Complete Axiomatizations for Reasoning About Knowledge and Time

Kaiyang Lin

2020.10.13

Introduction

When analyzing a system in terms of knowledge, not only is the current state of knowledge of the agents in the system relevant, but also **how that state of knowledge changes over time**. A formal propositional logic of knowledge and time was first proposed by Sato[5].

In [3] Halpern logics for knowledge and time were categorized along two major dimensions: **the language used and the assumptions made on the underlying distributed system**. The properties of knowledge in a system turn out to depend in subtle ways on these assumptions.

Introduction

The assumptions considered in [3] concern **whether agents have unique initial states(uis), operate synchronously(sync) or asynchronously, have perfect recall(pr), and whether they satisfy a condition called no learning(nl)**. There are $2^4 = 16$ possible combinations of these assumptions on the underlying system. Together with 6 choices of language, this gives us 96 logics in all.

Of these 96 logics, 48 involve linear time and 48 involve branching time. We focus here on the linear time logics, and provide axiomatic characterizations of all the linear time logics for which an axiomatization is possible at all.

Language

Definition (Language)

The set of formulas CKL_m is defined inductively as follows:

$$\varphi ::= p \mid \varphi \wedge \psi \mid \neg\varphi \mid true \mid K_i\varphi \mid E\varphi \mid C\varphi \mid \bigcirc\varphi \mid \varphi U\psi$$

where $p \in \mathbf{Var}$.We use the abbreviation $true = \neg(p \wedge \neg p)$, $L_i\varphi := \neg K_i \neg\varphi$, $\Diamond\varphi = true\, U\varphi$, $\Box\varphi = \neg\Diamond\neg\varphi$.

Remark

We take CKL_m to be the language for m agents with all the modal operators for knowledge and linear time discussed above;KL_m is the restricted version without the common knowledge operator.

System

Definition (System)

A system for m agents consists of a set $\mathcal{R}$ of runs, where each run $r \in \mathcal{R}$ is a function from $\mathbb{N}$ to L^{m+1}, where L is some set of local states. Formally, we could view a system as a tuple $(\mathcal{R}, L, m)$, making the L and m explicit.

Remark

There is a local state for each agent, together with a local state for the **environment***;intuitively, the environment keeps track of all the relevant features of the system not described by the agents' local states, such as messages in transit but not yet delivered.Thus, $r(n)$ has the form $\langle l_e, l_1, \ldots, l_m \rangle$, where l_e is the state of the environment, and l_i is the local state of agent i, for $i = 1, \ldots, m$. Such a tuple is called a global state.*

Interpreted System

Definition (Interpreted System)

An interpreted system $\mathcal{I}$ for m agents is a tuple $(\mathcal{R}, \pi)$ where $\mathcal{R}$ is a system for m agents, and π maps every point $(r, n) \in \mathcal{R} \times \mathbb{N}$ to a truth assignment $\pi(r, n)$ to the primitive propositions (so that $\pi(r, n)(p) \in \{\text{true,false}\}$ for each primitive proposition p).

We now give semantics to CKL_m and KL_m. Given an interpreted system $\mathcal{I} = (\mathcal{R}, \pi)$, we write $(\mathcal{I}, r, n) \models \varphi$ if the formula φ is true at (or satisfied by) the point (r, n) of interpreted system $\mathcal{I}$. We define $\models$ inductively for formulas of CKL_m.

Semantics

In order to give the semantics for formulas of the form $K_i\varphi$, we need to introduce one new notion. If $r(n) = \langle l_e, l_1, ..., l_m \rangle$, $r'(n') = \langle l'_e, l'_1, ..., l'_m \rangle$, and $l_i = l'_i$, then we say that $r(n)$ and $r'(n')$ are **indistinguishable to agent** i and write $(r, n) \sim_i (r', n')$.

$\sim_i$ is an equivalence relation on global states. $K_i\varphi$ is defined to be true at (r, n) exactly if φ is true at all the points whose associated global state is indistinguishable to i from that of (r, n).

Semantics

We proceed as follows:

- $(\mathcal{I}, r, n) \models p$ for a primitive proposition p iff $\pi(r, n)(p) = \text{true}$
- $(\mathcal{I}, r, n) \models \varphi \wedge \psi$ iff $(\mathcal{I}, r, n) \models \varphi$ and $(\mathcal{I}, r, n) \models \psi$
- $(\mathcal{I}, r, n) \models \neg\varphi$ iff $(\mathcal{I}, r, n) \not\models \varphi$
- $(\mathcal{I}, r, n) \models K_i\varphi$ iff $(\mathcal{I}, r', n') \models \varphi$ for all (r', n') such that $(r, n) \sim_i (r', n')$
- $(\mathcal{I}, r, n) \models E\varphi$ iff $(\mathcal{I}, r, n) \models K_i\varphi$ for $i = 1, ..., m$
- $(\mathcal{I}, r, n) \models C\varphi$ iff $(\mathcal{I}, r, n) \models E^k\varphi$ for $k = 1, 2, ...$ (where $E^1\varphi = E\varphi$ and $E^{k+1}\varphi = EE^k\varphi$)

Semantics

- $(\mathcal{I}, r, n) \models \bigcirc\varphi$ iff $(\mathcal{I}, r, n+1) \models \varphi$
- $(\mathcal{I}, r, n) \models \varphi U \psi$ iff there is some $n' \geq n$ such that $(\mathcal{I}, r, n') \models \psi$, and for all n'' with $n \leq n'' < n'$, we have $(\mathcal{I}, r, n'') \models \varphi$.

Remark

Since $L_i\varphi = \neg K_i \neg\varphi$, we have $(\mathcal{I}, r, n) \models L_i\varphi$ iff there exists (r', n') such that $(r, n) \sim_i (r', n')$ such that $(\mathcal{I}, r', n') \models \varphi$.

Since $\Diamond\psi = true\ U\psi$, we have $(\mathcal{I}, r, n) \models \Diamond\psi$ iff there is some $n' \geq n$ such that $(\mathcal{I}, r, n') \models \psi$.

Semantics

There is a graphical interpretation of the semantics of C which we shall find useful in the sequel. Fix an interpreted system $\mathcal{I}$. A point (r', n') in $\mathcal{I}$ is **reachable** from a point (r, n) if there exist points $(r_0, n_0), \ldots, (r_k, n_k)$ such that $(r, n) = (r_0, n_0)$, $(r', n') = (r_k, n_k)$, and for all $j = 0, \ldots, k-1$ there exists i such that $(r_j, n_j) \sim_i (r_{j+1}, n_{j+1})$. Then we have

Lemma (2.1)

$(\mathcal{I}, r, n) \models C\varphi$ iff $(\mathcal{I}, r', n') \models \varphi$ for all points (r', n') reachable from (r, n).

Semantics

Lemma (2.1)

$(\mathcal{I}, r, n) \models C\varphi$ iff $(\mathcal{I}, r', n') \models \varphi$ for all points (r', n') reachable from (r, n).

Proof.

($\Rightarrow$) Assume $(\mathcal{I}, r, n) \models C\varphi$, since (r', n') is reachable from (r, n), then there exists points $(r_0, n_0), \ldots, (r_k, n_k)$ such that $(r, n) = (r_0, n_0)$, $(r', n') = (r_k, n_k)$, and for all $j = 0, \ldots, k-1$ there exists i_j such that $(r_j, n_j) \sim_{i_j} (r_{j+1}, n_{j+1})$.By $(\mathcal{I}, r, n) \models C\varphi$ we have $(\mathcal{I}, r_0, n_0) \models E^k\varphi$. Hence $(\mathcal{I}, r_0, n_0) \models K_{i_0}E^{k-1}\varphi$.By $(r_0, n_0) \sim_{i_0} (r_1, n_1)$, we have $(\mathcal{I}, r_1, n_1) \models E^{k-1}\varphi$.Then $(\mathcal{I}, r_1, n_1) \models K_{i_1}E^{k-2}\varphi$. By $(r_1, n_1) \sim_{i_1} (r_2, n_2)$, we have $(\mathcal{I}, r_2, n_2) \models E^{k-2}\varphi$.Similarly by iteration, we can get $(\mathcal{I}, r_3, n_3) \models E^{k-3}\varphi$,...,$(\mathcal{I}, r_k, n_k) \models \varphi$. That is $(\mathcal{I}, r', n') \models \varphi$.

□

Semantics

Lemma (2.1)

$(\mathcal{I}, r, n) \models C\varphi$ iff $(\mathcal{I}, r', n') \models \varphi$ for all points (r', n') reachable from (r, n).

Proof.

($\Leftarrow$) Assume $(\mathcal{I}, r', n') \models \varphi$ for all points (r', n') reachable from (r, n) but $(\mathcal{I}, r, n) \not\models C\varphi$. Then there exists k such that $(\mathcal{I}, r, n) \not\models E^k\varphi$.So there exists some i_k such that $(\mathcal{I}, r, n) \not\models K_{i_k}(E^{k-1}\varphi)$. Then there exists some point (r_1, n_1) such that $(r, n) \sim_{i_k} (r_1, n_1)$ but $(\mathcal{I}, r_1, n_1) \not\models E^{k-1}\varphi$.So there exists some i_{k-1} such that $(\mathcal{I}, r_1, n_1) \not\models K_{i_{k-1}}(E^{k-2}\varphi)$. Then there exists some point (r_2, n_2) such that $(r_1, n_1) \sim_{i_{k-1}} (r_2, n_2)$ but $(\mathcal{I}, r_2, n_2) \not\models E^{k-2}\varphi$.Similarly by iteration, we can get a chain of points $(r_1, n_1), \ldots, (r_k, n_k)$ such that $(r_j, n_j) \sim_{i_{k-j}} (r_{j+1}, n_{j+1})$ and $(\mathcal{I}, r_k, n_k) \not\models \varphi$. A contradiction to the assumption. □

Semantics

As usual, we define a formula φ to be valid with respect to a class $\mathcal{C}$ of interpreted systems iff $(\mathcal{I}, r, n) \models \varphi$ for all interpreted systems $\mathcal{I} \in \mathcal{C}$ and points $(r, n) \in \mathcal{I}$. A formula φ is satisfiable with respect to $\mathcal{C}$ iff for some $\mathcal{I} \in \mathcal{C}$ and some point $(r, n) \in \mathcal{I}$, we have $(\mathcal{I}, r, n) \models \varphi$.

We now turn our attention to formally defining the classes of interpreted systems of interest.

Perfect recall

Define agent i's **local-state sequence** at the point (r, n) to be the sequence $l_0, ..., l_k$ of states that agent i takes on in run r up to and including time n, with consecutive repetitions omitted.

Example

If from time 0 through 4 in run r agent i goes through the sequence l, l, l', l, l of states, its history at $(r, 4)$ is just l, l', l.

Definition (pr)

We say that agent i has perfect recall (alternatively, agent i does not forget) in system $\mathcal{R}$ if at all points (r, n) and (r', n') in $\mathcal{R}$, if $(r, n) \sim_i (r', n')$, then r has the same local-state sequence at both (r, n) and (r', n').

Perfect recall

Definition ($\sim$-concordant)

Let $S = (s_0, s_1, s_2, ...)$ and $T = (t_0, t_1, t_2, ...)$ be two (finite or infinite) sequences and let $\sim$ be a relation on the elements of S and T. Then we say that S and T are $\sim$-concordant if there is some k (k may be ∞) and nonempty consecutive intervals $S_1, ..., S_k$ of S and $T_1, ..., T_k$ of T such that for all $s \in S_j$ and $t \in T_j$, we have $s \sim t$ for $j = 1, ..., k$.

Perfect recall

Lemma (2.2)

The following are equivalent.

(a) *Agent i has perfect recall in system $\mathcal{R}$.*

(b) *For all points $(r, n) \sim_i (r', n')$ in $\mathcal{R}$, $((r, 0), \ldots, (r, n))$ is $\sim_i$-concordant with $((r', 0), \ldots, (r', n'))$.*

(c) *For all points $(r, n) \sim_i (r', n')$ in $\mathcal{R}$, if $n > 0$, then either $(r, n-1) \sim_i (r', n')$ or there exists a number $l' < n'$ such that $(r, n-1) \sim_i (r', l')$ and for all k with $l' < k' \leq n'$ we have $(r, n) \sim_i (r', k')$.*

(d) *For all points $(r, n) \sim_i (r', n')$ in $\mathcal{R}$, if $k \leq n$, then there exists $k' \leq n'$ such that $(r, k) \sim_i (r', k')$.*

Perfect recall

Proof.

$(a) \Rightarrow (b)$ For all points $(r,n) \sim_i (r',n')$ in $\mathcal{R}$, since i has perfect recall in system $\mathcal{R}$, then i has the same local-state sequence at both (r,n) and (r',n'), noted as $l_1, l_2, ..., l_k$.Let $S = ((r,0),...,(r,n))$ and $T = ((r',0),...,(r',n'))$. Since their local-state sequence is $l_1, l_2, ..., l_k$,there exists k and $j_1, j_2, ..., j_k$ and $j'_1, j'_2, ..., j'_k$ such that for all $s \in ((r,0),...,(r,j_1))$ and $t \in ((r',0),...,(r',j'_1))$, the local state of i is l_1.For all $s \in ((r,j_1+1),...,(r,j_2))$ and $t \in ((r',j'_1+1),...,(r',j'_2))$, the local state of i is l_2.Hence $((r,0),...,(r,n))$ is $\sim_i$-concordant with $((r',0),...,(r',n))$.

□

Perfect recall

Proof.

$(b) \Rightarrow (c)$ Assume $((r,0),\ldots,(r,n))$ is $\sim_i$-concordant with $((r',0),\ldots,(r',n))$, then there is some k and nonempty consecutive intervals $S_1,\ldots,S_k$ of S and $T_1,\ldots,T_k$ of T such that for all $s \in S_j$ and $t \in T_j$, we have $s \sim t$ for $j = 1,\ldots,k$.Obviously $(r,n) \in S_k$ and $(r',n') \in T_k$.If $(r,n-1) \in S_k$, then we have $(r,n-1) \sim_i (r',n')$. Otherwise, $S_k = ((r,n))$ and $(r,n-1) \in S_{k-1}$.Assume $T_k = ((r',l'+1),\ldots,(r',n'))$. Then $(r',l') \in T_{k-1}$. So $(r,n-1) \sim_i (r',l')$. And for all k with $l' < k' \leq n'$ we have $(r,n) \sim_i (r',k')$.

□

Perfect recall

Proof.

$(c) \Rightarrow (d)$ Assume $(r, n) \sim_i (r', n')$ in $\mathcal{R}$ and $k \leq n$, then by (c) we have either $(r, n-1) \sim_i (r', n')$ or there exists a number $l' < n'$ such that $(r, n-1) \sim_i (r', l')$.Started from $(r, n-1) \sim_i (r', n')$ or $(r, n-1) \sim_i (r', l)$ and using condition (c) for $n-k-1$ times, we can get $(r, k) \sim_i (r', n')$ or $(r, k) \sim_i (r', l')$ for some $l' < n'$.

$(d) \Rightarrow (a)$ We prove by induction on $n + n'$. Assume $(r, n) \sim_i (r', n')$. If $n + n' = 0$, by $(r, 0) \sim_i (r', 0)$, then i has the same local-state sequence at (r, n) and (r', n'). Assume if $(r, k) \sim_i (r', k')$, then i has the same local-state sequence at (r, k) and (r', k') for $k + k' < n + n'$. We aim to prove i has the same local-state sequence at (r, n) and (r', n'). □

Perfect recall

Proof.

First if $(r, n) \sim_i (r, n-1)$, then $(r, n-1) \sim_i (r', n')$. By induction hypothesis we have that i has the same local-state sequence at both $(r, n-1)$ and (r', n'). By $(r, n) \sim_i (r, n-1)$, then i has the same local-state sequence at (r, n) and (r', n'). Similarly for the case of $(r, n) \sim_i (r', n'-1)$. Consider $(r, n-1)$, by (d) we have there exists $k' < n'$ such that $(r, n-1) \sim_i (r', k')(*)$. Consider $(r', n'-1)$, by (d) we have there exists $k < n$ such that $(r, k) \sim_i (r', n'-1)$. Then i has the same local-state sequence at both (r, k) and $(r', n'-1)$. Combined with $(*)$ we have $(r, n-1) \sim_i (r', n'-1)$. By induction hypothesis we have i has the same local-state sequence at both $(r, n-1)$ and $(r', n'-1)$. Since $(r, n) \sim_i (r', n')$, so i has the same local-state sequence at both (r, n) and (r', n'). □

No learning

We define an agents **future local-state sequence** at (r, n) to be the sequence of local states $l_0, l_1, \ldots$ that the agent takes on in run r, starting at (r, n), with consecutive repetitions omitted. We say agent i **does not learn** in system $\mathcal{R}$ if at all points (r, n) and (r', n') in $\mathcal{R}$, if $(r, n) \sim_i (r', n')$, then r has the same future local-state sequence at both (r, n) and (r', n').

Lemma (2.3)

The following are equivalent.

(a) *Agent i does not learn in system $\mathcal{R}$.*

(b) *For all points $(r, n) \sim_i (r', n')$ in $\mathcal{R}$, $((r, n), (r, n+1), \ldots)$ is $\sim_i$-concordant with $((r', n'), (r, n'+1), \ldots)$.*

(c) *For all points $(r, n) \sim_i (r', n')$ in $\mathcal{R}$, if $n > 0$, then either $(r, n+1) \sim_i (r', n')$ or there exists a number $l > n$ such that $(r, n+1) \sim_i (r', l)$ and for all k with $l > k \geq n$ we have $(r, n) \sim_i (r', k)$.*

Sync,uis

Definition (sync)

We say that a system $\mathcal{R}$ is synchronous if for all agents i and all points (r, n) and (r', n'), if $(r, n) \sim_i (r', n')$ then $n = n'$.

Observe that in a synchronous system where $(r, n) \sim_i (r', n')$, an easy induction on n shows that if i has perfect recall and $n > 0$, then $(r, n-1) \sim_i (r', n'-1)$, while if i does not learn, then $(r, n+1) \sim_i (r', n'+1)$.

Definition (uis)

We say that a system $\mathcal{R}$ has a unique initial state if for all runs r, $r' \in \mathcal{R}$, we have $r(0) = r'(0)$.

If $\mathcal{R}$ is a system with a unique initial state, then we have $(r, 0) \sim_i (r', 0)$ for all runs r, r' in $\mathcal{R}$ and all agents i.

Axiom system

We describe the axioms and inference rules that we need for reasoning about knowledge and time for various classes of systems, and state the completeness results.

- $K1$. All tautologies of propositional logic
- $K2$. $K_i\varphi \wedge K_i(\varphi \rightarrow \psi) \rightarrow K_i\psi, i = 1, ..., m$
- $K3$. $K_i\varphi \rightarrow \varphi, i = 1, ..., m$
- $K4$. $K_i\varphi \rightarrow K_iK_i\varphi, i = 1, ..., m$
- $K5$. $\neg K_i\varphi \rightarrow K_i\neg K_i\varphi, i = 1, ..., m$
- $R1$. From φ and $\varphi \rightarrow \psi$ infer ψ
- $R2$. From φ infer $K_i\varphi, i = 1, ..., m$

This axiom system is known as $S5_m$, which is a sound and complete system for reasoning about knowledge alone.

Axiom system

For reasoning about the temporal operators individually, the following system (together with K1 and R1), is well known to be sound and complete.

$$T1. \quad \bigcirc(\varphi) \wedge \bigcirc(\varphi \rightarrow \psi) \rightarrow \bigcirc(\psi)$$

$$T2. \quad \bigcirc(\neg\varphi) \rightarrow \neg\bigcirc(\varphi)$$

$$T3. \quad \varphi U \psi \leftrightarrow \psi \vee (\varphi \wedge \bigcirc(\varphi U \psi))$$

$$RT1. \quad \text{From } \varphi \text{ infer } \bigcirc\varphi$$

$$RT2. \quad \text{From } \varphi' \rightarrow \neg\psi \wedge \bigcirc\varphi' \text{ infer } \varphi' \rightarrow \neg(\varphi U \psi)$$

Axiom system

The system containing the above axioms and inference rules for both knowledge and time is called $S5_m^U$. $S5_m^U$ is easily seen to be sound for $\mathcal{C}_m$, the class of all systems for m agents.Given that there is no necessary connection between knowledge and time in $\mathcal{C}_m$, it is perhaps not surprising that $S5C_m^U$ should be complete with respect to $\mathcal{C}_m$ as well.Interestingly, even if we impose the requirements of synchrony or uis, $\mathcal{C}_m$ remains complete; **our language is not rich enough to capture these conditions.**

Theorem (3.1)

$S5_m^U$ is a sound and complete axiomatization for the language KL_m with respect to $\mathcal{C}_m$, $\mathcal{C}_m^{sync}$, $\mathcal{C}_m^{uis}$, and $\mathcal{C}_m^{sync,uis}$, for all m.

Axiom system

It is well known that the following two axioms and inference rule characterize common knowledge:

$$C1.\quad E\varphi \leftrightarrow \wedge_{i=i}^{m} K_i\varphi$$

$$C2.\quad C\varphi \rightarrow E(\varphi \wedge C\varphi)$$

$$RC1.\quad \text{From } \varphi \rightarrow E(\psi \wedge C\varphi) \text{ infer } \varphi \rightarrow C\psi$$

Let $S5C_m^U$ be the result of adding C1, C2, and RC1 to $S5_m^U$.

Theorem (3.2)

$S5C_m^U$ is a sound and complete axiomatization for the language CKL_m with respect to $\mathcal{C}_m$, $\mathcal{C}_m^{sync}$, $\mathcal{C}_m^{uis}$, and $\mathcal{C}_m^{sync,uis}$, for all m.

Axiom system

Consider extending $S5_m^U$ by adding the following five axioms:

$KT1.\ K_i\Box\varphi \rightarrow \Box K_i\varphi, i = 1, ..., m$

$KT2.\ K_i \bigcirc \varphi \rightarrow \bigcirc K_i\varphi, i = 1, ..., m$

$KT3.\ K_i\varphi_1 \wedge \bigcirc(K_i\varphi_2 \wedge \neg K_i\varphi_3) \rightarrow L_i((K_i\varphi_1)U[(K_i\varphi_2)U\neg\varphi_3])$

$KT4.\ K_i\varphi_1 U K_i\varphi_2 \rightarrow K_i(K_i\varphi_1 U K_i\varphi_2), i = 1, ..., m$

$KT5.\ \bigcirc K_i\varphi \rightarrow K_i \bigcirc \varphi, i = 1, ..., m$

Axiom system

Theorem (3.4)

$S5^U_m + KT3$ is a sound and complete axiomatization for the language KL_m with respect to $\mathcal{C}^{pr}_m$ and $\mathcal{C}^{pr,uis}_m$, for all m.

KT3 characterise the condition of perfect recall.

Theorem (3.5)

$S5^U_m + KT2$ is a sound and complete axiomatization for the language KL_m with respect to $\mathcal{C}^{pr,sync}_m$ and $\mathcal{C}^{pr,sync,uis}_m$, for all m.

KT2 suffices for completeness in $\mathcal{C}^{pr,sync}_m$. We do not need the complications of KT3.

Axiom system

Theorem (3.6)

$S5^U_m + KT4$ is a sound and complete axiomatization for the language KL_m with respect to $\mathcal{C}^{nl}_m$ for all m.

KT4 characterise the condition of no learning.

Theorem (3.7)

$S5^U_m + KT3 + KT4$ is a sound and complete axiomatization for the language KL_m with respect to $\mathcal{C}^{nl,pr}_m$ for all m. Moreover, it is a sound and complete axiomatization for the language KL_1 with respect to $\mathcal{C}^{nl,pr,uis}_1$.

Theorem (3.8)

$S5^U_m + KT5$ is a sound and complete axiomatization for the language KL_m with respect to $\mathcal{C}^{nl,sync}_m$.

Axiom system

Theorem (3.9)

$S5_m^U + KT2 + KT5$ is a sound and complete axiomatization for the language KL_m with respect to $\mathcal{C}_m^{nl,pr,sync}$ for all m.

Finally, it can be shown that when we combine no learning, synchrony, and uis, then not only do all agents consider the same worlds possible initially, but they consider the same worlds possible at all times. As a result, the axiom $K_i\varphi \leftrightarrow K_j\varphi$ is valid in this case. This allows us to reduce to the single-agent case.

Theorem (3.10)

$S5_m^U + KT2 + KT5 + \{K_i\varphi \leftrightarrow K_1\varphi\}$ is a sound and complete axiomatization for the language KL_m with respect to $\mathcal{C}_m^{nl,sync,uis}$ and $\mathcal{C}_m^{nl,pr,sync,uis}$ for all m.

A Framework for Completeness Proofs

A finite sequence $\sigma = i_1 i_2 ... i_k$ of agents, possibly equal to the null sequence ϵ, is called an **index** if $i_l \neq i_{l+1}$ for all $l < k$. We write $|\sigma|$ for the length k of such a sequence; the null sequence has length equal to 0.

If S is a set, and S^* is the set of all finite sequences over S, we define the absorptive concatenation function $\#$ from $S^* \times S$ to S^* as follows. Given a sequence σ in S^* and an element x of S, we take $\sigma \# x = \sigma$ if the final element of σ is x. If the final element of σ is not equal to x then we take $\sigma \# x$ to be σx, i.e. the result of concatenating x to σ.

A Framework for Completeness Proofs

If $\psi \in CKL_m$, for each $k \geq 0$, we define the k-closure $cl_k(\psi)$, and for each agent i, we define the k, i-closure $cl_{k,i}(\psi)$. The definition of these sets proceeds by mutual recursion:First, we let the basic closure $cl_0(\psi)$ be the smallest set containing ψ that is closed under subformulas, contains $\neg\varphi$ if it contains φ and φ is not of the form $\neg\varphi'$, contains $EC\varphi$ if it contains $C\varphi$, and contains $K_1\varphi, ..., K_n\varphi$ if it contains $E\varphi$.If i is a agent, we take $cl_{k,i}(\psi)$ to be the union of $cl_k(\psi)$ with the set of formulas of the form $K_i(\varphi_1 \vee ... \vee \varphi_n)$ or $\neg K_i(\varphi_1 \vee ... \vee \varphi_n)$, where the φ_l are distinct formulas in $cl_k(\psi)$.Finally, $cl_{k+1}(\psi)$ is defined to be $\cup_{i=1}^{m} cl_{k,i}(\psi)$.

A Framework for Completeness Proofs

If X is a finite set of formulas we write φ_X for the conjunction of the formulas in X. A finite set X of formulas is said to be consistent if φ_X is consistent.If X is a finite set of formulas and φ is a formula we write $X \Vdash \varphi$ when $\vdash \varphi_X \rightarrow \varphi$. Clearly if $X \Vdash \varphi_1$ and $\vdash \varphi_1 \rightarrow \varphi_2$ then $X \Vdash \varphi_2$.

Suppose Cl is a finite set of formulas with the property that for all $\varphi \in Cl$, either $\neg\varphi \in Cl$ or φ is of the form $\neg\varphi'$ and $\varphi' \in Cl$.Note that the sets $cl_k(\psi)$ and $cl_{k,i}(\psi)$ have this property.We define an **atom** of Cl to be a maximal consistent subset of Cl. Evidently, if X is an atom of Cl and $\varphi \in Cl$, then either $X \Vdash \varphi$ or $X \Vdash \neg\varphi$.

Lemma (4.1)

$\vdash \bigvee_{X \text{ an atom of } Cl} \varphi_X$.

A Framework for Completeness Proofs

We begin the construction of the model of ψ by first constructing a **pre-model**, which is a structure $\langle S, \rightarrow, \approx_1, ..., \approx_n \rangle$ consisting of a set S of states, a binary relation $\rightarrow$ on S, and for each agent i an equivalence relation $\approx_i$ on S.

For $\varphi \in KL_m$, we define $ad(\varphi)$ to be the greatest number of alternations of distinct $K_i's$ along any branch in $\varphi's$ parse tree. Let $d = ad(\psi)$ if $\psi \in KL_m$; otherwise let $d = 0$.

The set S consists of all the pairs (σ, X) such that σ is an index,$|\sigma| \leq d$, and

(1) if $\sigma = \epsilon$ then X is an atom of $cl_d(\psi)$, and

(2) if $\sigma = \tau$ then X is an atom of $cl_{k,i}(\psi)$, where $k = d - |\sigma|$.

A Framework for Completeness Proofs

The relation $\to$ is defined so that $(\sigma, X) \to (\tau, Y)$ iff $\tau = \sigma$ and the formula $\varphi_X \wedge \bigcirc\varphi_Y$ is consistent.If X is an atom we write X/K_i for the set of formulas φ such that $K_i\varphi \in X$.We say that states (σ, X) and (τ, Y) are **i-adjacent** if $\sigma\#i = \tau\#i$.The relation $\approx_i$ is defined so that $(\sigma, X) \approx_i (\tau, Y)$ iff σ and τ are i-adjacent and $X/K_i = Y/K_i$. Clearly, i-adjacency is an equivalence relation, as is the relation $\approx_i$.

A σ-**state** (for ψ) is a pair (σ, X) as above. If $s = (\sigma, X)$ is a state, we define φ_s to be the formula φ_X, and write $s \Vdash \varphi$ for $\vdash \varphi_s \to \varphi$.We say that the state s **directly decides** a formula φ if either (a) $\varphi \in X$ or (b) $\neg\varphi \in X$ or (c) $\varphi = \neg\varphi'$ and $\varphi' \in X$.

Note that if $\sigma = \tau i$ then each σ-state directly decides every formula in $cl_{d-|\sigma|,i}(\psi)$. Also, every ϵ-state directly decides every formula in $cl_d(\psi)$.And every state directly decides all formulas in the basic closure, $cl_0(\psi)$.

A Framework for Completeness Proofs

Lemma (4.2)

If s and t are i-adjacent states, then the same formulas of the form $K_i\varphi$ are directly decided by s and t.

Proof.

Suppose that s and t are i-adjacent, s is a σ-state, t is a τ-state. Clearly if $\sigma = \tau$, then s and t directly decide the same formulas, since they are both maximal consistent subsets of the same set of formulas.If $\sigma \neq \tau$, then either $\sigma = \tau i$ or $\tau = \sigma i$. By symmetry, it suffices to deal with the case $\sigma = \tau i$. Then $|\sigma| = |\tau| + 1$. By definition, s directly decides the K_i-formulas in $cl_{d-|\sigma|,i}(\psi)$, while t directly decides the K_i-formulas in $cl_{d-|\tau|,i}(\psi)$ if $\tau = \tau' j$ or $cl_d(\psi)$ if $\tau = \epsilon$. □

A Framework for Completeness Proofs

Proof.

If $\tau = \tau' j$, then the K_i-formulas directly decided by t are precisely those in $cl_{d-|\tau|,j}(\psi)$. $cl_{d-|\tau|,j}(\psi) = \underset{\sim\sim\sim}{cl_{d-|\tau|}(\psi)} \cup K_j(\varphi_1 \vee \ldots \vee \varphi_n)$ $\cup \neg K_j(\varphi_1 \vee \ldots \vee \varphi_n), \varphi_i \in cl_{d-|\tau|}(\psi)$. And $\underset{\sim\sim\sim}{cl_{d-|\tau|}(\psi) = \cup_{k=1}^{m} cl_{d-|\sigma|,k}(\psi)}$. Hence the K_i-formulas directly decided by t are precisely those in $\cup_{k=1}^{m} cl_{d-|\sigma|,k}(\psi) \cup \underline{K_j(\varphi_1 \vee \ldots \vee \varphi_n)} \cup \underline{\neg K_j(\varphi_1 \vee \ldots \vee \varphi_n)}$, $\varphi_i \in cl_{d-|\tau|}(\psi)$. By the definition of $cl_{d-|\sigma|,i}(\psi)$, we can see that the K_i-formulas directly decided by t are precisely those in $cl_{d-|\sigma|,i}(\psi)$. If $\tau = \epsilon$, the process is similar. □

A Framework for Completeness Proofs

If s is a σ-state, we take $\Phi_{s,i}$ to be the disjunction of the formulas φ_t, where t ranges over the σ-states satisfying $s \approx_i t$,and we take $\Phi^+_{s,i}$ to be the disjunction of the formulas φ_t, where t ranges over the $(\sigma\#i))$-states satisfying $s \approx_i t$.Observe that because $\approx_i$ is an equivalence relation we have that if $s \approx_i t$ then $\Phi_{s,i} = \Phi_{t,i}$ and $\Phi^+_{s,i} = \Phi^+_{t,i}$. We can lists a number of knowledge formulas decided by states.

A Framework for Completeness Proofs

Lemma (4.3)

(a) *If s is a σ-state and t is a σ-state or $(\sigma\#i)$-state such that $s \approx_i t$, then $s \Vdash K_i \neg\varphi_t$.*

(b) *For all σ-state s, we have $s \Vdash K_i \Phi_{s,i}$; in addition, if $\sigma\#i \leq d$, then $s \Vdash K_i \Phi^+_{s,i}$.*

(c) *For all σ-state s and $(\sigma\#i)$-state t with $s \approx_i t$, we have $s \Vdash L_i \varphi_t$.*

(d) *If s is a σ-state and t is a $(\sigma\#i)$-state such that $s \approx_i t$, then $t \Vdash \neg K_i \Phi^+_{s,i}$.*

A Framework for Completeness Proofs

Lemma (4.4)

If $s \not\rightarrow t$, then $\vdash \varphi_s \rightarrow \neg \bigcirc \varphi_t$.

Proof.

If $s \not\rightarrow t$, then $\varphi_s \wedge \bigcirc \varphi_t$ is not consistent. That is $\vdash \neg(\varphi_s \wedge \bigcirc \varphi_t)$, which is equivalent to $\vdash \varphi_s \rightarrow \neg \bigcirc \varphi_t$. □

If T is a set of states, then we write φ_T for the disjunction of the formulas φ_t for t in T.By $\vdash \bigvee_{X \text{ an atom of } Cl} \varphi_X$, we have $\vdash \bigvee_{s \text{ a } \sigma-\text{state}} \varphi_s$. Combined with above lemma, we derive that

Lemma (4.5)

Let s be a state and let T be the set of states t such that $s \rightarrow t$. Then $s \Vdash \bigcirc \varphi_T$.

A Framework for Completeness Proofs

Lemma (4.6)

For all formulas $\alpha, \beta . \gamma$, *if* $\vdash \alpha \rightarrow \neg\gamma$ *and* $\vdash \alpha \rightarrow \bigcirc(\alpha \vee (\neg\beta \wedge \neg\gamma))$, *then* $\vdash \alpha \rightarrow \neg(\beta U \gamma)$

This lemma provides a useful way to derive formulas containing the until operator.

Definition

We define a $\rightarrow$-sequence of states to be a (finite or infinite) sequences $s_1, s_2, \ldots$ such that $s_1 \rightarrow s_2 \rightarrow \ldots$

A Framework for Completeness Proofs

Lemma (4.7)

(a) *if* $\bigcirc\varphi \in cl_0(\psi)$, *then for all states* t *such that* $s \rightarrow t$, *we have* $s \Vdash \bigcirc\varphi$ *iff* $t \Vdash \varphi$.

(b) *If* $K_i\varphi \in cl_0(\psi)$, *then* $s \Vdash \neg K_i\varphi$ *iff there is some* σ*-state such that* $s \approx_i t$ *and* $t \Vdash \neg\varphi$. *Moreover, if* $|\sigma\#i| \leq d$, *then* $s \Vdash \neg K_i\varphi$ *iff there is some* $|\sigma\#i|$*-state* t *such that* $s \approx_i t$.

(c) *if* $\varphi_1 U \varphi_2 \in cl_0(\psi)$ *then* $s \Vdash \varphi_1 U \varphi_2$ *iff there exists a* $\rightarrow$*-sequence* $s = s_0 \rightarrow s_1 \ldots \rightarrow s_n$, *where* $n \geq 0$, *such that* $s_n \Vdash \varphi_2$, *and* $s_k \Vdash \varphi_1$ *for all* $k < n$.

(d) *If* $C\varphi \in cl_0(\psi)$, *then* $s \Vdash \neg C\varphi$ *iff there is a state* t *reachable from* s *through the relation* $\approx_i$ *such that* $t \Vdash \neg\varphi$.

This lemma shows that the pre-model **almost** satisfies the truth definitions for formulas in the basic closure.

A Framework for Completeness Proofs

We say that an infinite $\rightarrow$-sequence of states $(s_0, s_1, \ldots)$, where $s_n = (\sigma, X_n)$ for all n, is **acceptable** if for all $n \geq 0$, if $\varphi_1 U \varphi_2 \in X_n$ then there exists an $m \geq n$ such that $s_m \Vdash \varphi_2$ and $s_k \Vdash \varphi_1$ for all k with $n \leq k < m$.

Definition (enriched system)

An enriched system for ψ is a pair $(\mathcal{R}, \Sigma)$, where $\mathcal{R}$ is a set of runs and Σ is a partial function mapping points in $\mathcal{R} \times \mathbb{N}$ to states for ψ such that the following hold, for all runs $r \in \mathcal{R}$

(1) If $\Sigma(r, n)$ is defined then $\Sigma(r, n')$ is defined for all $n' > n$, and $\Sigma(r, n), \Sigma(r, n+1), \ldots$ is an acceptable $\rightarrow$-sequence.

(2) For all points $(r, n) \sim_i (r', n')$, if $\Sigma(r, n)$ is defined then $\Sigma(r', n')$ is defined and $\Sigma(r, n) \approx_i \Sigma(r', n')$.

A Framework for Completeness Proofs

Definition (enriched system)

(3) If $\Sigma(r,n)$ and s are σ-states such that $\Sigma(r,n) \approx_i s$, then there exists a point (r',n') such that $(r,n) \sim_i (r',n')$ and $\Sigma(r',n') = s$.

(4) If $C\varphi \in cl_0(\psi)$ and $\Sigma(r,n) \Vdash \neg C\varphi$, then there exists a point (r',n') reachable (r,n) such that $\Sigma(r',n') \Vdash \neg\varphi$.

An **enriched**$^+$ system for ψ is a pair $(\mathcal{R},\Sigma)$ satisfying conditions 1,2, and the following modification of 3:

(3') If $\Sigma(r,n)$ is a σ-states and s is a $(\sigma\#i)$-state such that $\Sigma(r,n) \approx_i s$, then there exists a point (r',n') such that $(r,n) \sim_i (r',n')$ and $\Sigma(r',n') = s$.

A Framework for Completeness Proofs

Given an enriched (resp., enriched+) system $(\mathcal{R}, \Sigma)$, we obtain an interpreted system $\mathcal{I} = (\mathcal{R}, \pi)$ by defining the valuation π on basic propositions p by $\pi(r, n)(p) = true$ just when $\Sigma(r, n)$ is defined and $\Sigma(r, n) \Vdash p$. If σ is the index $i_1\#...\#i_k$, let $K_\sigma\varphi$ be an abbreviation for $K_{i_1}...K_{i_k}\varphi$. (If $\sigma = \epsilon$, then we take $K_\sigma\varphi$ to be φ.)

Theorem (4.10)

(a) *If $(\mathcal{R}, \Sigma)$ is an enriched system for ψ, $\mathcal{I}$ is the associated interpreted system, φ is in the basic closure $cl_0(\psi)$, and $\Sigma(r, n)$ is defined, then $(\mathcal{I}, r, n) \models \varphi$ iff $\Sigma(r, n) \Vdash \varphi$.*

(b) *If $(\mathcal{R}, \Sigma)$ is an enriched$^+$ system for $\psi \in KL_m$, $\mathcal{I}$ is the associated interpreted system, φ is in the basic closure $cl_0(\psi)$, and $\Sigma(r, n)$ is a σ-state, and $ad(K_\sigma\varphi) \leq d$, then $(\mathcal{I}, r, n) \models \varphi$ iff $\Sigma(r, n) \Vdash \varphi$.*

A Framework for Completeness Proofs

We prove part(a) by induction on the complexity of φ. If φ is a propositional constant then the result is immediate from the definition of $\mathcal{I}$. The cases where φ is of the form $\neg\varphi$ or $\varphi_1 \wedge \varphi_2$ are similarly trivial. This leaves five cases:

Proof.

(1) Suppose that φ is of the form $\bigcirc\varphi_1$. Then $(\mathcal{I}, r, n) \vDash \varphi$ if and only if $(\mathcal{I}, r, n+1) \vDash \varphi_1$.Note that $\Sigma(r, n+1)$ must be defined by Condition 1 of definition of enriched system. Since φ_1 is a subformula of φ it is in $cl_0(\psi)$, so it follows by the induction hypothesis that $(\mathcal{I}, r, n+1) \vDash \varphi_1$ holds precisely when $\Sigma(r, n+1) \Vdash \varphi_1$.By Condition 1, $\Sigma(r, n) \to \Sigma(r, n+1)$, so we obtain from Lemma 4.7(a) that $\Sigma(r, n+1) \Vdash \varphi_1$ if and only if $\Sigma(r, n) \Vdash \bigcirc\varphi_1$. Putting the pieces together, we get $(\mathcal{I}, r, n) \vDash \varphi$ if and only if $\Sigma(r, n) \Vdash \varphi_1$.

□

A Framework for Completeness Proofs

Proof.

(2) Suppose that φ is of the form $\varphi_1 U \varphi_2$. Then the subformulas φ_1 and φ_2 are also in $cl_0(\varphi)$. Note also that by Condition 1 of definition of enriched system, $\Sigma(r, n')$ is defined for all $n' \geq n$, and $\Sigma(r, n)$,$\Sigma(r, n+1)$,... is an admissible $\rightarrow$-sequence.($\Leftarrow$) Assume $\Sigma(r, n) \Vdash \varphi_1 U \varphi_2$, then by Lemma 4.7(c) there exists some $n' \geq n$ such that $\Sigma(r, n') \Vdash \varphi_2$ and $\Sigma(r, k) \Vdash \varphi_1$ for $n \leq k < n'$.By the induction hypothesis, this implies that $(\mathcal{I}, r, n') \vDash \varphi_2$ and $(\mathcal{I}, r, k) \vDash \varphi_1$ for $n \leq k < n'$. In other words, we have $(\mathcal{I}, r, n) \vDash \varphi_1 U \varphi_2$. ($\Rightarrow$) Assume $(\mathcal{I}, r, n) \vDash \varphi_1 U \varphi_2$, then by the induction hypothesis and the semantics of U we have that there exists some $n' \geq n$ such that $\Sigma(r, n') \Vdash \varphi_2$ and $\Sigma(r, k) \Vdash \varphi_1$ for $n \leq k < n'$.Since $\Sigma(r, n) \rightarrow \Sigma(r, n+1) \rightarrow ... \rightarrow \Sigma(r, n')$, it follows using Lemma 4.7(c) that $\Sigma(r, n) \Vdash \varphi_1 U \varphi_2$.

□

A Framework for Completeness Proofs

Proof.

(3) Suppose that φ is of the form $K_i\varphi_1$. $(\Leftarrow)$ Assume $\Sigma(r,n) \Vdash K_i\varphi_1$ and suppose that $(r,n) \sim_i (r',n')$. Then by Condition 2 of definition of enriched system, we have that that $\Sigma(r',n')$ is defined and $\Sigma(r,n) \approx_i \Sigma(r',n')$.Since $K_i\varphi_1 \in cl_0(\psi)$ we obtain $\Sigma(r,n') \Vdash K_i\varphi_1$. By $K3$ this implies $\Sigma(r,n') \Vdash \varphi_1$.Since $\varphi_1 \in cl_0(\psi)$, by the induction hypothesis, we obtain that $(\mathcal{I},r',n') \vDash \varphi_1$. This shows that $(\mathcal{I},r',n') \vDash \varphi_1$ for all points $(r,n) \sim_i (r',n')$. That is, we have $(\mathcal{I},r,n) \vDash K_i\varphi_1$.$(\Rightarrow)$ suppose that $\Sigma(r,n) \Vdash \neg K_i\varphi_1$ and that $\Sigma(r,n)$ is a σ-state. By Lemma 4.7(b), there exists a σ-state t such that $\Sigma(r,n) \approx_i t$ and $t \Vdash \neg\varphi_1$.By Condition 3 of definition of enriched system, there exists a point (r',n') such that $(r,n) \sim_i (r',n')$ and $\Sigma(r',n') = t$.Using the induction hypothesis we obtain that $(\mathcal{I},r',n') \vDash \neg\varphi_1$. It follows that $(\mathcal{I},r,n) \vDash \neg K_i\varphi_1$.

A Framework for Completeness Proofs

Proof.

(4) If φ is of the form $E\varphi_1$, the result follows easily from the induction hypothesis, using axiom C1.

(5) Suppose φ is of the form $C\varphi_1$. By Condition 2, we have that $\Sigma(r', n')$ is defined for all (r', n') reachable from (r, n). An easy induction on the length of the path from (r, n) to (r', n'), using the fact that $K_i C\varphi_1$ is in the basic closure and axioms C1, C2, and K3, can be used to show that $\Sigma(r', n') \Vdash C\varphi_1$ for each point (r', n') reachable from (r, n). Using C1, C2, and K3, it is easy to see that $\Sigma(r', n') \Vdash C\varphi_1$. By the induction hypothesis, this implies that $(\mathcal{I}, r', n') \models \varphi_1$. Thus, $(\mathcal{I}, r, n) \models C\varphi_1$. For the converse, suppose that $\Sigma(r, n) \Vdash \neg C\varphi_1$. Then by Condition 4, we have $\Sigma(r', n') \Vdash \neg\varphi_1$ for some point (r', n') reachable from (r, n). By the induction hypothesis, we have that $(\mathcal{I}, r', n') \models \neg\varphi_1$, and hence $(\mathcal{I}, r, n) \models \neg C\varphi_1$.

A Framework for Completeness Proofs

Corollary (4.11)

If $(\mathcal{R}, \Sigma)$ is an enriched (resp., enriched$^+$) system for ψ, $\mathcal{I}$ is the associated interpreted system, and (r, n) is a point of $\mathcal{I}$ such that $\Sigma(r, n)$ is an ϵ-state and $\Sigma(r, n) \Vdash \psi$, then $(\mathcal{I}, r, n) \models \psi$.

We apply this corollary in all our completeness proofs, constructing an appropriate enriched or enriched$^+$ system in all cases.

Dealing with $\mathcal{C}_m$,$\mathcal{C}_m^{sync}$,$\mathcal{C}_m^{uis}$, and $\mathcal{C}_m^{sync,uis}$

The fact that $S5C_m^U$ is sound for $\mathcal{C}_m$ the class of all systems is straightforward (see also [1]). To prove completeness of $S5_m^U$ for the language KL_m and of $S5C_m^U$ for the language CKL_m with respect to $\mathcal{C}_m$, $\mathcal{C}_m^{sync}$,$\mathcal{C}_m^{uis}$, and $\mathcal{C}_m^{sync,uis}$, we construct an enriched system, and use Corollary 4.11. The proof proceeds in the same way whether or not common knowledge is in the language.We assume here that the language includes common knowledge and that we are dealing with the axiom system $S5C_m^U$ when constructing the states in the enriched structure.Recall that in this case we work with ϵ-states only.

Dealing with $\mathcal{C}_m$,$\mathcal{C}_m^{sync}$,$\mathcal{C}_m^{uis}$, and $\mathcal{C}_m^{sync,uis}$

The following result suffices for the generation of the acceptable sequences required for the construction of an enriched system in the cases not involving no learning.

Lemma (5.1)

Every finite $\rightarrow$-sequence of states can be extended to an infinite acceptable sequence.

Lemma (5.2)

The pair $(\mathcal{R}^{sync}, \Sigma)$ is an enriched system.

Proof.

It can be proved by verifying the conditions of enriched system. □

Dealing with $\mathcal{C}_m$,$\mathcal{C}_m^{sync}$,$\mathcal{C}_m^{uis}$, and $\mathcal{C}_m^{sync,uis}$

Theorem

$S5C_m^U$ is complete for CKL_m (resp.,$S5_m^U$ is complete for KL_m)

Proof.

Clearly the system $\mathcal{R}^{sync}$ is synchronous, so the interpreted system $\mathcal{I}$ derived from $(\mathcal{R}^{sync}, \Sigma)$ is also synchronous. Let s be an σ-state such that $s \Vdash \psi$. Such a state must exist because ψ was assumed consistent.By Lemma 5.1 there exists an acceptable sequence $(s_0, s_1, ...)$ with $s = s_0$. Let r be the corresponding run in $\mathcal{R}^{sync}$ with $r(0) = s_0$. Corollary 4.11 implies that $(\mathcal{I}, r, 0) \models \psi$.This establishes the completeness of the axiomatization $S5C_m^U$ for the language CKL_m (resp.,of $S5_m^U$ for the language KL_m) with respect to the classes of systems $\mathcal{C}_m$ and $\mathcal{C}_m^{sync}$.

□

Dealing with $\mathcal{C}_m$,$\mathcal{C}_m^{sync}$,$\mathcal{C}_m^{uis}$, and $\mathcal{C}_m^{sync,uis}$

Lemma

Suppose x is a subset of $\{pr, sync\}$. If $\varphi \in CKL_m$ is satisfiable with respect to $\mathcal{C}_m^{x}$, then it is also satisfiable with respect to $\mathcal{C}_m^{x,uis}$.

Proof.

Suppose $\mathcal{I} = (\mathcal{R}, \pi) \in \mathcal{C}_m^{x}$. We define a system $\mathcal{I}'$ by adding a new initial state to each run in $\mathcal{R}$.Formally, we define the system $\mathcal{I}' = (\mathcal{R}', \pi')$ as follows. Let l be some local state that does not occur in $\mathcal{I}$ and let s_e be any state of the environment.For each run $r \in \mathcal{R}$, let r^+ be the run such that $r^+(0) = (s_e, l, ..., l)$ and $r^+(n+1) = r(n)$ for $n > 0$. Let $\mathcal{R}' = \{r^+ : r \in \mathcal{R}\}$.The valuation π' is given by $\pi'(r, 0)(p) = \mathit{false}$ and $\pi'(r, n+1)(p) = \pi(r, n)(p)$, for $n \geq 0$ and propositions p. It is clear that $\mathcal{I}'$ is a system with unique initial states.

□

Dealing with $\mathcal{C}_m$,$\mathcal{C}_m^{sync}$,$\mathcal{C}_m^{uis}$, and $\mathcal{C}_m^{sync,uis}$

Lemma

Suppose x is a subset of $\{pr, sync\}$. If $\varphi \in CKL_m$ is satisfiable with respect to $\mathcal{C}_m^x$, then it is also satisfiable with respect to $\mathcal{C}_m^{x,uis}$.

Proof.

Moreover, if $\mathcal{I}$ is synchronous, then so is $\mathcal{I}'$, and if $\mathcal{I}$ is a system with perfect recall then so is $\mathcal{I}'$.A straightforward induction on the construction of the formula $\varphi \in CKL_m$ now shows that, for all points (r, n) in $\mathcal{I}$, we have $(\mathcal{I}, r, n) \models \varphi$ iff $(\mathcal{I}', r^+, n+1) \models \varphi$. □

This lemma shows that sound and complete axiomatizations for the class of systems satisfying some subset of the properties of **perfect recall** and **synchrony** are also sound and complete axiomatizations for the class of systems with the same subset of these properties, but with unique initial states in addition.

Discussion

It is worth remarking that our results are very sensitive to the language studied. As we have seen, the language considered in this paper is too coarse to reflect some properties of systems. In the absence of the other properties, synchrony and unique initial states do not require additional axioms. This may no longer be true for richer languages. For example, if we allow **past-time** operators[4]. We can describe more interaction between knowledge and time.

Suppose that we add an operator $\ominus$ such that $(\mathcal{I}, r, n) \models \ominus\varphi$ if $n \geq 1$ and $(\mathcal{I}, r, n-1) \models \ominus\varphi$.Notice that "$\neg \ominus$ *true*" expresses the property "the time is 0" and " $\ominus \neg \ominus$ *true*" expresses the property "the time is 1".Similarly, we can inductively define formulas that express the property the time is m for each $m \geq 0$.If $time = m$ is an abbreviation for this formula, then $(time = m) \rightarrow K_i(time = m)$ is valid in $\mathcal{C}^{sync}$, for each time m.

Discussion

On the other hand, by adding past time operators we can simplify the axiom for perfect recall. Introducing the operator S for "since", we may show that the formula

$$(K_i\varphi)S(K_i\psi) \rightarrow ((K_i\varphi)S(K_i\psi))$$

is valid in $\mathcal{C}^{pr}$. This axiom very neatly expresses the meaning of perfect recall.

Besides changes to the language, there are also additional properties of systems worth considering. One case of interest is the class of **asynchronous message passing systems** of [1]. That extra axioms are required in such systems is known [1], but the question of complete axiomatization is still open.

References I

Ronald Fagin, Joseph Halpern, Yoram Moses, and Moshe Vardi.
Reasoning About Knowledge.
01 2003.

Halpern, Joseph, Y., van, der, Meyden, Ron, Vardi, Moshe, and Y.
Complete axiomatizations for reasoning about knowledge and time.
Siam Journal on Computing, 2004.

J. Y. Halpern and M. Y. Vardi.
The complexity of reasoning about knowledge and time, i: lower bounds.
Journal of Computer and System Sciences, 38(1):195–237, 1989.

References II

Orna Lichtenstein, Amir Pnueli, and Lenore D. Zuck.
The glory of the past.
In Logics of Programs, Conference, Brooklyn College, June, 1985.

M. Sato.
A study of kripke-style methods for some modal logics by gentzen's sequential method.
Publications Research Institute for Mathematical Sciences, Kyoto University, 13(2):381–468, 1977.