Arrow's Decisive Coalitions

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The theme of this paper

In this paper, Holliday and Pacuit's goal is to analyze "reasoning about decisive coalitions" and formalize "how the concept of a decisive coalition gives rise to a social theoretic language and logic all of its own". They give a correspondence between Arrow's two conditions and axioms about decisive coalitions. They demonstrate this correspondence with two type of results: representation theorem and completeness theorem.

Basics

- Let $\mathcal{V} = \{v_1, \ldots, v_n\}$ be a nonempty set of **individuals** (or voters, or agents).
- Let $\mathcal{X} = \{x_1, x_2, x_3, \ldots\}$ be a nonempty set of **alternatives** (or candidates).
- Each voter in $\mathcal{V}$ will be asked to express an **asymmetric** and **negative transitive** binary relation over the alternatives in $\mathcal{X}$.
- A **collective choice** is an **asymmetric** binary relation on $\mathcal{X}$.

Basics

- $\mathcal{O}(\mathcal{X})$ is the set of all **asymmetric** and **negative transitive** binary relations i.e strict weak orders on $\mathcal{X}$.
- $\mathcal{P}(\mathcal{X})$ is the set of all asymmetric binary relations on $\mathcal{X}$.
- $\mathcal{L}(\mathcal{X})$ is the set of all strict linear orders on $\mathcal{X}$.
- A **profile P** $= (P_1, \ldots, P_n) \in \mathcal{O}(\mathcal{X})^{\mathcal{V}}$.
- $\mathbf{P}(x, y) = \{i \in \mathcal{V} \mid xP_iy\}$
- $\mathbf{P}_{|\{x,y\}}$ = the function assigning to each $i \in \mathcal{V}$ the relation $P_i \cap \{x, y\}^2$
- **Collective Choice Rule** (CCR) for $\langle \mathcal{X}, \mathcal{V} \rangle$ is a function f from a subset of $\mathcal{O}(\mathcal{X})^{\mathcal{V}}$ to $\mathcal{P}(\mathcal{X})$.

Conditions of f

- Domain Conditions
  - universal domain (UD): $dom(f) = \mathcal{O}(\mathcal{X})^{\mathcal{V}}$
  - linear domain (LD): $dom(f) = \mathcal{L}(\mathcal{X})^{\mathcal{V}}$
- Codomain Conditions (rationality postulates)
  - transitive rationality (TR): for all $\mathbf{P} \in dom(f)$, $f(\mathbf{P})$ is transitive
  - full rationality (FR): for all $\mathbf{P} \in dom(f)$, $f(\mathbf{P})$ is a strict weak order
- Interprofile Conditions
  - independence of irrelevant alternatives (IIA): for all $\mathbf{P}, \mathbf{P}' \in dom(f)$ and $x, y \in \mathcal{X}$, if $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|\{x,y\}}$, then $xf(\mathbf{P})y$ iff $xf(\mathbf{P}')y$
- Decisiveness Conditions
  - Pareto (P): for all $\mathbf{P} \in dom(f)$ and $x, y \in \mathcal{X}$, if $\mathbf{P}(x, y) = \mathcal{V}$, then $xf(\mathbf{P})y$
  - dictatorship: there is an $i \in \mathcal{V}$ s.t. for all $\mathbf{P} \in dom(f)$ and $x, y \in \mathcal{X}$, if xP_iy, then $xf(\mathbf{P})y$

Arrow's Theorem

Theorem (Arrow, 1951)

Assume that $|\mathcal{X}| \geq 3$ *and* $\mathcal{V}$ *is finite. Then any CCR for* $\langle \mathcal{X}, \mathcal{V} \rangle$ *satisfying UD, IIA, FR, and P is a dictatorship.*

Proof of Arrow's Theorem

Proof Strategy

Our proof broadly follows Sen (1986) and is based on the idea of "decisive coalitions". The main idea of the proof is to show that, whenever some coalition G (with $|G| \geq 2$) is decisive, then there exists a nonempty $G' \subset G$ that is decisive as well. Given the finiteness of $\mathcal{V}$, this means that f dictatorial. Actually, $\mathcal{V}$ is decisive due to **Pareto**.

Decisive Coalition

Let us call a coalition $G \subseteq \mathcal{V}$ decisive on alternatives x, y if for any $\mathbf{P}$, $G \subseteq \mathbf{P}(x, y)$ entails $(x, y) \in f(\mathbf{P})$. When G is decisive on all pairs of alternatives, then we simply say that G is decisive according to f.

Another Proof via Ultrafilter

Several alternative proofs for Arrow's Theorem may be found in the literature (Geanakoplos, 2005). We want to briefly mention one such proof here, due to Kirman and Sondermann (1972), which reduces Arrow's Theorem to a well- known fact in the theory of ultrafilters. Given the importance of ultrafilters in model theory and set theory, this proof provides additional evidence for the close connections between logic and social choice theory.

Another Proof via Ultrafilter

Definition

An **ultrafilter** $\mathcal{G}$ for a set $\mathcal{V}$ is a set of subsets of $\mathcal{V}$ satisfying the following conditions:

1. The empty set is not included: $\emptyset \notin \mathcal{G}$.
2. If $G_1 \subseteq G_2$ and $G_1 \in \mathcal{G}$, then $G_2 \in \mathcal{G}$
3. $\mathcal{G}$ is closed under intersection: if $G_1 \in \mathcal{G}$ and $G_2 \in \mathcal{G}$, then $G_1 \cap G_2 \in \mathcal{G}$.
4. $\mathcal{G}$ is maximal: for all $G \subseteq \mathcal{V}$, either $G \in \mathcal{G}$ or $(\mathcal{V} \setminus G) \in \mathcal{G}$.

Let us now interpret $\mathcal{V}$ as a set of individuals and $\mathcal{G}$ as the set of decisive coalitions for a given CCR satisfying those demanding conditions. It turns out that $\mathcal{G}$ satisfies the four conditions above, i.e., it is an ultrafilter (principle).

What Does Arrow's Theorem Say?

Marc Pauly views the impossibility results as definability results of corresponding classes of models (Pauly, 2008).

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- Given a **semantic domain** $\mathcal{D}$ and a target class $\mathcal{T} \subseteq \mathcal{D}$
- Fix a language $\mathcal{L}$ and a satisfaction relation $\models \subseteq \mathcal{D} \times \mathcal{L}$
- $\Delta \subseteq \mathcal{L}$ be a set of axioms

Δ axiomatizes $\mathcal{T}$ iff for all $\mathcal{M} \in \mathcal{D}$, $\mathcal{M} \in \mathcal{D}$ iff $\mathcal{M} \models \Delta$.

Arrow's Theorem

$\mathcal{D}$ is the set of CCRs w.r.t. 3 or more candidates, $\mathcal{T}$ is the class of dictatorships, $\mathcal{L}$ is the given language. Δ is the properties of Arrow's theorem, then Δ axiomatizes $\mathcal{T}$.

What Does Arrow's Theorem Say?

A CCR is not the only way to characterize a social choice process. We can also characterize it by the set of decisive coalitions, in the sense that we describe a social choice process by listing its decisive coalitions.

Question

Is there any relation of correspondence between the set of decisive coalitions and CCRs with certain characteristics.

Representation in Economics

Representation Theorems

Some data is consistent with **some model** if and only if it satisfies **some axioms**.

Example

Under what circumstances can we think of a decision maker (DM) as a preference maximizer? In other words, when can the choices of a DM be represented as resulting from the maximization of a complete, transitive, reflexive binary relation on $\mathcal{X}$?

- If $x \in B \subseteq A$ and $x \in C(A)$, then $x \in C(B)$
- If $x, y \in C(A), A \subseteq B$ and $y \in C(B)$ then $x \in C(B)$

Representation

In the previous example, a choice function is consistent with a preference order of a DM iff it satifies above two axioms. In other words, the choice function is represented by a preference order.

Then, we can think of a set of decisive coalitions is consistent with a CCR under Arrow's conditions iff it is an ultrafilter. In other words, the set of decisive coalitions is represented by a CCR with certain characteristics.

Decisiveness Function

Given Arrow's conditions, a set $A \subseteq \mathcal{V}$ is decisive for x over y entails A is decisive for all pairs of alternatives. But it is not always the case under other conditions. In other words, we can not always show the decisive coalitions by simply giving a set of subsets of $\mathcal{V}$. We need to define a function to tell the decisive coalitions for each pair of alternatives.

Definition

Let f be a CCR for $\langle \mathcal{X}, \mathcal{V} \rangle$. We define a fuction $\overline{D}_f : \mathcal{X}^2 \rightarrow \wp(\wp(\mathcal{V}))$ as follws:

$\overline{D}_f(x, y)$ is the set of all $A \subseteq \mathcal{V}$ that are decisive for x over y according to f.

Decisively Representable

Definition

Let K be a class of CCRs for $\langle \mathcal{X}, \mathcal{V} \rangle$. A function $D : \mathcal{X}^2 \to \wp(\underline{\wp}(\mathcal{V}))$ is decisively representable in K iff there is an $f \in K$ s.t. $D = \overline{D}_f$, in which case we say that f decisively represents D.

Representation Theorem

Theorem

Let $\mathcal{X}$ and $\mathcal{V}$ be nonempty sets with $|\mathcal{X}| \geq 3$. A function $D : \mathcal{X}^2 \to \wp(\wp(V))$ is decisively representable in the class of CCRs for $\langle X, V \rangle$ satisfying UD, IIA, and FR if and only if for all $A, B, C \subseteq \mathcal{V}$ and $x, y, z \in \mathcal{X}$ with $x \neq y$, $y \neq z$, and $x \neq z$:

1. $A \in D(x, x)$ *if and only if* $A \neq \varnothing$;
2. *if* $A \in D(x, y)$ *and* $A \cap B = \varnothing$, *then* $B \notin D(y, x)$;
3. *if* $A \in D(x, y)$ *and* $B \cap C \subseteq A \subseteq B \cup C$, *then* $B \in D(x, z)$ *or* $C \in D(z, y)$;
4. *if* $A \in D(x, y)$ *and* $A \subseteq B$, *then* $B \in D(x, y)$.

Proof

Suppose D is decisively representable by a CCR f satisfying UD, IIA, and FR.

For property 1, for any profile $\mathbf{P}$, we have $\mathbf{P}(x, x) = \varnothing$. So if $A \neq \varnothing$, then we trivially have $A \in D(x, x)$. On the other hand, if $A = \varnothing$, then since not $xf(\mathbf{P})x$, we have $A \notin D(x, x)$.

For property 2, suppose $A \in D(x, y)$ and $A \cap B = \varnothing$. Then by UD, there is a profile $\mathbf{P}$ in which $\mathbf{P}(x, y) = A$ and $\mathbf{P}(y, x) = B$. since $A \in D(x, y)$, it follows that $xf(\mathbf{P})y$, which with asymmetry implies not $yf(\mathbf{P})x$. Therefore, $B \notin D(y, x)$.

Property 5 holds is immediate from the definition of decisiveness.

Proof

For property 3, suppose $B \notin D(x,z)$, so there is a profile $\mathbf{P}$ in which $B \subseteq \mathbf{P}(x,z)$ and not $xf(\mathbf{P})z$. To show that $C \in D(z,y)$, consider a profile $\mathbf{P}^\star$ such that $C \subseteq \mathbf{P}^\star(z,y)$. We claim that there is a profile $\mathbf{P}'$ such that:

$$A \subseteq \mathbf{P}'(x,y) \quad \mathbf{P}'_{|\{y,z\}} = \mathbf{P}^\star_{|\{y,z\}} \quad \mathbf{P}'_{|\{x,z\}} = \mathbf{P}_{|\{x,z\}}$$

Then since $\mathbf{P}_{|\{x,z\}} = \mathbf{P}'_{|\{x,z\}}$, from not $xf(\mathbf{P})z$ we have not $xf(\mathbf{P}')z$ by IIA, and since $A \in D(x,y)$ we have $xf(\mathbf{P}')y$. Therefore, $zf(\mathbf{P}')y$ by the negative transitivity condition of FR. Then since $\mathbf{P}'_{|\{y,z\}} = \mathbf{P}^\star_{|\{y,z\}}$, we have $zf(\mathbf{P}^\star)y$ by IIA. Thus, $C \in D(z,y)$

Proof

cell	strict weak order
$A \cap B \cap C$	linear: $xP_i^0 zP_i^0 y$
$A \cap B \cap C^C$	$xP_i^0 y, xP_i^0 z$, and P_i^0 relates y and z as $P_i^\star$ does
$A \cap B^C \cap C$	$xP_i^0 y, zP_i^0 y$, and P_i^0 relates x and z as P_i does
$A^C \cap B \cap C^C$	$\begin{cases} \text{linear}: xP_i^0 yP_i^0 z & \text{if } yP_i^\star z \\ \text{linear}: xP_i^0 zP_i^0 y & \text{if } zP_i^\star y \\ xP_i^0 y, xP_i^0 z, yN_i^0 z & \text{if } yN_i^\star z \end{cases}$
$A^C \cap B^C \cap C$	$xP_i^0 y, zP_i^0 y$, and P_i^0 relates x and z as P_i does
$A^C \cap B^C \cap C^C$	$\begin{cases} \text{linear}: xP_i^0 zP_i^0 y & \text{if } xP_i z, zP_i^\star y \\ \text{linear}: xP_i^0 yP_i^0 z & \text{if } xP_i z, yP_i^\star z \\ xP_i^0 z, xP_i^0 y, zN_i^0 y & \text{if } xP_i z, zN_i^\star y \\ \text{linear}: zP_i^0 xP_i^0 y & \text{if } zP_i x, zP_i^\star y \\ \text{linear}: yP_i^0 zP_i^0 x & \text{if } zP_i x, yP_i^\star z \\ yP_i^0 x, zP_i^0 x, zN_i^0 y & \text{if } zP_i x, zN_i^\star y \\ xP_i^0 y, zP_i^0 y, xN_i^0 z & \text{if } xN_i z, zP_i^\star y \\ \ldots \end{cases}$

Proof

Conversely, suppose properties 1-4 hold. We define a CCR f as follows: for any $\mathbf{P} \in \mathcal{O}(\mathcal{X})^{\mathcal{V}}$, let $xf(\mathbf{P})y$ if and only if $\mathbf{P}(x,y) \in D(x,y)$. First, we claim that f satisfies FR:

- asymmetry: if $xf(\mathbf{P})y$, then not $yf(\mathbf{P})x$. Suppose $xf(\mathbf{P})y$, so $\mathbf{P}(x,y) \in D(x,y)$. Since $\mathbf{P}(x,y) \cap \mathbf{P}(y,x) = \varnothing$, property 2 implies $\mathbf{P}(y,x) \notin D(y,x)$ and hence not $yf(\mathbf{P})x$.
- negative transitivity (assuming property 3): if $xf(\mathbf{P})y$, then $xf(\mathbf{P})z$ or $zf(\mathbf{P})y$. Suppose $xf(\mathbf{P})y$, so $\mathbf{P}(x,y) \in D(x,y)$. Then since $\mathbf{P}(x,z) \cap \mathbf{P}(z,y) \subseteq \mathbf{P}(x,y) \subseteq \mathbf{P}(x,z) \cup \mathbf{P}(z,y)$, it follows by property 3 that $\mathbf{P}(x,z) \in D(x,z)$ or $\mathbf{P}(z,y) \in D(z,y)$, which implies $xf(\mathbf{P})z$ or $zf(\mathbf{P})y$.

Proof

Next observe that f satisfies IIA. Suppose $\mathbf{P}_{|\{x,y\}} = \mathbf{P}'_{|x,y\}}$. Then $\mathbf{P}(x, y) = \mathbf{P}'(x, y)$ and $\mathbf{P}(y, x) = \mathbf{P}'(y, x)$. It follows by the definition of f that $f(\mathbf{P})$ and $f(\mathbf{P}')$ agree with respect to x and y.

Finally, we claim that D is decisively represented by f. Suppose $A \in D(x, y)$ and consider any profile $\mathbf{P} \in \mathcal{O}(\mathcal{X})^{\mathcal{V}}$ in which $A \subseteq \mathbf{P}(x, y)$. Then by property 4 $A \in D(x, y)$ implies $\mathbf{P}(x, y) \in D(x, y)$, which implies $xf(\mathbf{P})y$ by the definition of f. Hence $A \in \overline{D}_f(x, y)$. Conversely, suppose $A \notin D(x, y)$, so by property 1 we have $A = \varnothing$ if $x = y$. Consider a profile $\mathbf{P} \in \mathcal{O}(\mathcal{X})^{\mathcal{V}}$ in which $A = \mathbf{P}(x, y)$. Then $A = \mathbf{P}(x, y) \notin D(x, y)$, so by definition of f we have not $xf(\mathbf{P})y$. Therefore, $A \notin \overline{D}_f(x, y)$.

Representation Theorem

Theorem

Let $\mathcal{X}$ and $\mathcal{V}$ be nonempty sets with $|\mathcal{X}| \geq 3$. A function $D : \mathcal{X}^2 \to \wp(\wp(V))$ is decisively representable in the class of CCRs for $\langle X, V \rangle$ satisfying UD, IIA, and FR if and only if for all $A, B, C \subseteq \mathcal{V}$ and $x, y, z \in \mathcal{X}$ with $x \neq y$, $y \neq z$, and $x \neq z$:

1. *$A \in D(x, x)$ if and only if $A \neq \varnothing$;*
2. *if $A \in D(x, y)$ and $A \cap B = \varnothing$, then $B \notin D(y, x)$;*
3. *if $A \in D(x, y)$ and $B \cap C \subseteq A \subseteq B \cup C$, then $B \in D(x, z)$ or $C \in D(z, y)$;*
4. *if $A \in D(x, y)$ and $A \subseteq B$, then $B \in D(x, y)$.*

The Correspondence

Question

Is there any relation of correspondence between the set of decisive coalitions and CCRs with certain characteristics.

By Representation Theorem, we have answered the above question in some way. We show that given two of Arrow's axioms about group decision methods, namely the Independence of Irrelevant Alternatives (IIA) and Universal Domain (UD), rationality postulate of FR for group preference corresponds to axioms about decisive coalitions.

The axioms about decisive coalitions that we identify are indeed consequences of IIA, UD. But, are there any other consequences of decisive coalitions given two of Arrow's conditions? We need a logic.

Syntax

Fix a finite set $\mathcal{X}$ with $\mathcal{X} \geq 3$.

Definition

Let **Coal** be a nonempty set, called the set of coalition labels. The set **Term** of coalition terms is generated by the following grammmar where $a \in$ **Coal**:

$$t ::= a \mid 0 \mid 1 \mid -t \mid (t \sqcap t) \mid (t \sqcup t).$$

Let **Alt** be a set with $|\textbf{Alt}| = |\mathcal{X}|$, called the set of alternative labels. The set **Form** of formulas is generated by the following grammmar where $t \in$ **Term** and $x, y \in$ **Alt**:

$$\phi ::= t \equiv t \mid D_{x>y}(t) \mid \neg\phi \mid (\phi \rightarrow \phi).$$

Semantics

Definition

Given a nonempty set $\mathcal{V}$, a coalition labeling for $\mathcal{V}$ is a function α : **Coal** $\to \wp(\mathcal{V})$. We extend α to a function $\dot{\alpha}$: **Term** $\to \wp(\mathcal{V})$ as follows:

$$\begin{array}{lll} \dot{\alpha}(a) = \alpha(a) \text{ for } a \in \text{ Coal} & & \dot{\alpha}(-t) = \alpha(t)^c \\ \dot{\alpha}(0) = \varnothing & \dot{\alpha}(s \sqcap t) = \dot{\alpha}(s) \cap \dot{\alpha}(t) & \\ \dot{\alpha}(1) = \mathcal{V} & \dot{\alpha}(s \sqcup t) = \dot{\alpha}(s) \cup \dot{\alpha}(t) & \end{array}.$$

An alternative labeling is a bijection β : **Alt** $\to \mathcal{X}$.

Semantics

Models

A model is a tripet$\langle f, \alpha, \beta \rangle$ where f is a CCR for $\langle \mathcal{X}, \mathcal{V} \rangle$, α a coalition labeling and β an alternative labeling.

Satisfaction Relation

$$\begin{aligned} f \models_{\alpha,\beta} s \equiv t \quad &\Leftrightarrow \quad \dot{\alpha}(s) = \dot{\alpha}(t) \\ f \models_{\alpha,\beta} D_{x>y}(t) \quad &\Leftrightarrow \quad \dot{\alpha}(t) \in \overline{D}_f(\beta(x), \beta(y)) \end{aligned}$$

Boolean connectives are defined as usual.

Example

$$(s \sqsubseteq t) := s \sqcap t \equiv s, \qquad D(t) := \bigwedge_{x,y \in \mathbf{Alt}, x \neq y} D_{x>y}(t), \qquad D(1)$$

Logic

Fro any class K of CCRs we can ask the following key logical question:

Question

Is there a formal calculus for deriving all and the only formulas that are true of CCRs in K?

The answer is yes where K is the class of CCRs satifying UD, IIA, and FR.

Formal System

Axioms Part I

1. all valid equations of set, such as $\neg(0 \equiv 1)$, distribution law
2. $s \equiv t \rightarrow (\phi[s/u] \leftrightarrow \phi[t/u])$
3. all instances of tautologies of propositional logic
4. MP
5. if $\vdash \phi$, then $\vdash \phi[s/u]$ and $\vdash \phi(x/y)$ where x does not occur in ϕ

Formal System

Axioms in Representation Theorem

1. $A \in D(x,x)$ if and only if $A \neq \varnothing$;
2. if $A \in D(x,y)$ and $A \cap B = \varnothing$, then $B \notin D(y,x)$;
3. if $A \in D(x,y)$ and $B \cap C \subseteq A \subseteq B \cup C$, then $B \in D(x,z)$ or $C \in D(z,y)$;
4. if $A \in D(x,y)$ and $A \subseteq B$, then $B \in D(x,y)$.

Axioms Part II

1. $D(x,x)(a) \leftrightarrow \neg(a \equiv 0)$
2. $(D_{x>y}(a) \wedge ((a \sqcap b) \equiv 0)) \rightarrow \neg D_{y>x}(b)$
3. $(D_{x>y}(a) \wedge (b \sqcap c \sqsubseteq a) \wedge (a \sqsubseteq b \sqcup c)) \rightarrow (D_{x>z}(b) \vee D_{z>y}(c))$
4. $(D_{x>y}(a) \wedge (a \sqsubseteq b)) \rightarrow D_{x>y}(b)$

Soundness and Completeness

For logic we have shown before, it has "all" part completeness and "only" part soundness.

Theorem

- *Soundness: if ϕ is a theorem, then for any* **nonempty** *set $\mathcal{V}$, ϕ is true of all CCRs satisfying UD, IIA and FR.*
- *Completeness: if for any* **finite nonempty** *set $\mathcal{V}$, ϕ is true of all CCRs satisfying UD, IIA and FR, then ϕ is a theorem.*

Theorem

The set of theorems is decidable.

Arrow's Theorem Again

In this logic, Arrow's theorem may be presented in this way:

Theorem

1. $\vdash D(1) \to \neg D(0)$
2. $\vdash D(1) \to (D(a) \wedge (a \sqsubseteq b)) \to D(b)$
3. $\vdash D(1) \to ((D(a) \wedge D(b)) \leftrightarrow D(a \sqcap b))$
4. $\vdash D(1) \to (D(a) \vee D(-a))$

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