Decidable Bundled Fragments of First-Order Modal Logic

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2020.03.17

Some Results

Language	Decidability
P^1	undecidable
x, y, p, P^1	undecidable
$x, y, \Box_i$, single P^1	undecidable
single x	decidable

where P^1 refers to unary predicates, x, y refers to the two-variable fragment, $\Box_i$ is multi-modal logic.

Know-wh Logics

Recent years witnessed a growing interest in non-standard epistemic logics of knowing whether, knowing how, knowing what, knowing why and so on. The new epistemic modalities introduced in those logics all share, in their semantics, the general schema of $\exists x \Box(x)$:

"knowing how to achieve ϕ" roughly means that there exists a way such that you know that it is a way to ensure that ϕ;

"knowing why ϕ" means that there exists an explanation such that you know that it is an explanation to the fact ϕ.

Moreover, the resulting logics are decidable.

Inspiration

Inspired by those particular logics, [Wang, 2017] proposes a *bundled* modality $\exists x \Box$, which packs exactly $\exists x \Box$ together. The resulting language, though much more expressive, shares finite-tree-model property over increasing domain.

Bundled Modalities: $\forall x \Box, \Box \exists x, \Box \forall x$?
Domain: Increasing; Costant

B-FOML

We only consider the "pure" first order unimodal logic (no equality, no constants and no function symbols).

Definition (Language)

The bundled fragment of FOML is defined as follows:

$$\varphi ::= P\overline{x} \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \exists x \Box \phi \mid \forall x \Box \phi \mid \Box \forall x \phi \mid \Box \exists x \phi$$

where $\overline{x}$ denotes a finite sequence of variables.

We denote the fragment $\mathrm{B}^{\exists\Box}$ -FOML to be the formulas which contains only $\exists\Box$ (and its dual $\forall\Diamond$) formulas; $\mathrm{B}^{\forall\Box\exists\Box}$ -FOML which contains $\forall\Box$ (and its dual $\exists\Diamond$) and $\exists\Box$ (and $\forall\Diamond$) formulas.

Negation Normal Form

For technical reasons, we consider formulas given in negation normal form (NNF) (of $\mathsf{B}^{\exists\Box\forall\Box}$-FOML):

$$\varphi ::= P\bar{x} | \neg P\bar{x} | (\varphi \wedge \varphi) | (\varphi \vee \varphi) | \exists x \Box \varphi | \forall x \Diamond \varphi | \forall x \Box \varphi | \exists x \Diamond \varphi$$

Every B-FOML-formula φ can be rewritten into an equivalent formula in NNF. Formulas of the form $P\bar{x}$ and $\neg P\bar{x}$ are literals.

Clean Formulas

We call a formula clean if no variable occurs both bound and free in it and every use of a quantifier quantifies a distinct variable. A finite set of formulas is clean if their conjunction is clean.

Note that every B-FOML-formula can be rewritten into an equivalent clean formula.
For instance, $P(x) \wedge \exists x \Box Q(x)$ is unclean formula, whereas $P(x) \wedge \exists y \Box Q(y)$ is its clean equivalents; $\exists x \Box P(x) \vee \exists x \Box Q(x)$ is unclean formula, whereas $\exists x \Box P(x) \vee \exists y \Box Q(y)$ is its clean equivalent.

Model

Definition (Constant Domain Model)

An constant domain model $\mathcal{M}$ for B-FOML is a tuple (W, D, R, ρ) where

- W is a non-empty set of worlds;
- D is a non-empty domain;
- $R \subseteq (W \times W)$;
- $\rho : (W \times \mathbf{P}) \to \bigcup_{n \in \omega} 2^{D^n}$ such that ρ assigns to each n-ary predicate on each world an n-ary relation on D.

Model

Definition (Increasing Domain Model)

An increasing domain model $\mathcal{M}$ for B-FOML is a tuple (W, D, δ, R, ρ) where

- W is a non-empty set of worlds;
- D is a non-empty domain;
- $R \subseteq (W \times W)$;
- $\delta : W \to 2^D$ assigns to each $w \in W$ a non-empty local domain s.t. wRv implies $\delta(w) \subseteq \delta(v)$ for any $w, v \in W$;
- $\rho : (W \times \mathbf{P}) \to \bigcup_{n \in \omega} 2^{D^n}$ such that ρ assigns to each n -ary predicate on each world an n-ary relation on D.

Given a model $\mathcal{M}$, we write D_w for $\delta^{\mathcal{M}}(w)$.
Note that a constant domain model is one where $D_w = D^M$ for any $w \in W^M$.
A finite model is one with both $W^{\mathcal{M}}$ finite and $D^{\mathcal{M}}$ finite.

Semantics

Given $\mathcal{M} = (W, D, \delta, R, \rho)$, $w \in W$, and an assignment $\sigma : \text{Var} \to D$, define $\mathcal{M}, w, \sigma \vDash \varphi$ as follows:

$$\mathcal{M}, w, \sigma \vDash P(x_1 \cdots x_n) \iff (\sigma(x_1), \cdots, \sigma(x_n)) \in \rho(P, w)$$

$\mathcal{M}, w, \sigma \vDash \exists x \Box \varphi$ $\iff$ there is some $d \in \delta(w)$ such that $\mathcal{M}, w, \sigma[x \mapsto d] \vDash \Box \varphi$
$\iff$ there is some $d \in \delta(w)$ such that $\mathcal{M}, v, \sigma[x \mapsto d] \vDash \varphi$ for all v s.t. wRv

$\mathcal{M}, w, \sigma \vDash \Box \exists x \varphi$ $\iff$ for all v such that $wRv, \mathcal{M}, v, \sigma \vDash \exists x \varphi$
$\iff$ for all v such that wRv, there is some $d \in \delta(v)$ such that $\mathcal{M}, v, \sigma[x \mapsto d] \vDash \varphi$

$\mathcal{M}, w, \sigma \vDash \forall x \Diamond \varphi$ $\iff$ for all $d \in \delta(w)$, there is some $v \in W$ such that wRv and $\mathcal{M}, v, \sigma[x \mapsto d] \vDash \varphi$

$\mathcal{M}, w, \sigma \vDash \Diamond \forall x \varphi$ $\iff$ there is some v such that wRv and $\mathcal{M}, v, \sigma[x \mapsto d]$ for all $d \in \delta(v)$

Fragment	Increasing Domain	Constant Domain
$\exists\Box$	decidable	decidable
$\forall\Box$	decidable	undecidable
$\Box\exists$	decidable	?
$\Box\forall$	decidable	undecidable

Tableau

A tableau is a tree structure $T = (W, V, E, \lambda)$ where W is a finite set, (V, E) is a rooted tree and $\lambda : V \to L$ is a labelling map. Each element in L is of the form (w, Γ, F), where $w \in W, \Gamma$ is a finite set of formulas and $F \subseteq \mathrm{Var}$ is a finite set. The intended meaning of the label is that the node constitutes a world w that satisfies the formulas in Γ with the "assignment" F, with each variable in F denoting one that occurs free in Γ and as we will see, the assignment will be the identity.

Tableau Rules for $B^{\exists\Box\forall\Box}$-FOML

$$\frac{w:\varphi_1\vee\varphi_2,\Gamma,F}{w:\varphi_1,\Gamma,F\,|\,w:\varphi_2,\Gamma,F}(\vee)\qquad\frac{w:\varphi_1\wedge\varphi_2,\Gamma,F}{w:\varphi_1,\varphi_2,\Gamma,F}(\wedge)$$

(BR) Given $n_1\geq 1$ or $m_1\geq 1$; $n_2,m_2,s\geq 0$:

$$\frac{\begin{array}{l} w:\exists x_1\Diamond\alpha_1,\cdots,\exists x_{n_1}\Diamond\alpha_{n_1},\exists y_1\Box\beta_1,\cdots,\exists y_{n_2}\Box\beta_{n_2},\\ \quad\forall z_1\Diamond\varphi_1,\cdots,\forall z_{m_1}\Diamond\varphi_{m_1},\forall z'_1\Box\psi_1,\cdots,\forall z'_{m_2}\Box\psi_{m_2}\\ \quad l_1,\cdots,l_s,F\end{array}}{\begin{array}{c}\left\langle wv_{x_i}:\alpha_i,\{\beta_j|1\leq j\leq n_2\},\{\psi_l\left[z/z'_l\right]|z\in F',l\in[1,m_2]\},F'\right\rangle\cup\\ wv_{z_k}^{y'}:\varphi_k[y'/z_k],\{\beta_j|1\leq j\leq n_2\},\{\psi_l[z/z'_l]|z\in F',l\in[1,m_2]\},F'\end{array}}$$

where $y'\in F'=F\cup\{x_1,\cdots,x_{n_1}\}\cup\{y_1,\cdots,y_{n_2}\}$,
$i\in[1,n_1]$, $k\in[1,m_1]$

(END) Given $n_2\geq 1$ or $m_2\geq 1$; $s\geq 0$:

$$\frac{w:\exists y_1\Box\beta_1,\cdots,\exists y_{n_2}\Box\beta_{n_2},\forall z'_1\Box\psi_1,\cdots,\forall z'_{m_2}\Box\psi_{m_2},l_1,\cdots,l_s,F}{w:l_1,\cdots,l_s,F}$$

Proposition

The rule BR is well-defined. Specifically, if the label in the premise contains only clean formulas, then the label in the conclusion does the same.

Proof.

Note that a formula is clean if no variable occurs both bound and free in it and every use of a quantifier quantifies a distinct variable.

Every use of a quantifier in the conclusion quantifies a distinct variable.

Let Γ be a set of clean formulas. Let Δ, Δ' stand for any modality. If $\exists x \Delta \varphi$ and $\forall y \Delta' \psi$ are both in Γ, then x do not occur in $\forall y \Delta' \psi$. The formula of the form $\psi[z/y]$ is clean, since z is free in the premise or z is some x. □

Note that maintaining "cleanliness" allows us to treat existential quantifiers as giving their own witnesses.

A tableau is said to be *open* if it does not contain any node u such that its label contains a literal l as well as its negation.

We say a node $(w : \Gamma, F)$ is a *branching node* if it is branching due to the application of BR.
We call $(w : \Gamma, F)$ the *last node* of w, if it is a leaf node or a branching node. Given any label w appearing in any node of a tableau T, the last node of w uniquely exists. If it is a non-leaf node, every child of w is labelled wu for some u.

Let t_w denote the last node of w in tableau T and let $\lambda(t_w) = (w : \Gamma, F)$. If it is a non-leaf node, then it is a branching node with rule (BR) applying to it with F' as its conclusion. We let $\text{Dom}(t_w)$ denote the set F' in this case and $\text{Dom}(t_w) = F$ otherwise.

Theorem

For any clean $B^{\exists\Box\forall\Box}$-formula θ in NNF, let $F_r = \{x|x \text{ is free in } \theta\} \cup \{z\}$, where $z \in \text{Var}, z$ does not appear in θ. Then: There is an open tableau T from $(r : \{\theta\}, F_r)$ iff θ is satisfiable in an increasing domain model.

Proof.

$(\Rightarrow)$ Note that we include a new variable $z \in F_r$ to ensure that the domain is always non-empty.

Define $\mathcal{M} = (W, D, \delta, R, \rho)$ where:
$W = \{w|(w : \Gamma, F)$ occurs in some label of $T\}$; $D = \text{Var}$;
wRv iff $v = wv'$ for some v'; $\delta(w) = \text{Dom}\,(t_w)$;
$\bar{x} \in \rho(w, P)$ iff $P\bar{x} \in \Gamma$, where $\lambda\,(t_w) = (w, \Gamma, F)$.
Clearly, if wRv then $\text{Dom}\,(t_w) \subseteq \text{Dom}\,(t_v)$. Moreover ρ is well-defined due to openness of T.
Next we show that $\mathcal{M}, r, id$ is indeed a model of θ.

cont.

For any $w \in W$, if $\lambda(t_w) = (w : \Gamma, F)$ and $\alpha \in \Gamma$ then $(\mathcal{M}, w, id) \models \alpha$.
The proof proceeds by reverse induction on the height of the node at which w occurs as label.
The base case is when the node considered is a leaf node and hence it is also the last node with that label. The definition of ρ ensures that the literals are evaluated correctly. Hence the base case follows.
For the induction step, the conjunction and disjunction cases are trivial.

cont.

Next consider the applicaion of BR at a branching node t_w with label $(w : \Gamma, F)$. Let

$$\begin{aligned}\Gamma = \ & \{\exists x_i \Diamond \alpha_i | i \in [1, n_1]\} \cup \{\exists y_j \Box \beta_j | j \in [1, n_2]\} \\ & \cup \{\forall z_k \Diamond \varphi_k | k \in [1, m_1]\} \cup \{\forall z'_l \Box \psi_q | l \in [1, m_2]\} \\ & \cup \{r_1 \ldots r_s\}\end{aligned}$$

By IH, we have that for every $i \in [1, n_1]$ and every $y \in F'$,

$$\mathcal{M}, wv_{x_i}, id \vDash \alpha_i \wedge \Lambda \beta_j \wedge \Lambda \psi_l [y/z'_i]$$

and for every $k \in [1, m_1]$,

$$\mathcal{M}, wv^{y}_{z_k}, id \vDash \varphi_k \, [y/z_k] \wedge \Lambda \beta_j \wedge \Lambda \psi_l [y/z'_i].$$

Note that $D_w = \mathrm{Dom}\,(t_w) = F'$. Next we show that $\mathcal{M}, w, id \vDash \alpha$ for each $\alpha \in \Gamma$.

cont.

$$\mathcal{M}, wv_{x_i}, id \vDash \alpha_i \wedge \bigwedge\beta_j \wedge \bigwedge\psi_l[y/z'_i]$$

$$\mathcal{M}, wv^{y}_{z_k}, id \vDash \varphi_k\,[y/z_k] \wedge \bigwedge\beta_j \wedge \bigwedge\psi_l[y/z'_i]$$

Every such α is either a literal or a bundled formula. The assertion for literals follows from the definition of ρ.

For $\exists x_i \Diamond \alpha_i \in \Gamma$, we have the successor wv_{x_i} where $\mathcal{M}, wv_{x_i}, id \vDash \alpha_i$. Then $\mathcal{M}, w, id[x_i \mapsto x_i] \vDash \Diamond\alpha_i$. Since $x_i \in D_w$, $\mathcal{M}, w, id \vDash \exists x_i \Diamond \alpha_i$.

For every $\forall z_k \Diamond \varphi_k \in \Gamma$ and $y \in D_w$, we have the successor $wv^{y}_{z_k}$ where $\mathcal{M}, wv^{y}_{z_k}, id \vDash \varphi_k\,[y/z_k]$. Thus $\mathcal{M}, w, id \vDash \forall z_k \Diamond \varphi_k$.

For the case $\exists y_j \Box \beta_j$: by IH, for all $wv^{\#}_{z}$ ($\#$ is empty or $\# \in F'$), we have $\mathcal{M}, wv^{\#}_{z}, id \vDash \beta_j$, that is, $\mathcal{M}, wv^{\#}_{z}, id\,[y_j \mapsto y_j] \vDash \beta_j$. Since $y_i \in F' = D_w$, $\mathcal{M}, w, id \vDash \exists y_j \Box \beta_j$.

The case $\forall z'_l \Box \psi_l$ is similar.

The soundness of tableau construction is finished.

cont.

($\Leftarrow$) We show that all rule applications preserve the satisfiability of the formula sets in the labels. This would ensure that there is an open tableau since satisfiability of formula sets ensures lack of contradiction among literals.
$\wedge, \vee$ and END are trivial. It remains only to show that BR preserves satisfiability.
Consider a label set Γ of clean formulas at a branching node. Let

$$\Gamma = \{\exists x_i \Diamond \alpha_i\} \cup \{\exists y_j \Box \beta_j\} \cup \{\forall z_k \Diamond \varphi_k]\} \cup \left\{\forall z'_l \Box \psi_q\right\} \cup \{r_1 \ldots r_s\}$$

be satisfiable at a model $\mathcal{M} = \{W, D, \delta, R, \rho\}, w \in W$ and a relevant assignment η .

cont.

By the semantics, we have the following:

(A)($\exists x_i \Diamond \alpha_i$) There exist $a_1, \ldots, a_{n_1} \in D_w$ and $v_1 \ldots v_{n_1} \in W$ where wRv_i such that $\mathcal{M}, v_i, \eta[x_i \mapsto a_i] \vDash \alpha_i$ for all i

(B)($\exists y_j \Box \beta_j$) There exist $b_1, \ldots b_{n_2} \in D_w$ such that for all $v \in W$ if wRv then $\mathcal{M}, v, \eta[y_j \mapsto b_j] \vDash \beta_j$ for all j

(C)($\forall z_k \Box \phi_k$) For all $c \in D_w$ there exist $v_1^c \ldots v_{m_1}^c \in W$, where $wRv_{m_t}^c$ such that $\mathcal{M}, v_k^c, \eta[z_k \mapsto c] \vDash \varphi_k$ for all for all k

(D)($\forall z_l' \Box \psi_q$) For all $d \in D_w$ and for all $v \in W$ if wRv then $\mathcal{M}, v, \eta[z_l' \mapsto d] \vDash \psi_l$ for all l

Moreover, due to the fact that Γ is clean, we observe that:

(O) $\bar{x}, \bar{y}, \bar{z}$ and $\overline{z'}$ only occur in $\alpha_i, \beta_j, \varphi_k$ and ψ_l respectively.

We now need to show:

(1) $\{\alpha_i\} \cup \{\beta_j | 1 \leq j \leq n_2\} \cup \{\psi_i[f/z_i'] | f \in F', 1 \leq l \leq m_2\}$ is satisfiable for all i

(2) $\{\varphi_k[f'/z_k]\} \cup \{\beta_j | 1 \leq j \leq n_2\} \cup \{\psi_l[f/z_l'] | f \in F', 1 \leq l \leq m_2\}$ is satisfiable for all k and all $f' \in F'$

cont.

For (1): given i, we can pick an $a_i \in D_w$ and a successor v_i of w, and some $\bar{b} \in D_w$ such that
$\mathcal{M}, v_i, \eta\left[x_i \mapsto a_i; \bar{y} \mapsto \bar{b}\right] \vDash \alpha_i \wedge \bigwedge \beta_j \wedge \bigwedge \{\langle \psi_l [z/z_l'] | z \in F'\}$
For (2): Given k and $f' \in F'$. Suppose $\eta(f') = c \in D_w$, then we have a successor v_k^c of w such that

$$\mathcal{M}, v_k^c, \eta[\bar{y} \mapsto \bar{b}] \vDash \varphi_k \left[y'/z_k\right] \wedge \bigwedge \beta_j \wedge \bigwedge \left\{\psi_l \left[z/z_l'\right] | z \in F'\right\}$$

Theorem

$B^{\forall\Box\exists\Box}$*-FOML is decidable over increasing domain models.*

Theorem

$B^{\Box\forall\Box\exists}$*-FOML is decidable over increasing domain models.*

(BR) Given $n_1 \geq 1$ or $m_1 \geq 1$; $n_2, m_2, s \geq 0$:

$$\frac{\begin{array}{c} w : \Box\exists x_1\varphi_1, \cdots, \Box\exists x_{n_1}\varphi_{n_1}, \Box\forall y_1\psi_1, \cdots, \Box\forall y_{n_2}\psi_{n_2}, \\ \Diamond\forall z_1\alpha_1, \cdots, \Diamond\forall z_{m_1}\alpha_{m_1}, \Diamond\exists u_1\beta_1, \cdots, \Diamond\exists u_{m_2}\beta_{m_2}, \\ l_1, \ldots, l_s, F \end{array}}{\begin{array}{c} \langle wv_{z_i} : \varphi_i, \{\psi_j[x/y_j] \mid j \in [1, n_2], x \in \sigma'\}, \{\alpha_i[y/z_i] \mid y \in F'\}, F'\rangle \\ \cup\langle wv_{u_k} : \varphi_1, \cdots, \varphi_{n_1}, \{\psi_j[x/y_j] \mid j \in [1, n_2], x \in F'\}, \beta_k, F'\rangle \end{array}}$$

where $F' = F \cup \{x_1, \ldots, x_{n_1}, u_1, \ldots, u_{m_2}\}, i \in [1, n_1], k \in [1, m_2]$.

Proposition

The $B^{\exists\Box}$-FOML is decidable over constant domain models.

In these models, we need to fix the domain right at the start of the tableau construction and use only these elements as witnesses. We do this by calculating a precise bound on how many new elements need to be added for each subformula of the form $\exists x \Box \phi$ and include as many as needed at the beginning of the tableau construction.

Let Sub(θ) stand for the finite set of subformulas of θ. Given a clean formula $\theta \in \mathrm{B}^{\exists\Box}$ -FOML in NNF, for every $\exists x_j \Box\varphi \in \mathrm{Sub}(\theta)$ let $\mathsf{Var}^{\exists}(\theta) = \{x | \exists x \Box\varphi \in \mathsf{Sub}(\theta)\}$.

Fix a clean formula θ in NNF with modal depth h. For every $x \in \mathsf{Var}^{+}(\theta)$ define Var_x to be the set of h fresh variables $\{x^k | 1 \leq k \leq h\}$, and let $\mathsf{Var}^{+}(\theta) = \bigcup \{\mathsf{Var}_x \, | x \in \mathsf{Var}^{\exists}(\theta)\}$ be the set of new variables to be added.

Fix a variable z not occurring in $\mathsf{Var}^{+}(\theta)$. Define $D_\theta = \mathrm{Fv}(\theta) \cup \mathsf{Var}^{+}(\theta) \cup \{z\}$

Tableau Rules for $B^{\exists\Box}$-FOML

(BR$_c$) Given $n, s \geq 0$, $m \geq 1$:

$$\frac{w : \exists\Box x_1\varphi_1, \cdots, \exists\Box x_n\varphi_n, \forall\Diamond y_1\psi_1, \cdots, \forall\Diamond y_m\psi_m, l_1, \ldots, l_s, C}{\langle wv^y_{y_i} : \{\varphi_j[x_j^{k_j}/x_j] | 1 \leq j \leq n\}, \psi_i[y/y_i], C'\rangle}$$

where $y \in D_\theta$, $i \in [1, m]$, $C \subseteq D_\theta$ and $C' = C \cup \{x_j^{k_j} | 1 \leq j \leq n\}$, k_j is the smallest number such that $x_j^{k_j} \in \mathrm{Var}_{x_j} \setminus C$.

Theorem

For any clean $B^{\exists\Box}$ -FOML-formula θ in NNF, there is an open constant tableau from $(r, \{\theta\}, Fv(\theta))$ iff θ is satisfiable in a constant domain model.

proof

$(\Leftarrow)$ We shall consider only the (BR_c) rule in the proof here.
Suppose $(w : \Gamma, C)$ is a branching node where
$\Gamma = \{\exists x_1 \Box \varphi_1 \ldots \exists x_n \Box \varphi_n, \forall y_1 \Diamond \psi_1 \ldots \forall y_m \Diamond \psi_m, r_1, \ldots r_s\}$
By IH, $\mathcal{M}, wv^y_{y_i}, id \vDash \psi_i[y/y_i] \wedge \bigwedge \varphi_j[x_j^{k_j}/x_j]$ for every $y \in D_\theta$ and $i \in [1, m]$.
Next we show that $\mathcal{M}, w, id \vDash \Gamma$.
For each $\exists x_j \Box \varphi_j \in \Gamma$ and each $wv^y_{y_i}$, with $y \in D_\theta$, we have $\mathcal{M}, wv^y_{y_i}, id \vDash \varphi_j[x_j^{k_j}/x_j]$ by induction hypothesis. It is clear that $\{x_j^{k_j} | 1 \leq j \leq n\}$ are not free in φ_j since they are chosen to be new. Further, since $x_j^{k_j}$ are not free in φ_j, $\mathcal{M}, wv^y_{y_i}, id[x_j \mapsto x_j^{k_j}] \vDash \varphi_j$ for all $wv^y_{y_i}$. Therefore $\mathcal{M}, w, id \vDash \exists x_j \Box \varphi_i$.