W, N \rangle$ be a frame.

(i) Let V be an evaluation on K . If $V(\alpha) = W$, then we say $\langle K, V \rangle$ *satisfies* α , denoted by $\langle K, V \rangle \models \alpha$.

(ii) If for any evaluation V on K , we have $\langle K, V \rangle \models \alpha$, then we say K

satisfies α , denoted by $K \models \alpha$.

($K \models \alpha$, if for any evaluation V on K , we have $V(\alpha) = W$.)

Therefore, $K \not\models \alpha$, if there exists an evaluation V on K , such that

$$V(\alpha) \neq W.$$

The notion of satisfaction can be generalized to classes of frames. Let Σ be a class of frames. If for any $K \in \Sigma$, we have $K \models \alpha$, then we say Σ satisfies α , denoted by $\Sigma \models \alpha$.

Hence, $\Sigma \not\models \alpha$, if there exist $K \in \Sigma$ and an evaluation V on K , such that

$$V(\alpha) \neq W.$$

Unlike other semantics, in neighborhood semantics the minimal logic characterized by all frames makes no sense, since the neighborhood function N characterizes a most general ternary connective. Thus, we need those frames characterizing primary and secondary conditionals.

Definition 3.6 (Frames of primary and secondary conditionals) Let $K = \langle W, N \rangle$ be a frame. If K satisfies:

(i) **Truth preserving:** if $\langle S, P, Q \rangle \in N(x)$ and $x \in S \cap P$, then $x \in Q$,

(ii) **Monotonicity:** if $\langle S, P, Q_1 \rangle \in N(x)$ and $Q_1 \subseteq Q_2$, then

$$\langle S, P, Q_2 \rangle \in N(x),$$

(iii) **Conjunctivity of results:** if $\langle S, P, Q_1 \rangle \in N(x)$ and $\langle S, P, Q_2 \rangle \in N(x)$,

then $\langle S, P, Q_1 \cap Q_2 \rangle \in N(x)$,

(iv) **Completeness of the primary condition:** if $S \subseteq Q$, then

$$\langle S, P, Q \rangle \in N(x),$$

(v) **Basic property of primary and secondary conditions:** if

$$\langle S, P, Q \rangle \in N(x) \text{ and } \langle P, P, Q \rangle \in N(x), \text{ then } \langle S, S, Q \rangle \in N(x),$$

then we call K a frame of primary and secondary conditionals, or a **C2**-frame for short.

The class of all frames of primary and secondary conditionals is denoted by $\Sigma(\mathbf{C2})$.

The minimal system $\mathbf{C2L}_m$ of LPSC is sound with respect to $\Sigma(\mathbf{C2})$.

Lemma 3.7 Given any **C2**-frame K , and any axiom α of $\mathbf{C2L}_m$, $K \models \alpha$ holds.

Proof. Let $K = \langle W, N \rangle$ be given. We prove $V(\alpha) = W$, for any evaluation V on K .

Axiom (i): Clearly.

Axiom (ii): If $x \in V(\alpha \wedge \beta \wedge (\alpha \beta \supseteq \gamma))$, then $x \in V(\alpha \wedge \beta)$ and $x \in V(\alpha \beta \supseteq \gamma)$.

Therefore,

$$x \in V(\alpha \wedge \beta) \text{ and } \langle V(\alpha), V(\beta), V(\gamma) \rangle \in N(x).$$

By truth preserving of **C2**-frames, we have $x \in V(\gamma)$. Hence,

$$V(\alpha \wedge \beta \wedge (\alpha \beta \supseteq \gamma)) \subseteq V(\gamma).$$

Then by Lemma 3.4(iii), we obtain $V((\alpha \wedge \beta \wedge (\alpha \beta \geq \gamma) \rightarrow \gamma) = W$.

Axiom (iii): If $x \in V((\alpha \beta \geq \gamma) \wedge (\alpha \beta \geq \delta))$, then $x \in V(\alpha \beta \geq \gamma)$ and $x \in V(\alpha \beta \geq \delta)$. Therefore,

$$\langle V(\alpha), V(\beta), V(\gamma) \rangle \in N(x) \text{ and } \langle V(\alpha), V(\beta), V(\delta) \rangle \in N(x).$$

By conjunctivity of results of **C2**-frames, we have $\langle V(\alpha), V(\beta), V(\gamma) \cap V(\delta) \rangle \in N(x)$. Hence,

$$\langle V(\alpha), V(\beta), V(\gamma \wedge \delta) \rangle \in N(x).$$

That is $x \in V(\alpha \beta \geq \gamma \wedge \delta)$ and thus $V((\alpha \beta \geq \gamma) \wedge (\alpha \beta \geq \delta)) \subseteq V(\alpha \beta \geq \gamma \wedge \delta)$.

Then by Lemma 3.4(iii), we obtain $V((\alpha \beta \geq \gamma) \wedge (\alpha \beta \geq \delta) \rightarrow (\alpha \beta \geq \gamma \wedge \delta)) = W$.

Axiom (iv): For any $x \in W$, by $V(\alpha) \subseteq V(\alpha)$, and completeness of the primary condition of **C2**-frames, we have

$$\langle V(\alpha), V(\beta), V(\alpha) \rangle \in N(x).$$

Therefore $x \in V(\alpha \beta \geq \alpha)$, and hence $V(\alpha \beta \geq \alpha) = W$.

Axiom (v): If $x \in V((\alpha \beta \geq \gamma) \wedge (\beta > \gamma))$, then $x \in V(\alpha \beta \geq \gamma)$ and $x \in V(\beta > \gamma)$. Therefore,

$$\langle V(\alpha), V(\beta), V(\gamma) \rangle \in N(x) \text{ and } \langle V(\beta), V(\beta), V(\delta) \rangle \in N(x).$$

By the basic property of primary and secondary condition of **C2**-frames, we have

$$\langle V(\alpha), V(\alpha), V(\gamma) \rangle \in N(x),$$

and hence $x \in V(\alpha > \gamma)$. Therefore,

$$V((\alpha \beta \geq \gamma) \wedge (\beta > \gamma)) \subseteq V(\alpha > \gamma).$$

Then by Lemma 3.4(iii), we obtain $V((\alpha \beta \geq \gamma) \wedge (\beta > \gamma) \rightarrow (\alpha > \gamma)) = W$. ■

Lemma 3.8 Given any **C2**-frame K .

- (i) If $K \models \alpha$ and $K \models \alpha \rightarrow \beta$, then $K \models \beta$;
- (ii) if $K \models \gamma \rightarrow \delta$, then $K \models (\alpha \beta \geq \gamma) \rightarrow (\alpha \beta \geq \delta)$;
- (iii) if $K \models \alpha_1 \leftrightarrow \alpha_2$ and $K \models \beta_1 \leftrightarrow \beta_2$, then $K \models (\alpha_1 \beta_1 \geq \gamma) \leftrightarrow (\alpha_2 \beta_2 \geq \gamma)$.

Proof.

(i) For any evaluation V on K , we have $V(\alpha) = V(\alpha \rightarrow \beta) = W$. Then we get $V(\alpha) \subseteq V(\beta)$, and hence $V(\beta) = W$. Therefore, $K \models \beta$.

(ii) For any evaluation V on K , we have $V(\gamma \rightarrow \delta) = W$, and hence

$$V(\gamma) \subseteq V(\delta).$$

If $x \in V(\alpha \beta \geq \gamma)$, then $\langle V(\alpha), V(\beta), V(\gamma) \rangle \in N(x)$. By monotonicity of **C2**-frames, we have

$$\langle V(\alpha), V(\beta), V(\delta) \rangle \in N(x).$$

Thus $x \in V(\alpha \beta \geq \delta)$.

That proved $V(\alpha \beta \geq \gamma) \subseteq V(\alpha \beta \geq \delta)$. Therefore,

$$V((\alpha \beta \geq \gamma) \rightarrow (\alpha \beta \geq \delta)) = W,$$

and hence, $K \models (\alpha \beta \geq \gamma) \rightarrow (\alpha \beta \geq \delta)$.

(iii) For any evaluation V on K , we have $V(\alpha_1 \leftrightarrow \alpha_2) = W$ and $V(\beta_1 \leftrightarrow \beta_2) = W$. Therefore,

$$V(\alpha_1) = V(\alpha_2) \text{ and } V(\beta_1) = V(\beta_2).$$

For any x ,

$$\begin{aligned} x \in V(\alpha_1 \beta_1 \supseteq \gamma) & \quad \text{iff} & \quad \langle V(\alpha_1), V(\beta_1), V(\gamma) \rangle \in N(x) \\ & \quad \text{iff} & \quad \langle V(\alpha_2), V(\beta_2), V(\delta) \rangle \in N(x) \\ & \quad \text{iff} & \quad x \in V(\alpha_2 \beta_2 \supseteq \gamma). \end{aligned}$$

That proved $V(\alpha_1 \beta_1 \supseteq \gamma) = V(\alpha_2 \beta_2 \supseteq \gamma)$. Therefore,

$$V((\alpha_1 \beta_1 \supseteq \gamma) \leftrightarrow (\alpha_2 \beta_2 \supseteq \gamma)) = W,$$

and hence, $K \models (\alpha_1 \beta_1 \supseteq \gamma) \leftrightarrow (\alpha_2 \beta_2 \supseteq \gamma)$. ■

Theorem 3.9 (Soundness theorem) If $\vdash \alpha$, then $\Sigma(\mathbf{C2}) \models \alpha$.

Proof. For any proof $\alpha_1, \dots, \alpha_n$ of α , given any $K \in \Sigma(\mathbf{C2})$, we prove $K \models \alpha_i$ by induction. Lemma 3.7 and 3.8 will be used in the proof, and the details are omitted here. ■

4 Canonical models

We define derivation as in an ordinary modal logic.

Definition 4.1 (Derivation) Let u be a set of formulas, and α be a formula. If there exist $\varphi_1, \dots, \varphi_n \in u$, such that $\vdash \varphi_1 \wedge \dots \wedge \varphi_n \rightarrow \alpha$, then we say α is derived from u , denoted by $u \vdash \alpha$.

Note that n can be 0, and if so, $\vdash \varphi_1 \wedge \dots \wedge \varphi_n \rightarrow \alpha$ is just $\vdash \alpha$. The derivation defined above satisfies basic properties of derivation.

Theorem 4.2 (Basic properties of derivation)

- (i) **Monotonicity:** If $u \vdash \alpha$ and $u \subseteq v$, then $v \vdash \alpha$;
- (ii) **Transitivity:** If $v \vdash \alpha$ and $u \vdash \alpha$ for all $\alpha \in v$, then $u \vdash \alpha$;
- (iii) **Deduction theorem:** $u \cup \{\varphi\} \vdash \alpha$ iff $u \vdash \varphi \rightarrow \alpha$.

Although the definition of derivation is not the same as that of classical propositional logic, we can likewise use the notion of derivation to define consistency and maximal consistency.

Definition 4.3 (Consistency)

- (i) Let u be a set of formulas. If there is no formula α , such that both $u \vdash \alpha$ and $u \vdash \neg \alpha$ hold, then we say that u is *consistent*.
- (ii) Let u be a consistent set of formulas. If for any $\alpha \notin u$, $u \cup \{\alpha\}$ is not consistent, then we say that u is *maximal consistent*.

Lemma 4.4 (Properties of consistent sets)

- (i) u is *inconsistent* iff there exists a formula α , such that $u \vdash \alpha$ and $u \vdash \neg \alpha$.

- (ii) u is inconsistent iff for any formula α , $u \vdash \alpha$.
- (iii) u is consistent iff there exists a formula α , such that $u \not\vdash \alpha$.
- (iv) u is consistent iff every finite subset of u is consistent.
- (v) $u \not\vdash \alpha$ iff $u \cup \{\neg\alpha\}$ is consistent. ■

Lemma 4.5 (Properties of maximal consistent sets) Let u be a maximal consistent set, and α, β be formulas.

- (i) If $u \vdash \alpha$, then $\alpha \in u$. Particularly, if $\vdash \alpha$, then $\alpha \in u$.
- (ii) $\neg\alpha \in u$ iff $\alpha \notin u$.
- (iii) $\alpha \wedge \beta \in u$ iff $\alpha \in u$ and $\beta \in u$.
- (iv) If $\alpha, \alpha \rightarrow \beta \in u$, then $\beta \in u$.
- (v) If $\alpha, \beta, \alpha\beta \supseteq \gamma \in u$, then $\gamma \in u$. ■

Every consistent set can be extended to be maximal consistent. Hence,

Lemma 4.6

- (i) If $\not\vdash \alpha$, then there exists a maximal consistent set u , such that $\alpha \notin u$.
- (ii) If $\not\vdash \alpha \rightarrow \beta$, then there exists a maximal consistent set u , such that $\alpha \in u$ and $\beta \notin u$. ■

Assume $W = \{u \mid u \text{ is a maximal consistent set}\}$, $|\alpha| = \{u \in W \mid \alpha \in u\}$, then we have:

Lemma 4.7

- (i) $u \in |\alpha|$ iff $\alpha \in u$.
- (ii) $|\neg\alpha| = W \setminus |\alpha|$.
- (iii) $|\alpha \wedge \beta| = |\alpha| \cap |\beta|$.
- (iv) $|\alpha| \subseteq |\beta|$ iff $\vdash \alpha \rightarrow \beta$.
- (v) $|\alpha| = |\beta|$ iff $\vdash \alpha \leftrightarrow \beta$. ■

Unlike other semantics, canonical models in neighborhood semantics are not unique. They are not concretely constructed, but are actually defined by some properties. The models satisfying those properties are all called canonical models.

Definition 4.8 (Canonical models) Assume $K = \langle W, N \rangle$. Let V be an evaluation on K . If:

- (i) $W = \{u \mid u \text{ is a maximal consistent set}\}$,
- (ii) $\langle |\alpha|, |\beta|, |\gamma| \rangle \in N(u)$ iff $\alpha\beta \supseteq \gamma \in u$,

and (iii) for any propositional variable p , $V(p) = |p|$ holds, then we call $\langle K, V \rangle$ a *canonical model*.

The essential feature of canonical models can be seen in the following theorems:

Theorem 4.9 (Basic theorem on canonical models) Let $\langle K, V \rangle$ be a canonical model. For any formula φ , $|\varphi| = V(\varphi)$ holds. That is, $u \in V(\varphi)$ iff $\varphi \in u$.

Proof. We prove the theorem by induction on formulas. We omit the proof of φ being a propositional variable, $\varphi = \neg\alpha$ and $\varphi = \alpha \wedge \beta$.

Assume $\varphi = \alpha\beta \supseteq \gamma$. By induction hypothesis, $|\alpha| = V(\alpha)$, $|\beta| = V(\beta)$, $|\gamma| = V(\gamma)$. Therefore,

$$\begin{aligned}
 u \in V(\varphi) & \text{ iff } & u \in V(\alpha\beta \supseteq \gamma) \\
 & \text{ iff } & \langle V(\alpha), V(\beta), V(\gamma) \rangle \in N(u) & \text{ by 3.3(iii)} \\
 & \text{ iff } & \langle |\alpha|, |\beta|, |\gamma| \rangle \in N(u) & \text{ by induction hypothesis} \\
 & \text{ iff } & \alpha\beta \supseteq \gamma \in u & \text{ by 4.8(ii)} \\
 & \text{ iff } & \varphi \in u. & \blacksquare
 \end{aligned}$$

From the basic theorem on canonical models and the properties of maximal consistent sets, we get:

Theorem 4.10 Let $\langle K, V \rangle$ be a canonical model. For any formula α , $\langle K, V \rangle \models \alpha$ iff $\vdash \alpha$.

Proof. Assume $\vdash \alpha$. For any $u \in W$, we get $\alpha \in u$ by lemma 4.5 (i). Therefore,

$$|\alpha| = W.$$

By Theorem 4.9, $V(\alpha) = |\alpha| = W$, and hence $\langle K, V \rangle \models \alpha$.

Assume $\not\vdash \alpha$. By lemma 4.6(i), there exists $u \in W$, such that $\alpha \notin u$. Therefore,

$$|\alpha| \neq W.$$

Then by theorem 4.9, $V(\alpha) = |\alpha| \neq W$, and hence $\langle K, V \rangle \not\models \alpha$. ■

5 Completeness

Completeness says: if $\Sigma(\mathbf{C2}) \models \alpha$, then $\vdash \alpha$; that is to say, if $\not\vdash \alpha$, then $\Sigma(\mathbf{C2}) \not\models \alpha$. Let $\langle K, V \rangle$ be a canonical model. For every α , $\not\vdash \alpha$ implies $\langle K, V \rangle \not\models \alpha$, and if we have $K \in \Sigma(\mathbf{C2})$, then $\Sigma(\mathbf{C2}) \not\models \alpha$ can be obtained. Thereby in order to prove the completeness by the canonical model method, it will be sufficient to prove that we can find a canonical model $\langle K, V \rangle$, such that $K \in \Sigma(\mathbf{C2})$.

We construct a model $\langle K^*, V^* \rangle$ where $K^* = \langle W, N \rangle$ as follows:

- (i) $W = \{ u \mid u \text{ is a maximal consistent set} \}$;
- (ii) $N_0(u) = \{ \langle |\alpha|, |\beta|, Q \rangle \mid \text{there exists } \alpha\beta \supseteq \gamma \in u, \text{ such that } |\gamma| \subseteq Q \}$,
 $N_1(u) = \{ \langle S, P, Q \rangle \mid S \subseteq Q \}$,
 $N(u) = N_0(u) \cup N_1(u)$;
- (iii) for any propositional variable p , $V^*(p) = |p|$ holds.

We prove first that $\langle K^*, V^* \rangle$ is a canonical model.

Theorem 5.1 $\langle K^*, V^* \rangle$ is a canonical model.

Proof. According to the definition of canonical models, it's sufficient to prove:

$$\langle |\alpha|, |\beta|, |\gamma| \rangle \in N(u) \text{ iff } \alpha\beta \geq \gamma \in u.$$

Assume $\alpha\beta \geq \gamma \in u$. Since $|\gamma| \subseteq |\gamma|$, $\langle |\alpha|, |\beta|, |\gamma| \rangle \in N_0(u)$, and thus

$$\langle |\alpha|, |\beta|, |\gamma| \rangle \in N(u).$$

Assume $\langle |\alpha|, |\beta|, |\gamma| \rangle \in N(u)$. There are two cases:

(i) $\langle |\alpha|, |\beta|, |\gamma| \rangle \in N_0(u)$. Then by the definition, there exists $\alpha'\beta' \geq \gamma' \in u$, such that

$$|\alpha'| = |\alpha|, |\beta'| = |\beta|, |\gamma'| \subseteq |\gamma|.$$

By Lemma 4.7(iv)&(v), we have

$$\vdash \alpha' \leftrightarrow \alpha, \vdash \beta' \leftrightarrow \beta \text{ and } \vdash \gamma' \rightarrow \gamma.$$

By $\vdash \alpha' \leftrightarrow \beta, \vdash \beta' \leftrightarrow \beta$, and the rule of replacement of conditions, we have then

$$\vdash (\alpha'\beta' \geq \gamma') \leftrightarrow (\alpha\beta \geq \gamma').$$

Then, by $\alpha'\beta' \geq \gamma' \in u$, we get $\alpha\beta \geq \gamma' \in u$.

By $\vdash \gamma' \rightarrow \gamma$ and the rule of monotonicity of results, we obtain

$$\vdash (\alpha\beta \geq \gamma') \rightarrow (\alpha\beta \geq \gamma),$$

and finally by $\alpha\beta \geq \gamma' \in u$ we get $\alpha\beta \geq \gamma \in u$.

(ii) $\langle |\alpha|, |\beta|, |\gamma| \rangle \in N_1(u)$. We get $|\alpha| \subseteq |\gamma|$ by the definition, and then by Lemma 4.7(iv), we have $\vdash \alpha \rightarrow \gamma$. By the completeness of the primary condition (theorem 2.2), we get $\vdash \alpha\beta \geq \gamma$, and finally $\alpha\beta \geq \gamma \in u$. ■

We now prove that K^* is a **C2**-frame.

Theorem 5.2 K^* is a **C2**-frame.

Proof. We verify all five properties of **C2**-frames.

(i) Truth preserving.

If $\langle |\alpha|, |\beta|, Q \rangle \in N_0(u)$ and $u \in |\alpha| \cap |\beta|$, then

there exists $\alpha\beta \geq \gamma \in u$, such that $|\gamma| \subseteq Q$.

By $u \in |\alpha| \cap |\beta|$ we get $\alpha, \beta \in u$, and then by $\alpha\beta \geq \gamma \in u$ and lemma 4.5(v), $\gamma \in u$ holds.

Therefore, $u \in |\gamma|$, and hence $u \in Q$.

If $\langle S, P, Q \rangle \in N_1(u)$ and $u \in S \cap P$, then $S \subseteq Q$, and hence $u \in Q$.

(ii) Monotonicity.

If $\langle |\alpha|, |\beta|, Q_1 \rangle \in N_0(x)$ and $Q_1 \subseteq Q_2$, then

there exists $\alpha\beta \geq \gamma \in u$, such that $|\gamma| \subseteq Q_1$.

Therefore,

there exists $\alpha\beta \geq \gamma \in u$, such that $|\gamma| \subseteq Q_2$.

Hence, $\langle |\alpha|, |\beta|, Q_2 \rangle \in N_0(x)$.

If $\langle S, P, Q_1 \rangle \in N_1(x)$ and $Q_1 \subseteq Q_2$, then $S \subseteq Q_1$. So $S \subseteq Q_2$, and thus

$\langle S, P, Q_2 \rangle \in N_1(x)$.

(iii) Conjunctivity of results.

(a) If $\langle S, P, Q_1 \rangle \in N_0(x)$ and $\langle S, P, Q_2 \rangle \in N_0(x)$, then

there exist $\alpha_1\beta_1 \supseteq \gamma_1$, $\alpha_2\beta_2 \supseteq \gamma_2 \in u$, such that $|\gamma_1| \subseteq Q_1$, $|\gamma_2| \subseteq Q_2$,

and

$$S = |\alpha_1| = |\alpha_2|, P = |\beta_1| = |\beta_2|.$$

Therefore, $\vdash \alpha_1 \leftrightarrow \alpha_2$, $\vdash \beta_1 \leftrightarrow \beta_2$. By $\vdash \alpha_1 \leftrightarrow \alpha_2$, $\vdash \beta_1 \leftrightarrow \beta_2$ and the rule of replacement of conditions,

$$\vdash (\alpha_1\beta_1 \supseteq \gamma_1) \leftrightarrow (\alpha_2\beta_2 \supseteq \gamma_1).$$

Then by $\alpha_1\beta_1 \supseteq \gamma_1 \in u$, we have

$$\alpha_2\beta_2 \supseteq \gamma_1 \in u.$$

By $\alpha_2\beta_2 \supseteq \gamma_1$, $\alpha_2\beta_2 \supseteq \gamma_2 \in u$ and the axiom of conjunctivity of results,

$$\alpha_2\beta_2 \supseteq \gamma_1 \wedge \gamma_2 \in u.$$

Hence,

there exists $\alpha_2\beta_2 \supseteq \gamma_1 \wedge \gamma_2 \in u$, such that $|\gamma_1 \wedge \gamma_2| \subseteq Q_1 \cap Q_2$.

Therefore, $\langle S, P, Q_1 \cap Q_2 \rangle \in N_0(x)$.

(b) If $\langle S, P, Q_1 \rangle \in N_1(x)$ and $\langle S, P, Q_2 \rangle \in N_1(x)$, then

$$S \subseteq Q_1 \text{ and } S \subseteq Q_2.$$

Therefore, $S \subseteq Q_1 \cap Q_2$, and hence $\langle S, P, Q_1 \cap Q_2 \rangle \in N_1(x)$.

(c) If $\langle S, P, Q_1 \rangle \in N_0(x)$ and $\langle S, P, Q_2 \rangle \in N_1(x)$, then

there exists $\alpha\beta \supseteq \gamma \in u$, such that $|\gamma| \subseteq Q_1$, and $S = |\alpha|$, $P = |\beta|$, $S \subseteq Q_2$.

Therefore,

$$|\gamma \wedge \alpha| = |\gamma| \cap S \subseteq Q_1 \cap Q_2.$$

By $\alpha\beta \supseteq \gamma$, $\alpha\beta \supseteq \alpha \in u$ and the axiom of conjunctivity of results, we get $\alpha\beta \supseteq \gamma \wedge \alpha \in u$. Hence,

there exists $\alpha\beta \supseteq \gamma \wedge \alpha \in u$, such that $|\gamma \wedge \alpha| \subseteq Q_1 \cap Q_2$.

Therefore, $\langle S, P, Q_1 \cap Q_2 \rangle \in N_0(x)$.

(d) If $\langle S, P, Q_1 \rangle \in N_1(x)$ and $\langle S, P, Q_2 \rangle \in N_0(x)$, then the proof is similar to the case (c).

(iv) Completeness of the primary condition.

If $S \subseteq Q$, then $\langle S, P, Q \rangle \in N_1(x)$.

(v) The basic property of primary and secondary conditions.

(a) If $\langle S, P, Q \rangle \in N_0(x)$ and $\langle P, P, Q \rangle \in N_0(x)$, then there exist

$\alpha_1\beta_1 \supseteq \gamma_1$, $\alpha_2\beta_2 \supseteq \gamma_2 \in u$, such that $|\gamma_1| \subseteq Q$, $|\gamma_2| \subseteq Q$,

and

$$S = |\alpha_1|, P = |\beta_1| = |\alpha_2| = |\beta_2|, Q = |\gamma_1| = |\gamma_2|.$$

Therefore,

$$|\gamma_1 \vee \gamma_2| \subseteq Q, \vdash \beta_1 \leftrightarrow \alpha_2, \vdash \beta_1 \leftrightarrow \beta_2, \vdash \gamma_1 \leftrightarrow \gamma_2.$$

By $\vdash \beta_1 \leftrightarrow \alpha_2$, $\vdash \beta_1 \leftrightarrow \beta_2$ and the rule of replacement of conditions, we have

$$\vdash (\alpha_2\beta_2 \supseteq \gamma_2) \leftrightarrow (\beta_1\beta_2 \supseteq \gamma_2) \text{ and } \vdash (\beta_1\beta_2 \supseteq \gamma_2) \leftrightarrow (\beta_1\beta_1 \supseteq \gamma_2).$$

By $\vdash \gamma_1 \leftrightarrow \gamma_2$ and the replacement principle of results (Theorem 2.7),

$$\vdash (\beta_1\beta_1 \supseteq \gamma_2) \leftrightarrow (\beta_1\beta_1 \supseteq \gamma_1).$$

Then by the three bi-conditionals above, we get

$$\vdash (\alpha_2\beta_2 \supseteq \gamma_2) \leftrightarrow (\beta_1\beta_1 \supseteq \gamma_1).$$

By $\alpha_2\beta_2 \supseteq \gamma_2 \in u$,

$$\beta_1\beta_1 \supseteq \gamma_1 \in u.$$

By $\alpha_1\beta_1 \supseteq \gamma_1$, $\beta_1\beta_1 \supseteq \gamma_1 \in u$ and the rule of monotonicity of results,

$$\alpha_1\beta_1 \supseteq \gamma_1 \vee \gamma_2, \beta_1\beta_1 \supseteq \gamma_1 \vee \gamma_2 \in u.$$

Then by the basic axiom of primary and secondary conditions,

$$\alpha_1\alpha_1 \supseteq \gamma_1 \vee \gamma_2 \in u.$$

Hence, there exists $\alpha_1\alpha_1 \supseteq \gamma_1 \vee \gamma_2 \in u$, such that $|\gamma_1 \vee \gamma_2| \subseteq Q$.

Thereby $\langle S, S, Q \rangle \in N_0(x)$.

(b) If $\langle S, P, Q \rangle \in N_0(x)$ and $\langle P, P, Q \rangle \in N_1(x)$, then

there exists $\alpha\beta \supseteq \gamma \in u$, such that $|\alpha| = S, |\beta| = P, |\gamma| \subseteq Q$, and $P \subseteq Q$.

Therefore,

$$|\gamma \vee \beta| \subseteq Q.$$

By $\alpha\beta \supseteq \gamma \in u$ and the rule of monotonicity of results,

$$\alpha\beta \supseteq \gamma \vee \beta \in u.$$

By Theorem 2.2, $\beta\beta \supseteq \gamma \vee \beta \in u$, and then by the basic axiom of primary and secondary conditionals,

$$\alpha\alpha \supseteq \gamma \vee \beta \in u.$$

Hence, there exists $\alpha\alpha \supseteq \gamma \vee \beta \in u$, such that $|\gamma \vee \beta| \subseteq Q$.

Thus $\langle S, S, Q \rangle \in N_0(x)$.

(c) If $\langle S, P, Q \rangle \in N_1(x)$, then $S \subseteq Q$, and hence $\langle S, S, Q \rangle \in N_1(x)$. ■

Theorem 5.3 (Completeness theorem) If $\Sigma(\mathbf{C2}) \models \alpha$, then $\vdash \alpha$.

Proof. It's sufficient to prove: if $\not\vdash \alpha$, then $\Sigma(\mathbf{C2}) \not\models \alpha$.

If $\not\vdash \alpha$, by Theorem 5.1, $\langle K^*, V^* \rangle$ is a canonical model, and then by theorem 4.10, $\langle K^*, V^* \rangle \not\models \alpha$. Hence $K^* \not\models \alpha$. Since $K^* \in \Sigma(\mathbf{C2})$ by Theorem 5.2, we have $\Sigma(\mathbf{C2}) \not\models \alpha$ then. ■

6 Discussions

6.1 Monotonicity of conditions

Among the properties of condition implication, monotonicity is of great importance. Actually, there are two different monotonicities.

Monotonicity a: If a single condition α can lead to a result γ , then two conditions α and β can also do.

Monotonicity b: If β logically implies α (viz. $\beta \rightarrow \alpha$ is a tautology) and condition α can lead to result γ , then condition β can also lead to γ .

Lots of examples falsify Monotonicity a. In the logic of conditionals with

only single conditions, we put two conditions α and β together as a single condition $\alpha\wedge\beta$. Therefore, the falsity of Monotonicity a implies that of Monotonicity b . Nevertheless, it is not suitable to conjunct α and β as $\alpha\wedge\beta$, since \wedge is an extensive connective.

If we do not conjunct two conditions, we will not have an example falsifying Monotonicity b . Hence, in the logic of conditionals with only single conditions, we can consider those systems in which Monotonicity b holds, in spite of those examples falsifying Monotonicity a .

Our system certainly distinguishes the two-conditionals of the form $\alpha\beta\geq\gamma$ and the one-conditionals of the form $\alpha\wedge\beta>\gamma$. One can easily prove: there exists $K\in\Sigma(\mathbf{C2})$, such that $K\nvDash\alpha\beta\geq\gamma\leftrightarrow\alpha\wedge\beta>\gamma$. Hence, by the soundness theorem, $\nvDash\alpha\beta\geq\gamma\leftrightarrow\alpha\wedge\beta>\gamma$.

6.2 Implication paradox

An example of the so-called “implication paradox” is “a false proposition implies any proposition”, which also holds in the logic of conditionals. A similar example “an inconsistent proposition implies any proposition”, however, leads to two different understandings. One is to take the inconsistent proposition as a single proposition, namely, the constant false proposition. The other is to regard it as two conflicting propositions, and the example now becomes “two conflicting propositions imply any propositions”. In our system, those two understandings are different: “two conflicting propositions imply any propositions” can be formalized as $\alpha(-\alpha)\geq\gamma$, which can be proved not a theorem of our system.

6.3 Existence of primary conditions

Assume that conditions α and β can lead to a result γ . α and β can both be primary conditions in our system. However, it has not been demanded that one of α and β is a primary condition. If we introduce the existence of primary conditions,

If two conditions can lead to a result, then one of the conditions is a primary condition.

we can use $(\alpha\beta\geq\gamma)\vee(\beta\alpha\geq\gamma)$ to mean “ α and β can lead to γ ”, which is a more general method. As far as whether the existence of primary conditions holds or not, we need to study further into the two-conditionals of daily languages.

6.4 Extensions of our system

We have only constructed a minimal system of LPSC in this article. However, it can be extended to more complicated systems. There are two different ways:

(i) Consider similar axioms to those of the logic of conditionals with single conditions. For example, of the axiom of disjunction of conditions, we can consider the primary and secondary conditions respectively. Another example is the axiom of elimination of unnecessary conditions:

$$(\alpha\beta\geq\gamma)\wedge(\alpha>\beta)\rightarrow(\alpha>\gamma).$$

(ii) Consider properties different from those of the logic of conditionals with single conditions. For example, we can add to the system axioms of Monotonicity b with respect to primary and secondary conditions respectively, and we can also add to it an axiom saying “two conflicting propositions imply any propositions”, and so on.

References

- Liu Zhuanghu, Implication in logical systems, in The Chinese Association of Logic (ed.), *Researches on Logic Today*, Beijing: Social Sciences Academic Press, 1999
- Liu Zhuanghu, Logical analysis of Necessity, *Zhexue Yanjiu* (Philosophical Research), 2002 (2)
- Liu Zhuanghu, Neighborhood semantics and model completeness, *Beijing Daxue Xuebao* (Journal of Peking University) (Philosophy and Social Sciences Edition), 1995 (5)
- Mao Yi, Neighborhood semantics and model completeness for the logic of conditionals, in: Institute of Philosophy of The Chinese Academy of Social Sciences (ed.) *Reason and Necessity*, Beijing: Social Sciences Academic Press, 1995