Independence-friendly Logic

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April, 7th, 2020

Agenda

- Background
- Syntax of IF logic.
- Semantic of IF logic
  - Extensive game definition of imperfect information game
  - Game theory semantic of IF logic
  - Skolem semantic of IF logic
- Basic property
  - Compactness
  - Löwenheim-Skolem
  - Completeness

Motivation:information dependencies

Independence friendly logic (IF logic, IF first-order logic) is an extension of first-order logic. It was introduced by Jaakko Hintikka and Gabriel Sandu in their article 'Informational Independence as a Semantical Phenomenon' (1989).

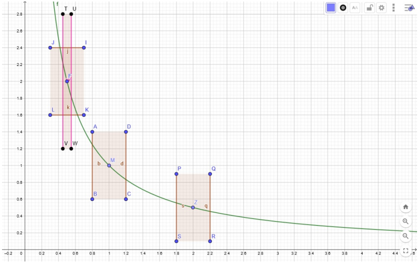
In it, more **quantifier dependencies and independencies** can be expressed than in first-order logic. Quantifier dependencies/independencies is actually information dependencies/independencies relation. The phenomenon is widespread in language. For example:

Someone loves everybody.($\exists x \forall y love(x, y)$)

Everyone loves someone.($\forall x \exists y love(x, y)$)

Another example from calculus

Let's see a concrete example from calculus:
A function f:$D \to R$ is **continuous**, if for all x_0 in the set D and for all $\epsilon > 0$, there exists $\delta > 0$ such that for all x in D, if $| x - x_0 | < \delta$,then $| f(x) - f(x_0) | < \epsilon$.
The definition of **uniform continuity** is obtained from that of continuity by specifying that the quantifer 'there exists δ' depends only on the quantifier 'for all ϵ', not on the quantifier 'for all x_0'.
Example: $f(x) = 1/x$



An example from calculus

The function f is said to be continuous on D iff

$$\forall x_0 \forall \varepsilon \exists \delta \forall x \left[|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon \right]$$

The function f is said to be uniformly continuous on D

$$\forall x_0 \forall \varepsilon \exists \delta / x_0 \forall x \left[|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon \right]$$

IF first-order logic is an extension of first- order logic, involving a specific syntactic device '/' (slash, independence indicator).

Background

For sentence $(\forall x)(\exists y)R(x, y)$.The dependence of $\exists y$ on $\forall x$ means that the witness of $\exists y$ may vary with the value interpreting $\forall x$. This dependence is made explicit in the language of **Skolem functions**, whose use in general in the semantics requires the Axiom of Choice. That is to say there be a function f such that $R(a, f(a))$ holds in M , for any a interpreting $\forall x$.
So for formula

$$(\forall x)(\forall y)(\exists z/\forall y)R(x, y, z)$$

to be true in M , there must be a function f of one argument such that $R(a, b, f(a))$ holds in M. We will show the strict definition of Skolem function later.

What is gained with the slash notation?

$(\forall x)(\forall y)(\exists z/\forall y)R(x,y,z)$ is true in a model M iff the first-order sentence $(\forall x)(\exists z)(\forall y)R(x,y,z)$ is true.

What is gained with the slash notation?

Consider the sentence $(\forall x)(\exists y)(\forall z)(\exists w/\{x,y\})R(x,y,z,w)$.Its truth-condition is of the following form: there are one-argument functions f and g such that $R(a,f(a),b,g(b))$ holds in M.So the sentence is true iff the following sentence (*) containing a finite partially ordered quantifier is true:

$$\begin{matrix} \forall x \exists y \\ \forall z \exists w \end{matrix} \; R(x,y,z,w)$$

Consider the quantifier dependence relations as partial order, FO can only have one partial order chain, but IFL can have finite partial order chains. So we can see IFL can express more dependent or independent relations among quantifiers.

The deepest reason for IF logic, as Hintikka sees it, is that the relations of dependence and independece between quantifiers are the only way of expressing relations of dependence and independece between variables on the first-order level . Relations of quantifier (in)dependence are semantic relations, but syntactically expressed. More precisely, in IF logic the (in)dependence relations are syntactically expressed by the interplay of two factors: syntactic scope and the independence indicator '/'.
"The additional expressive power of independence-friendly logic was the main reason why Hintikka advocated its superiority over first-order logic for the foundations of mathematics."

Motivation: game-theoretical semantics

Motivated in part by games with imperfect information, Hintikka and Sandu (1989) proposed consideration of semantic games where Eloise's choices do not depend on all (or any) of Abelard's prior choices. Recall first-order logic meets game theory as soon as one considers sentences with alternating quantifiers. For example a game between Abelard and Eloise that tests the truth of formula as below:

$$\forall x \exists y (x < y)$$

First Abelard picks an object m. Then Eloise observes which object Abelard chose, and picks another object n. If $m < n$, we declare that Eloise has won the game; otherwise we declare Abelard the winner. If we restrict Eloise's abilities of seeing objects Abelard has chosen. This kind of game is imperfect information game. The formula can be written as :$\forall x \exists y/x(x < y)$. Hintikka used the imperfect game theoretic interpreted the IF logic.

1,dependence logic why friendly?

Syntax of IF Logic

Let L be a vocabulary. L-terms are defined as for **first-order logic**. The independence-friendly language IFL is generated from L according to the following rules:

- If t_1 and t_2 are L -terms, then $(t_1 = t_2) \in \mathrm{IF}_L$ and $\neg(t_1 = t_2) \in \mathrm{IF}_L$
- If R is an n -ary relation symbol in L and $t_1, \ldots, t_n$ are L -terms, then $R(t_1, \ldots, t_n) \in \mathrm{IF}_L$ and $\neg R(t_1, \ldots, t_n) \in \mathrm{IF}_L$.
- If $\varphi, \varphi' \in \mathrm{IF}_L$, then $(\varphi \vee \varphi') \in \mathrm{IF}_L$ and $(\varphi \wedge \varphi') \in \mathrm{IF}_L$
- If $\varphi \in \mathrm{IF}_L, x$ is a variable, and W is a finite set of variables, then $(\exists x/W)\varphi \in \mathrm{IF}_L$ and $(\forall x/W)\varphi \in \mathrm{IF}_L$.

Syntax of IF logic

- To simplify the presentation, we only allow the negation symbol $\neg$ to appear in front of atomic formulas.We will see later that this restriction is not essential; it simply allows us to assume that Eloise is always the verifier. Formulas of the form $(t1 = t2)$, $\neg(t1 = t2)$,$R(t1, ..., tn)$, or $\neg R(t1, ..., tn)$ are called **literals**.
- The finite set of variables W in $\exists x/W$ and $\forall x/W$ is called a slash set.

Subformula

Let ϕ be an IF formula. The *subformulas* of ϕ are defined recursively:

$$\text{Subf}(\psi) = \{\psi\} \quad (\psi \text{ literal })$$

$$\text{Subf}\,(\psi \circ \psi') = \{\psi \circ \psi'\} \cup \text{Subf}(\psi) \cup \text{Subf}\,(\psi')$$

$$\text{Subf}((Qx/W)\psi) = \{(Qx/W)\psi\} \cup \text{Subf}(\psi)$$

IF logic semantics

- **Game-theoretic Sematics**
  We interpret IF formulas as specifying a game with imperfect information.
- Skolem Semantics
- Compositional Semantics

Defining the semantics of a logic using GTS is a two-step process. The first step is to define the relevant imperfect semantic games. The second step is to define the notions of 'true' and 'false' in terms of the semantic games; this happens by reference to the notion of winning strategy.

Extensive game with perfect information

An extensive game form with perfect information has the following components:

- N, a set of players.
- H, a set of finite sequences called histories or plays.
  - If $(a_1, \ldots, a_\ell) \in H$ and $(a_1, \ldots, a_n) \in H$, then for all $\ell < m < n$ we must have $(a_1, \ldots, a_m) \in H$. We call $(a_1, \ldots, a_\ell)$ an initial segment and $(a_1, \ldots, a_n)$ an extension of $(a_1, \ldots, a_m)$
  - A sequence $(a_1, \ldots, a_m) \in H$ is called an initial history (or minimal play) if it has no initial segments in H, and a terminal history (or maximal play) if it has no extensions in H. We require every history to be either terminal or an initial segment of a terminal history. The set of terminal histories is denoted Z

Extensive game with perfect information

- $P : (H - Z) \to N$, the player function, which assigns a player $p \in N$ to each nonterminal history.
  - We imagine that the transition from a nonterminal history $h = (a_1, \ldots, a_m)$ to one of its successors $h^\frown a = (a_1, \ldots, a_m, a)$ in H is caused by an action. We will identify actions with the final member of the successor.
  - The player function indicates whose turn it is to move. For every nonterminal history $h = (a_1, \ldots, a_m)$, the player $P(h)$ chooses an action a' from the set

$$A(h) = \{a : (a_1, \ldots, a_m, a) \in H\}$$

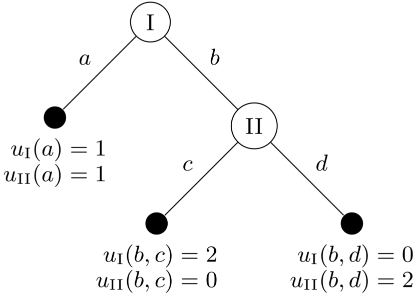
    and play proceeds from $h' = (a_1, \ldots, a_m, a')$

An extensive game with perfect information has the above components, plus:

- $\cdot u_p : Z \to \mathbb{R}$, a utility function (also called a payoff function) for each player $p \in N$

Background of game theory

When drawing extensive game forms, we label decision points with the active player, and edges with actions. Filled-in nodes represent terminal histories. Picture below shows the extensive form of a simple two-player game. First, player I chooses between two actions a and b. If she chooses a the game ends. If she chooses b, player II chooses between actions c and d.



Background of game theory

A two-player extensive game is **strictly competitive** if the players have no incentive to cooperate, that is, if for all $h, h' \in Z$

$$u_{\mathrm{I}}(h) \geq u_{\mathrm{I}}(h') \quad \text{iff} \quad u_{\mathrm{II}}(h') \geq u_{\mathrm{II}}(h)$$

A **constant-sum** game is one in which the sum of the players' payoffs is constant, i.e., there exists a $c \in \mathbb{R}$ such that for every terminal history h we have $u_{\mathrm{I}}(h) + u_{\mathrm{II}}(h) = c$. When $c = 0$ the game is called **zero sum**. Every constant-sum game is strictly competitive, but not vice versa.

An extensive game is **win-lose** if exactly one player wins each terminal history, in which case we can replace the players' utility functions with

$$u : Z \to N, \text{the winner function}$$

which indicates the winner of each terminal history.

Strategy

Let $H_p = P^{-1}(p)$ denote the set of histories where it is player p 's turn to move. A strategy for player p is a choice function

$$\sigma \in \prod_{h \in H_p} A(h)$$

that tells the player how to move whenever it is his or her turn. A player follows a strategy σ during a history $h' = (a_1, \ldots, a_n)$ if, whenever $h = (a_1, \ldots, a_m) \in H_p$ is an initial segment of h', the history

$$h \sim \sigma(h) = (a_1, \ldots, a_m, \sigma(h))$$

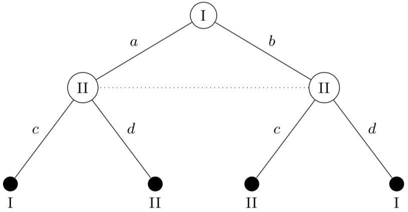
is either h' or an initial segment of h'

Extensive game with imperfect information

An extensive game form with imperfect information is a tuple

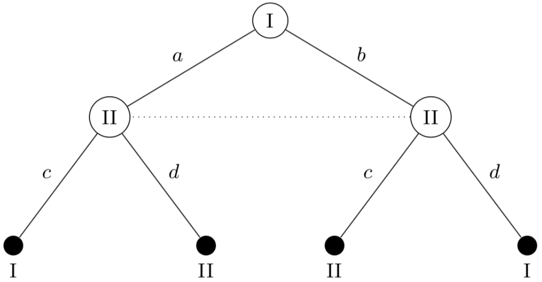
$$(N, H, P, \{\sim_p : p \in N\}, u_p)$$

$\sim_p$ is an equivalence relation on H_p with the property that $A(h) = A(h')$ whenever $h \sim_p h'$. When $h \sim_p h'$ we say that h and h' are indistinguishable for player p. (Without it, we can somehow know the previous step).



Strategy restriction of imperfect information game

A strategy σ for player p in an extensive game with imperfect information is defined as for an extensive game with perfect information, with the restriction that $\sigma(h) = \sigma(h')$ whenever $h \sim_p h'$.



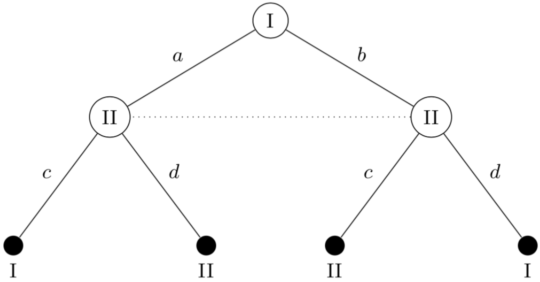
δ is a choice function, choose next step from action function $A(h)$.

Gale-Stewart theorem?

Gale-Stewart theorem

Every two-player, win-lose, extensive game with perfect information that has finite horizon is determined.

But this theorem fails on imperfect information games.



Neither player I nor player II has winning strategy.

Game-theoretic semantics for IF logic

We will define the semantic game for an IF formula as an extensive game with imperfect information by restricting the players' access to the current assignment. That is, a player may be forced to choose an action without knowing the current assignment in its entirety.

- Two assignments, s and s', such that $W \subseteq \operatorname{dom}(s) = \operatorname{dom}(s')$ are *equivalent modulo* W (or W -equivalent), denoted $s \approx_W s'$ if for every variable $x \in \operatorname{dom}(s) - W$ we have $s(x) = s'(x)$.

Game-theoretic semantics for IF logic

Let φ be an IF formula, $\mathbb{M}$ a suitable structure, and s an assignment whose domain contains Free (φ). The semantic game $G(\mathbb{M}, s, \varphi)$ is a win-lose extensive game with imperfect information:

- There are two players, Eloise ($\exists$) and Abelard (V).
- The set of histories is $H = \bigcup \{H_\psi : \psi \in \mathrm{Subf}(\varphi)\}$, where H_ψ is defined recursively:
    - $H_\varphi = \{(s, \varphi)\}$
    - if ψ is $\chi_1 \circ \chi_2$, then $H_{\chi_i} = \{h \frown \chi_i : h \in H_{\chi_1 \circ \chi_2}\}$
    - if ψ is $(Qx/W)\chi$, then $H_\chi = \left\{h^\frown(x, a) : h \in H_{(Qx/W)\chi}, a \in M\right\}$

Observe that (s, φ) is the unique initial history. The assignment s is called the initial assignment. Every history h' induces an assignment $s_{h'}$ extending and/or modifying the initial assignment:

$$s_{h'} = \begin{cases} s & \text{if } h' = (s, \varphi) \\ s_h & \text{if } h' = h' \sim \chi \\ s_h(x/a) & \text{if } h' = h^\frown(x, a) \end{cases}$$

- Once play reaches a literal, the game ends:

$$Z = \bigcup \{H_\chi : \chi \in \text{Lit}(\varphi)\}$$

- Disjunctions and existential quantifiers are decision points for Eloise, while conjunctions and universal quantifiers are decision points for Abelard:

$$P(h) = \begin{cases} \exists & \text{if } h \in H_{\chi \vee \chi'} \text{ or } h \in H_{(\exists x/W)\chi} \\ \forall & \text{if } h \in H_{\chi \wedge \chi'} \text{ or } h \in H_{(\forall x/W)\chi} \end{cases}$$

- The indistinguishability relations $\sim_\exists$ and $\sim_\forall$ are defined as follows. Certainly $\forall h, h' \in H$, we have $A(h) = A(h')$
For all $h, h' \in H_{\chi \vee \chi'}$: we have $h \sim_\exists h'$ if and only if $s_h = s_{h'}$.
For all $h, h' \in H_{(\exists x/W)\chi}$
$$h \sim_\exists h' \text{ iff } \quad s_h \approx_W s_{h'}$$
Similarly, for all $h, h' \in H_{\chi \wedge \chi'}$ we have $h \sim_\forall h'$ only if $s_h = s_{h'}$ and for all $h, h' \in H_{(\forall x/W)\chi}$
$$h \sim_\forall h' \text{ iff } s_h \approx_W s_{h'}$$

- Eloise wins if the literal χ reached at the end of play is satisfied by the current assignment; Abelard wins if it is not:

$$u(h) = \begin{cases} \exists & \text{if } \mathbb{M}, s_h \models \chi \\ \forall & \text{if } \mathbb{M}, s_h \not\models \chi \end{cases}$$

- Let φ be an IF formula, M a suitable structure, and s an assignment whose domain contains Free (φ).
$\mathbb{M}, s \models^{+}_{\mathrm{GTS}} \varphi$ iff Eloise has a winning strategy for $G(\mathbb{M}, s, \varphi)$.
$\mathbb{M}, s \models^{-}_{\mathrm{GTS}} \varphi$ iff Abelard has a winning strategy for $G(\mathbb{M}, s, \varphi)$.

Why we define satisfaction and dissatisfaction unlike Tarski Semantics? Because it fails of bivalence shown as next example.

Example 1 In the game Matching Pennnies there are two players. Each player has a coin that he or she secretly turns to heads or tails. The coins are revealed simultaneously. The first player wins if the coins are both heads or both tails; the second player wins if they differ. We can express the game Matching Pennies using the IF sentence:

$$\forall x(\exists y/x)x = y$$

interpreted in the two-element structure $\mathbb{M} = \{a, b\}$. Call the original sentence ϕ_{MP}, and let ψ be the subformula: $(\exists y/\{x\})x = y$.Then $H_{\varphi_{\mathrm{MP}}}$ includes only the initial history $(\varnothing, \varphi_{\mathrm{MP}})$, while H_ψ includes two histories: $h_a = (\varnothing, \varphi_{\mathrm{MP}}, (x, a))$ and $h_b = (\varnothing, \varphi_{\mathrm{MP}}, (x, b))$. Let σ be a strategy for Eloise. since $h_a \sim_\exists h_b$ she must choose the same value for y in both cases:

$$\sigma(h_a) = (y, c) = \sigma(h_b)$$

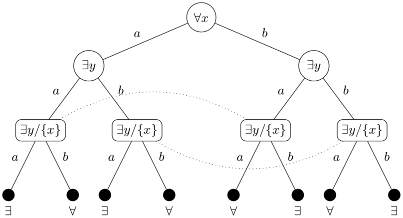
No matter $c = a$ or $c = b$, δ can not be a winning strategy.
Now let τ be a strategy for Abelard such that $\tau(\varnothing, \varphi_{\mathrm{MP}}) = (x, c)$ Then τ is a winning strategy if and only if Abelard wins both maximal plays $(\varnothing, \varphi_{\mathrm{MP}}, (x, c), (y, a))$ and $(\varnothing, \varphi_{\mathrm{MP}}, (x, c), (y, b))$ which is again impossible. In IFL, falsity does not ensue from non-truth. That is, bivalence fails in IFL.

Example 2 We add one dummy quantifier $\exists y$ to the sentence in Example 1 to get the irregular IF sentence:

$$\forall x \exists y (\exists y / x) x = y$$

Here is a winning strategy:

$$\sigma(h_a) = (y, a) \quad \text{and} \quad \sigma(h_{aa}) = \sigma(h_{ba}) = (y, a)$$
$$\sigma(h_b) = (y, b) \quad \text{and} \quad \sigma(h_{ab}) = \sigma(h_{bb}) = (y, b)$$



Signaling

Such phenomena are common in games of imperfect information. In bridge, skilled partners can communicate to each other about their hands using only the cards they play. Playing according to a predetermined convention in order to circumvent informational restrictions is called signaling.

Skolem semantics

Definition Let φ be an IF $_L$ formula, let U be a finite set of variables containing Free(φ), and let

$$L^* = L \cup \{f_\psi : \psi \in \mathsf{Subf}_\exists(\varphi)\}$$

be the expansion of L obtained by adding a fresh function symbol for every existentially quantified subformula of φ.
The Skolem form $\psi \in \mathsf{Subf}(\varphi)$ with variables in U is defined recursively:

$$\begin{array}{lll}
\mathrm{Sk}_U(\psi) & \textit{is} & \psi(\psi \text{ literal }) \\
\mathrm{Sk}_U(\psi \circ \psi') & \textit{is} & \mathrm{Sk}_U(\psi) \circ \mathrm{Sk}_U(\psi') \\
\mathsf{Sk}_U((\exists x/W)\psi) & \textit{is} & \mathsf{Subst}\left(\mathsf{Sk}_{U\cup\{x\}}(\psi), x, f_{(\exists x/W)\psi}(y_1,\ldots,y_n)\right) \\
\mathsf{Sk}_U((\forall x/W)\psi) & \textit{is} & \forall x\, \mathsf{Sk}_{U\cup\{x\}}(\psi)
\end{array}$$

where $y_1, \ldots, y_n$ enumerates the variables in $U - W$. Every FO formulas can be skolemized.

Example 1

Example Examine the Skolem form of the Matching Pennies sentence $\forall x(\exists y/\{x\})x = y$. Proceeding inside-out

$$\begin{aligned} \mathrm{Sk}_{\{x,y\}}(x = y) &\text{ is } x = y \\ \mathrm{Sk}_{\{x\}}[(\exists y/\{x\})x = y] &\text{ is } x = c \\ \mathrm{Sk}[\forall x(\exists y/\{x\})x = y] &\text{ is } \forall x(x = c) \end{aligned}$$

where c is a fresh constant symbol.

Example 2

Example Now consider the Skolem form of the Matching Pennies sentence augmented with a dummy quantifier, $\forall x \exists y (\exists y / \{x\}) x = y$:

$$\begin{array}{rlcl} \mathrm{Sk}_{\{x,y\}}(x & = y) & \text{is} & x = y \\ \mathrm{Sk}_{\{x,y\}}[(\exists y/\{x\})x & = y] & \text{is} & x = g(y) \\ \mathrm{Sk}_{\{x\}}[\exists y(\exists y/\{x\})x & = y] & \text{is} & x = g(f(x)) \\ \mathrm{Sk}[\forall x \exists y(\exists y/\{x\})x & = y] & \text{is} & \forall x[x = g(f(x))] \end{array}$$

We can consider y as a signal, like in bridge poker. From Signaling perspective, we can consider $f(x)$ as an encoder, and $g(x)$ as a decoder.

Skolem semantics

Definition Let φ be an IF $_L$ formula, $\mathbb{M}$ a suitable structure, and s an assignment whose domain contains Free (φ). Define

$$\mathbb{M}, s \models^{+}_{\mathrm{Sk}} \varphi \quad \text{iff} \quad \mathbb{M}^{*}, s \models SK_{dom(s)}(\varphi)$$

for some expansion $\mathbb{M}^{*}$ of $\mathbb{M}$ to the vocabulary

$$L^{*} = L \cup \{f_{\psi} : \psi \in \mathrm{Subf}_{\exists}(\varphi)\}$$

Skolem semantics

Theorem Let φ be an IF $_L$ formula, $\mathbb{M}$ a suitable structure, and s an assignment whose domain contains Free(φ) Then:

$$\mathbb{M}, s \models^{+}_{\mathrm{GTS}} \varphi \quad \textit{iff} \quad \mathbb{M}, s \models^{+}_{\mathrm{Sk}} \varphi$$

Proof $\Rightarrow$ Suppose Eloise has a winning strategy σ for $G(\mathbb{M}, s, \varphi)$. Let $\mathbb{M}^*$ be an expansion of $\mathbb{M}$ to the vocabulary

$$L^* = L \cup \{f_\psi : \psi \in \mathrm{Subf}_\exists(\varphi)\}$$

such that for every existential subformula $(\exists x/W)\psi'$ of φ and every history $h \in H_{(\exists x/W)\psi'}$

$$f^{\mathbb{M}^*}_{(\exists x/W)\psi'}\left(s_h\left(y_1\right), \ldots, s_h\left(y_n\right)\right) = a$$

where $y_1, \ldots, y_n$ enumerates $\mathrm{dom}\left(s_h\right) - W$, and $\sigma(h) = (x, a)$. To show the function is well defined, suppose $h, h' \in H_{(\exists x/W)\psi'}$ are two histories such that

$$\sigma(h) = (x, a) \neq (x, a') = \sigma\left(h'\right)$$

Then $s_h \not\approx_W s_{h'}$ which means $s_h\left(y_i\right) \neq s_{h'}\left(y_i\right)$ for some $y_i \in \{y_1, \ldots, y_n\}$

Proof $\Rightarrow$: We show by induction on ϕ.Suppose ϕ is literal. ...

Suppose ϕ is $\psi_1 \wedge \psi_2$.If Eloise follows σ in $h \in H_{\psi_1 \wedge \psi_2}$, then she follows σ in both $h_1 = h^\frown \psi_1$ and $h_2 = h^\frown \psi_2$. By inductive hypothesis $\mathbb{M}^*, s_{h_1} \models \mathrm{Sk}_{\mathrm{dom}(s_{h_1})}(\psi_1)$ and $\mathbb{M}^*, s_{h_2} \models \mathrm{Sk}_{\mathrm{dom}(s_{h_2})}(\psi_2)$, whence

$$\mathbb{M}^*, s_h \models \mathsf{SK}_{\mathsf{dom}(s_h)}(\psi_1) \wedge \mathsf{Sk}_{\mathsf{dom}(s_h)}(\psi_2)$$

It follows that $\mathbb{M}^*, s_h \models SK_{dom(s_h)}(\psi_1 \wedge \psi_2)$.

Suppose ϕ is $(\exists x/W)\psi'$. If Eloise follows σ in $h \in H_{(\exists x/W)\psi'}$, and $\sigma(h) = (x, a)$, then Eloise follows σ in $h' = h^\frown(x, a)$. By inductive hypothesis $\mathbb{M}^*, s_{h'} \models Sk_{dom(s_{h'})}(\psi')$, which is to say

$$\mathbb{M}^*, s_h(x/a) \models Sk_{dom(s_h(x/a))}(\psi')$$

By construction $f^{\mathbb{M}^*}_{(\exists x/W)\psi'}(s_h(y_1), \ldots, s_h(y_n)) = a$, where $y_1, \ldots, y_n$ enumerates $\mathrm{dom}(s_h) - W$, so an application of the substitution lemma yields

$$\mathbb{M}^*, s_h \models \left(\mathsf{Sk}_{\mathsf{dom}(s_h(x/a))}(\psi'), x, f_{(\exists x/W)\psi'}(y_1, \ldots, y_n)\right)$$

Hence $\mathbb{M}^*, s_h \models SK_{dom(s_h)}((\exists x/W)\psi')$ Boolean cases of $\vee$ and $\forall$ are similar.

Proof $\Leftarrow$: Conversely, suppose there is an expansion $\mathbb{M}^*$ of $\mathbb{M}$ such that

$$\mathbb{M}^*, s \models SK_{dom(s)}(\varphi)$$

Let σ be the strategy for Eloise defined as follows. If $h \in H_{\psi_1 \vee \psi_2}$, then

$$\sigma(h) = \begin{cases} \psi_1 & \text{if } \mathbb{M}^*, s_h \models \mathrm{Sk}_{\mathrm{dom}(s_h)}(\psi_1) \\ \psi_2 & \text{otherwise} \end{cases}$$

If $h \in H_{(\exists x/W)\psi'}$, then

$$\sigma(h) = \left(x, f^{\mathbb{M}^*}_{(\exists x/W)\psi'}\left(s_h\left(y_1\right), \ldots, s_h\left(y_n\right)\right)\right)$$

where $y_1, \ldots, y_n$ enumerates $\mathrm{dom}\left(s_h\right) - W$.
We show by induction on the length of h that if Eloise follows σ in $h \in H_\psi$, then $\mathbb{M}^*, s_h \models Sk_{dom(s_h)}(\psi)$. The basis step follows from the original supposition. For the inductive step, suppose Eloise follows σ in

$$h' = (s, \varphi, a_1, \ldots, a_m, a_{m+1})$$

Then she certainly follows σ in $h = (s, \varphi, a_1, \ldots, a_m)$.
Suppose $h \in H_{\psi_1 \wedge \psi_2}$. Then by inductive hypothesis

$$\mathbb{M}^*, s_h \models SK_{dom(s_h)}\left(\psi_1 \wedge \psi_2\right)$$

from which it follows that $\mathbb{M}^*, s_{h'} \models Sk_{dom(s_{h'})}(\psi_i)$

Proof "$\Leftarrow$": Suppose $h \in H_{\psi_1 \vee \psi_2}$ and $a_{m+1} = \psi_i$. Then by inductive hypothesis $\mathbb{M}^*, s_h \models Sk_{dom(s_h}\left(\psi_1 \vee \psi_2\right)$, so by construction $\mathbb{M}^*, s_{h'} \models Sk_{dom(s_{h'})}(\psi_i)$.

Suppose $h \in H_{(\exists x/W)\psi'}$ and $a_{m+1} = (x, a)$. By inductive hypothesis $\mathbb{M}^*, s_h \models Sk_{dom(s_h)}((\exists x/W)\psi')$, which is to say

$$\mathbb{M}^*, s_h \models Subst(\mathrm{Sk}_{dom(s_h)\cup\{x\}}\left(\psi'\right), x, f_{(\exists x/W)\psi}\left(y_1, \ldots, y_n\right))$$

where $y_1, \ldots, y_n$ enumerates $\mathrm{dom}\left(s_h\right) - W$. By the substitution lemma,

$$\mathbb{M}^*, s_h(x/a) \models SK_{(s_h)\cup\{x\}}\left(\psi'\right)$$

which implies $\mathbb{M}^*, s_{h'} \models SK_{(s_{h'})}(\psi')$.

Finally, observe that if Eloise follows σ in a terminal history $h \in H_\chi$ then $\mathbb{M}^*, s_h \models \mathrm{Sk}_{\mathrm{dom}(s_h)}(\chi)$. It follows that $\mathbb{M}, s_h \models \chi$, so Eloise wins h.Therefore σ is a winning strategy for Eloise.

Skolem function: example

An involution is a function f that satisfies $f(f(x)) = x$ for all x in its domain. A finite structure has an even number of elements if and only if there is a way of pairing the elements without leaving any element out, i.e., if there exists an involution without a fixed point. Let ϕ_{even} be the IF sentence:

$$\forall x \forall y (\exists u/\{y\})(\exists v/\{x, u\}) \\ [(x = y \to u = v) \wedge (u = y \to v = x) \wedge u \neq x]$$

The Skolem form of ϕ_{even} is

$$\forall x \forall y [(x = y \to f(x) = g(y)) \wedge (f(x) = y \to g(y) = x) \wedge f(x) \neq x]$$

Since f and g denote the same function, we can simplify $Sk(\phi_{even})$ to

$$\forall x [f(f(x)) = x \wedge f(x) \neq x],$$

which asserts that f is an involution without a fixed point. Therefore $Sk(\phi_{even})$ is satisfiable by an expansion of a finite structure if and only if the universe of the structure has an even number of elements.

Falsity and Kreisel counterexamples

Skolem functions encode Eloise's strategies for the relevant semantic game. In the next section, we show how to use Kreisel's counterexamples to encode Abelard's strategies.

Definition Let φ be an IF_L formula, and let

$$L^* = L \cup \{f_\psi : \psi \in \mathrm{Subf}_\forall(\varphi)\}$$

be the expansion of L obtained by adding a fresh function symbol for every universally quantified subformula of φ. The Kreisel form (or Kreiselization of $\psi \in \mathrm{Subf}(\varphi)$ with variables in U is defined recursively:

Falsity and Kreisel counterexamples

$$
\begin{array}{lll}
\mathrm{Kr}_U(\psi) & \text{is} & \neg\psi \quad (\psi \text{ literal })\\
\mathrm{Kr}_U(\psi \vee \psi') & \text{is} & \mathrm{Kr}_U(\psi) \wedge \mathrm{Kr}_U(\psi')\\
\mathrm{Kr}_U(\psi \wedge \psi') & \text{is} & \mathrm{Kr}_U(\psi) \vee \mathrm{Kr}_U(\psi')\\
\mathsf{Kr}_U((\exists x/W)\psi) & \text{is} & \forall x\, \mathsf{Kr}_{U\cup\{x\}}(\psi)\\
\mathsf{Kr}_U((\forall x/W)\psi) & \text{is} & \mathsf{Subst}\left(\mathsf{Kr}_{U\cup\{x\}}(\psi), x, f_{(\forall x/W)\psi}(y_1, \ldots, y_n)\right)
\end{array}
$$

where $y_1, \ldots y_n$ enumerates the variables in $U - W$.An interpretation of $f(\forall x/W)\psi$ is called a Kreisel counterexample. We abbreviate $Kr_\emptyset(\psi)$ by $Kr(\psi)$.

Falsity and Kreisel counterexamples

Definition Let ϕ be an *IFL* formula, $\mathbb{M}$ a suitable structure, and s an assignment whose domain contains $Free(\phi)$. Define

$$M, s \models^{-}_{Sk} \phi \quad iff \quad M^{*}, s \models Kr_{dom(s)}(\phi)$$

for some expansion $\mathbb{M}^{*}$ of $\mathbb{M}$ to the vocabulary

$$L^{*} = L \cup \{f_{\psi} : \psi \in Subf_{\forall}(\phi)\}$$

Theorem Let ϕ be an IFL formula, M a suitable structure, and s an assignment whose domain contains Free(ϕ). Then

$$M, s \models^{-}_{GTS} \phi \quad iff \quad M, s \models^{-}_{Sk} \phi$$

Proof The proof is dual of the previous proof.

Compactness

Compactness An IF theory Γ is satisfiable if every finite subtheory of Γ is satisfiable.

Proof Observe that an IF theory Γ is satisfiable if and only if

$$\Gamma^* = \{Sk(\phi) : \phi \in \Gamma\}$$

is satifiable. Hence, if every finite subtheory $\delta \subseteq \Gamma$ is satisfiable, then so is every finite subtheory $\Gamma^* \subseteq \Gamma^*$. By the compactness theorem for first-order logic, Γ^* must be satisfiable, which implies Γ is satisfiable too.

Compactness theorem?

Definition When $\Gamma \cup \{\phi\}$ is an IF theory, Γ *truth entail* ϕ, denoted $\Gamma \models^{+} \phi$, if:

$$\mathbb{M} \models^{+} \Gamma \quad \text{implies} \quad \mathbb{M} \models^{+} \phi$$

Γ *false entail* ϕ, denoted $\Gamma \models^{-} \phi$, if:

$$\mathbb{M} \models^{-} \Gamma \quad \text{implies} \quad \mathbb{M} \models^{-} \phi$$

An alternative formulation of the compactness theorem for first-order logic is the following: Every first-order theory $\Gamma \cup \{\varphi\}$ has the property that $\Gamma \vDash \varphi$ if and only if there exists a finite $\Delta \subseteq \Gamma$ such that $\Delta \models \varphi$ In contrast, when Γ is an IF theory it is possible to have $\Gamma \vDash^{+} \varphi$ even if $\Delta \nvDash^{+} \varphi$ for every finite $\Delta \subseteq \Gamma$. We will show it from next example.

An example of Compactness theorem fails

Example Let φ_n denote the IF sentence

$$\exists x_1 \ldots \exists x_n \left(\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \right)$$

which asserts that the universe has at least n elements. Then

$$\{\varphi_n : n \geq 2\} \models^+ \varphi_\infty$$

where φ_∞ is the IF sentence that asserts the universe is infinite (the previous example). However, there is no finite subtheory $\Delta \subseteq \{\varphi_n : n \geq 2\}$ such that $\Delta \models^+ \varphi_\infty$
It follows immediately from the previous example that IF cannot have a complete proof system in which proofs have finite length.

Completeness Theorem fails

Theorem There is no sound and semantically complete proof system for IF logic. That is, there is no proof system F_{IF} such that for every IF theory $\Gamma \cup \{\varphi\}$

$$\Gamma \models^+ \varphi \quad \text{iff} \quad \Gamma \vdash_{\mathrm{IF}} \varphi$$

Proof Suppose for the sake of a contradiction that $\vdash_{\mathsf{IF}}$ is such a proof system. Then by Example 5.52 we must have

$$\{\varphi_n : n \geq 2\} \vdash_{\mathsf{IF}} \varphi_\infty$$

since a proof of φ_∞ from $\{\varphi_n : n \geq 2\}$ can use at most finitely many premises, there must be a finite subtheory $\Delta \subseteq \{\varphi_n : n \geq 2\}$ such that $\Delta \vdash_{\mathrm{IF}} \varphi_\infty$, which would imply $\Delta \models^+ \varphi_\infty$

Weak completeness fails

A proof system $\vdash$ for a logical language is weakly complete if for every sentence φ in the language we have $\vdash \varphi$ if and only if $\models \varphi$. In other words, every valid sentence is provable. One naturally wonders whether IF logic might have a proof system that is complete in this weaker sense.

Theorem There is no proof system t $_{\text{IF}}$ such that for every IF sentence φ we have $\vdash_{\text{IF}} \varphi$ if and only if $\models^{+} \varphi$

Proof Observe that for every IF sentence φ we have $\models^{+} \varphi \vee \varphi_{\infty}$ if and only if φ is true in every (suitable) finite model. Thus, if there were such a proof system, the set of first-order sentences that are true in every (suitable) finite model would be recursively enumerable, contrary to Trakhtenbrot's theorem,according to which the set of FO sentences true in all finite models is not recursively enumerable.

Löwenheim-Skolem theorem

The Löwenheim-Skolem theorem states that if a countable first-order theory has an infinite model, then it has models of every infinite cardinality. Like the compactness theorem, we can extend the Löwenheim-Skolem theorem to IF logic.

Theorem (Löwenheim-Skolem) Let Γ be a countable IF theory. If there is an infinite structure $\mathbb{M}$ such that $\mathbb{M} \models^{+} \Gamma$, then for all infinite cardinalities κ there is a structure $\mathbb{M}'$ of size κ such that $\mathbb{M}' \models^{+} \Gamma$.

Proof Suppose $\mathbb{M}$ is an infinite structure such that $\mathbb{M} \vDash^{+} \Gamma$ and that κ is an infinite cardinal. Then there is an expansion $\mathbb{M}^{*}$ of $\mathbb{M}$ such that $\mathbb{M}^{*} \models \{\mathsf{Sk}(\varphi) : \varphi \in \Gamma\}$. By the Löwenheim-Skolem theorem for first- order logic, $\{\mathrm{Sk}(\varphi) : \varphi \in \Gamma\}$ has a model of cardinality κ, the reduct of which to the vocabulary of Γ is a structure $\mathbb{M}'$ of size κ such that $\mathbb{M}' \models^{+} \Gamma$

Reference