QS4, Intuitionistic Logic and Forcing

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- Some background knowledge about forcing
- Forcing via QS4
- Forcing via First-order intuitionistic Logic
- The relation between these two methods

Some background knowledge about forcing

Some Definitions and Theorems needed in this part

Ordinal hierarchy

Let $\mathcal{O}$ be an *On*-sequence $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_\alpha, \ldots$ of subsets of a class A. We shall call $\mathcal{O}$ an ordinal hierarchy on A if the following conditions hold:

(1) $\mathcal{O}_0 = \emptyset$

(2) For each ordinal α, $\mathcal{O}_\alpha \subseteq \mathcal{O}_{\alpha+1}$

(3) For each limit ordinal λ, $\mathcal{O}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{O}_\alpha$

Some Definitions and Theorems needed in this part

Δ_0 and Σ formulas

Δ_0-formulas are defined as follows:

(1) Every atomic formula $x \in y$ is Δ_0

(2) If ϕ and ψ are Δ_0, so are $\neg\phi$, $\phi \wedge \psi$

(3) If ϕ is a Δ_0, then for distinct variables x, y, $\exists x \in y\phi$ and $\forall x \in y\phi$ are Δ_0.

Σ-formulas are defined as follows:

(1) Every Δ_0 formula is Σ-formula

(2) If ϕ and ψ are Σ-formulas, so is $\phi \wedge \psi$ and $\phi \vee \psi$

(3) If ϕ is a Σ-formula, so is $\exists x\phi$

(4) If ϕ is a Σ-formula, so are $\forall x \in y\phi$ and $\exists x \in y\phi$

Some Definitions and Theorems needed in this part

Absoluteness

Consider a class K and a sentence X whose constants are in K. To say that the truth value of X over K means that the quantifiers only consider elements in K, while the truth value of X in V means that the quantifiers consider elements in V, if these truth values are the same, we say that X is absolute over K.
Consider a formula $\phi(x_1, \dots, x_n)$ with no constants. We say that the formula is absolute over a class K if for any elements $a_1, \dots, a_n$ of K, the sentence $\phi(a_1, \dots, a_n)$ is absolute over K.

Standard model

A standard model is a relational system (A, E) where all the axioms of ZF are true, A is transitive and E is the actual $\in$-relation.

Why we need forcing

With the method of inner models, the relative consistency of the continuum hypothesis and ZF was shown by the formula $L(x)$, which defines a transitive class L of V which is a first order universe, in which the continuum hypothesis, the axiom of choice, and the axiom of constructibility are true.

However, this method cannot be used to show that ZF and the negation of the continuum hypothesis are relatively consistent. Here's why:

Suppose that we have such a formula $M(x)$ which defines a transitive proper class M s.t M is a first-order universe in which the negation of the continuum hypothesis is true. Since L is the smallest proper class that is a first-order universe, $L \subseteq M$. Realize that the continuum hypothesis is provable in ZF relativized to L and its negation is provable in ZF relativized to M, this will result in the conclusion that $L \neq M$ can be proved in ZF, which means that not every set is constructable, which contradicts the result that the axiom of constructibility is consistent with ZF.

Why we need forcing

So we need a new technique to peove such results, which is called forcing.

Forcing via QS4

QS4 models

A QS4 model is a structure $(\mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{V})$ where $\mathcal{G}$ is a non-empty set (of possible worlds), $\mathcal{R}$ is a binary relation that is reflexive and transitive, $\mathcal{D}$ is a non-empty set (or class), called the domain of the frame, and $\mathcal{V}$ is a mapping from possible worlds to sets of closed atomic sentences (with constants from the domain).

It's easy to see why we choose QS4 models here. We will view the possible worlds $\mathcal{G}$ as forcing conditions and the relation $p\mathcal{R}q$ as q is as strong as or stronger than p. As a result, the relation would be a partial ordering, which means that the frame is an S4 frame, and the model is a QS4 model.

A classical embedding

When we have such a QS4 model, the problem is that how can we say that a forcing condition p forces a formula X, which is, to ensure the truth of X when strengthening the condition? Obviously, for most of the formulas, $p \models X$ is not enough, since stronger conditions may cause X to be false.

An attempt is to add $\Box$ before every subformula to ensure the truth of the formula (as well as its subformulas) when the forcing condition is strengthened. However, some classical valid formulas may be not S4 valid after the translation depending on how we choose the primitive ($A \vee \neg A$ for example).

Instead of ensuring the truth in every condition stronger than the initial condition, we can ensure the case that for every condition stronger than p, we can always find a even stronger condition that ensures the truth of X and its subformulas, which can also achieve our goal. So we have the following translation:

A classical embedding

We specify a translation from non-modal formulas to modal formulas as follows:

(1) For A atomic, $[[A]] = \Box\Diamond A$

(2) $[[\neg X]] = \Box\Diamond\neg[[X]]$

(3) $[[X \wedge Y]] = \Box\Diamond([[X]] \wedge [[y]])$

(4) $[[\exists x\phi]] = \Box\Diamond\exists x[[\phi]]$

A classical embedding

With this translation we can prove the following proposition:

Proposition

Let X be a closed formula in the language of classical logic, X is classical valid if and only if $[[X]]$ is valid in all S4 models.

Proof

The $\leftarrow$ part is rather trivial. For every classical model M, we can constuct a QS4 model with s single world p based on the classical model. By induction we have that for every closed formula X, X is true in M iff $p \models [[X]]$.
For the $\rightarrow$ part, first we use the completeness theorem for classical logic that for a classical valid formula X, X can be proved by the axiom system. Then we complete the proof by proving the translation of the axioms are all valid in QS4 models and the validity of translation of the premises of the rules of inference implies validity of the translation of the conclusions of the rules in QS4 models, thus end the proof.

The embedding properties

For $\mathcal{M}$ a QS4 model and $p \in \mathcal{G}$,

- P_1 if $[[X]]$ is true at p, it is true at any worlds accessible from p
- P_2 if $[[X]]$ is not true at p, then for some q s.t pRq, $[[\neg X]]$ is true at q
- P_3 if $[[\neg X]]$ is true at p, $[[X]]$ is not
- P_4 if $[[\neg X]]$ is not true at p, then for some q s.t pRp, $[[X]]$ is true at q
- P_5 $[[X \wedge Y]]$ is true at p iff $[[X]]$ and $[[Y]]$ are true at p
- P_6 if either $[[X]]$ or $[[Y]]$ is true at p, so is $[[X \vee Y]]$
- P_7 if $[[X \vee Y]]$ is true at p, then for some q s.t pRq, at least one of $[[X]]$ and $[[Y]]$ is true at q
- P_8 if $[[X \to Y]]$ and $[[X]]$ are true at p, so is $[[Y]]$
- P_9 if $[[X \to Y]]$ is not true at p, then for some q s.t pRq, $[[X]]$ and $[[\neg Y]]$ are true at q
- P_{10} if some instance of $[[\phi(x)]]$ is true at p, so is $[[\exists x \phi(x)]]$
- P_{11} if $[[\exists x \phi(x)]]$ is true at p, then for some q s.t pRq, some instance of $[[\phi(x)]]$ is true at q
- P_{12} $[[\forall x \phi(x)]]$ is true at p iff each instance of $[[\phi(x)]]$ is true at p

The basic idea

Definition

We call a QS4 model $\mathcal{M}$ an S4-ZF model if, for each axiom A of ZF, and for A being the axiom of well foundednedd, $[[A]]$ is valid in $\mathcal{M}$.

Now, suppose that we can construct an S4-ZF model $\mathcal{M}$ in which $[[CH]]$ is not valid, then the classical independence is immediate, by the following argument:

If CH were provable from the axioms of ZF and the axiom of well foundedness, then it is provable from finite axioms say $A_1, \ldots, A_n$, which means that $(A_1 \wedge \cdots \wedge A_n) \rightarrow CH$ is classically valid, which means that $[[(A_1 \wedge \cdots \wedge A_n) \rightarrow CH]]$ is QS4 valid, which means that $([[A_1]] \wedge \cdots \wedge [[A_n]]) \rightarrow [[CH]]$ is QS4 valid, which is impossible since $\mathcal{M}$ is a contermodel.

Construction of QS4 models for ZF

This work will be based on M, which is a transitive subclass of V.

We will first create $(\mathcal{G}, \mathcal{R})$, which is an S4 frame, and also a member of M.

Now we begin to definethe domain of the model as follows:

For each ordinal α (in M) we define a set $R^{\mathcal{G}}_{\alpha}$ as follows:

(1) $R^{\mathcal{G}}_{0} = \emptyset$

(2) $R^{\mathcal{G}}_{\alpha+1}$ is the set of all subsets (in M) od $\mathcal{G} \times R^{\mathcal{G}}_{\alpha}$

(3) For a limit ordinal λ, $R^{\mathcal{G}}_{\lambda} = \bigcup_{\alpha<\lambda} R^{\mathcal{G}}_{\alpha}$

Now, let

$$\mathcal{D}^{\mathcal{G}} = \bigcup_{\alpha} R^{\mathcal{G}}_{\alpha}$$

There are several important items concerning this definition:

(1) Each $R^{\mathcal{G}}_{\alpha}$ is a member of M, $\mathcal{D}^{\mathcal{G}}$ is a subclass of M
(2) The sequence $R^{\mathcal{G}}_{0}, R^{\mathcal{G}}_{1}, \ldots$ is a strict ordinal hierarchy, so $\mathcal{D}^{\mathcal{G}}$ is a proper class in M
(3) If $f \subseteq \mathcal{F}^{\mathcal{G}}$ and f is an M-set then $f \in \$\mathcal{D}^{\mathcal{G}}$, ehich is a consequence of axiom of substitution

Since there is an ordinal hierarchy, we can introduce the notion of $\mathcal{G}$-rank:

We say $f \in \mathcal{D}^{\mathcal{G}}$ has $\mathcal{G}$-rank α if $f \in R^{\mathcal{G}}_{\alpha+1}$ but $f \notin R^{\mathcal{G}}_{\alpha}$.

Now we begin to define the truth assignment, we first introduce the notion of ε:

For $p \in \mathcal{G}$ and $f, g \in \mathcal{D}^{\mathcal{G}}$, $f \varepsilon g$ is true at p if $(p, f) \in g$.

However, there is a problem about this definition, which makes this difinition cannot act as $\in$ in the truth assignment:

Example

Consider $f = \{(q, a)\}$, $g = \{(p, a), (q, a)\}$, $h = \{(q, f)\}$, we realize that f and g have the same elements in q but also $f \varepsilon h$ holds in q and $g \varepsilon h$ does not hold in q.

Now we turn to a new notion of $\approx$, which serves as the equality in the model. To achieve this, we define a sequence of $\approx_\alpha$:

For $p \in \mathcal{G}$ and $f, g \in \mathcal{D}^{\mathcal{G}}$,

(1) $p \not\models f \approx_0 g$

(2) $p \models f \approx_{\alpha+1} g$ if
$p \models [[\forall x(x\varepsilon f \to \exists y(y\varepsilon g \wedge y \approx_\alpha x)) \wedge \forall x(x\varepsilon g \to \exists y(y\varepsilon f \wedge y \approx_\alpha x))$

(3) For a limit ordinal λ, $p \models f \approx_\lambda g$ if $p \models f \approx_\alpha g$ for some $\alpha < \lambda$

Now we have the following proposition:

If $p \models [[f \approx_\alpha g]]$ and $\alpha < \beta$ then $p \models [[f \approx_\beta g]]$.

Proof

Call α good if $p \models [[f \approx_\alpha g]]$ implies $p \models [[f \approx_\beta g]]$ for every $\beta > \alpha$. we show that every ordinal is good by transfinite induction.
Trivially 0 and limit ordinals are good.
We now turn to the case of successor ordinals. Assume that α is good and $p \models [[f \approx_{\alpha+1} g]]$ and $\alpha < \gamma$. Without the loss of generality, we can set γ as $\beta + 1$. Now we suppose that $p \not\models [[f \approx_{\beta+1} g]]$ and derive a contradiction.
Since $p \not\models [[f \approx_{\beta+1} g]]$, we can assume that $p \not\models [[\forall x(x \varepsilon f \rightarrow \exists y(y \varepsilon g \wedge y \approx_\beta x))]]$ without loss of generality, which means that for some p' s.t pRp' and for some a, $p' \models [[a \varepsilon f]]$ and $p' \models \forall y[[y \varepsilon g \rightarrow \neg y \approx_\beta a]]$.
Since $p \models [[f \approx_{\alpha+1} g]]$, $p' \models [[f \approx_{\alpha+1} g]]$, which means that $p' \models [[\forall x(x \varepsilon f \rightarrow \exists y(y \varepsilon g \wedge y \approx_\alpha x))]]$, so $p' \models [[\exists y(y \varepsilon g \wedge y \approx_\alpha a))]]$. It follows that for some p'' s.t $p'Rp''$ and some $b \in \mathcal{D}^{\mathcal{G}}$, $p'' \models [[b \varepsilon g]]$ and $p'' \models [[b \approx_\alpha a]]$, by induction hypothesis, $p'' \models [[b \approx_\beta a]]$.
Realize that $p' \models [[b \varepsilon g \rightarrow \neg b \approx_\beta a]]$, so $p'' \models [[b \varepsilon g \rightarrow \neg b \approx_\beta a]]$, which means that $p'' \models [[\neg b \approx_\beta a]]$, but $p'' \models [[b \approx_\beta a]]$, which is a contradiction.

Now we define $\approx$ as follows:

For $f, g \in \mathcal{D}^{\mathcal{G}}$, $f \approx g$ is true at p if $f \approx_\alpha g$ is true at p for some ordinal α in M.

Now we can define the assignment of $\in$:

For $p \in \mathcal{G}$ and $f, g \in \mathcal{D}^{\mathcal{G}}$, $f \in g$ is true at p if for some $h \in \mathcal{D}^{\mathcal{G}}$, $p \models [[h \approx f]]$ and $p \models [[h \varepsilon g]]$.
This can be also written as:

$$f \in g \leftrightarrow \exists x([[x \approx f]] \wedge [[x \varepsilon g]]) \leftrightarrow \exists x[[x \approx f \wedge x \varepsilon g]]$$

Now we've constructed an S4-ZF model. Next we'll show that the axioms are valid in such models.

The validity of the axioms

About equality

We have defined equality in the last part, which arises a question that whether this definition has the logical property and the set-theoritic properties of equality, which will lead to the validity of the translate of the axiom of extensionality.

We'll start from the logical properties.

Transitivity, reflexivity and symmetry

Lemma

For each ordinal α, if $p \models [[f \approx_\alpha g]]$ and $p \models [[g \approx_\alpha h]]$ then $p \models [[f \approx_\alpha h]]$.

Proof

By induction on α, the case of 0 and limit ordinals are simple. Now we consider the successor case.
suppose that the result is known for α, and suppose that $p \models [[f \approx_{\alpha+1} g]]$ and $p \models [[g \approx_{\alpha+1} h]]$ but $p \not\models [[f \approx_{\alpha+1} h]]$.
Since $p \not\models [[f \approx_{\alpha+1} h]]$, without loss of generality, we say that $p \not\models \forall x[[x \varepsilon f \to \exists y(y \varepsilon h \wedge y \approx_\alpha x)]]$. Then for some p' s.t pRp' and soma $a \in \mathcal{D}^{\mathcal{G}}$, $p' \models [[a \varepsilon f]]$ and $p' \models [[\neg \exists y(y \varepsilon h \wedge y \approx_\alpha a)]]$.
Next, since $p \models [[f \approx_{\alpha+1} g]]$, also $p' \models [[f \approx_{\alpha+1} g]]$, so $p' \models \forall x[[x \varepsilon f \to \exists y(y \varepsilon g \wedge y \approx_\alpha x)]]$. Since $p' \models [[a \varepsilon f]]$, then $p' \models [[\exists y(y \varepsilon g \wedge y \approx_\alpha a]]$, so for some p'' s.t $p'Rp''$ and some $b \in \mathcal{D}^{\mathcal{G}}$, $p'' \models [[b \varepsilon g]]$ and $p'' \models [[b \approx_\alpha a]]$.

Transitivity, reflexivity and symmetry

Proof

Similarly, we have that for some p''' s.t $p''Rp'''$ and $c \in \mathcal{D}^{\mathcal{G}}$, $p''' \models [[c \varepsilon h]]$ and $p''' \models [[c \approx_\alpha b]]$. Realize that also $p'' \models [[b \approx_\alpha a]]$, so $p''' \models [[b \approx_\alpha a]]$, we have $p''' \models [[c \approx_\alpha a]]$ by induction hypothesis. We also have $p' \models [[\neg \exists y (y \varepsilon h \wedge y \approx_\alpha a)]]$, so $p''' \models [[\neg \exists y (y \varepsilon h \wedge y \approx_\alpha a)]]$, which is equivalent to $p''' \models \forall y [[y \varepsilon h \rightarrow \neg y \approx_\alpha a]]$. Since we have $p''' \models [[c \varepsilon h]]$, we have $p''' \models [[\neg c \approx_\alpha a]]$, which is a contradiction.

Transitivity, reflexivity and symmetry

Theorem

The equality relation is transitive.

Proof

Suppose $p \models [[f \approx g]]$, then for some α, $p \models [[f \approx_\alpha g]]$. Likewise we suppose that $p \models [[g \approx h]]$, then for some β, $p \models [[g \approx_\beta h]]$. Let γ be the larger of α and β, and by the lemma, $p \models [[f \approx_\gamma h]]$, so $p \models [[f \approx h]]$.

Transitivity, reflexivity and symmetry

Theorem

The equality relation is reflexive and symmetric.

Proof

Symmetry is obvious from the definition.

For the reflexivity part, we prove that for arbitrary f and p, if $f \in R^{\mathcal{G}}_{\alpha}$ then $p \models [[f \approx_{\alpha} f]]$ by transfinite induction.
The case of 0 is obvious. Consider the successor case of $\alpha + 1$.
Assume that $f \in R^{\mathcal{G}}_{\alpha+1}$, suppose that $p \not\models [[f \approx_{\alpha+1} f]]$, which means that $p \not\models \forall x[[x \varepsilon f \rightarrow \exists y(y \varepsilon f \wedge y \approx_{\alpha} x)]]$, which means that for some p' s.t pRp' and some a, $p' \models [[a \varepsilon f]]$ and $p' \models \forall y[[y \varepsilon f \rightarrow \neg y \approx_{\alpha} a]]$. So $p' \models [[\neg a \approx_{\alpha} a]]$, realize that $p' \models [[a \varepsilon f]]$, $a \in R^{\mathcal{G}}_{\alpha}$, by induction hypothesis, $p' \models [[a \approx_{\alpha} a]]$, which is a contradiction.
The case of limit ordinal λ is an easy consequence of the definition of $R^{\mathcal{G}}_{\lambda}$ and $\approx_{\lambda}$.

Substitutivity

We start from the atomic level:

Lemma

If $p \models [[f \approx g]]$ and $p \models [[f \in h]]$ then $p \models [[g \in h]]$.

Proof

Suppose $p \models [[f \approx g]]$ and $p \models [[f \in h]]$ but $p \not\models [[g \in h]]$. We derive a contradiction.
Since $p \not\models [[g \in h]]$, $p \not\models [[\exists x(x \approx g \wedge x \varepsilon h)]]$, then for some p' s.t pRp', $p' \models \forall x[[x \approx g \to \neg x \varepsilon h]]$.
Now, $p \models [[f \in h]]$, $p' \models [[f \in h]]$. Then for some a, $p' \models [[a \approx f]]$ and $p' \models [[a \varepsilon h]]$. We also have $p \models [[f \approx g]]$, so $p' \models [[f \approx g]]$, so $p' \models [[a \approx g]]$, so $p' \models [[\neg a \approx h]]$, which is a contradiction.

Substitutivity

Lemma

If $p \models [[f \approx g]]$ and $p \models [[h \in f]]$ then $p \models [[h \in g]]$.

Proof

Suppose $p \models [[f \approx g]]$ and $p \models [[h \in f]]$ but $p \not\models [[h \in g]]$. We derive a contradiction.
Since $p \not\models [[h \in g]]$, For some p' s.t pRp', $p' \models [[\neg h \in g]]$, realize that $p' \models [[f \approx g]]$ and $p' \models [[h \in f]]$.
Since $p' \models [[h \in f]]$, for some a, $p' \models [[a \approx h]]$ and $p' \models [[a \varepsilon f]]$.
Now, $[[f \approx g]]$ is true at p', so for some α, $[[f \approx_{\alpha+1} g]]$ is true at p', so $p' \models \forall x[[x \varepsilon f \to \exists y(y \varepsilon g \wedge y \approx_\alpha x)]]$. So $p' \models [[\exists y(y \varepsilon g \wedge y \approx_\alpha a)]]$.
Then for some p'' s.t $p'Rp''$ and some b, $p'' \models [[b \varepsilon g]]$ and $p'' \models b \approx_\alpha a]]$, so $p'' \models [[b \approx a]]$. Realize that $[[a \approx h]]$ is true at p', it is also true at p'', so $p'' \models [[b \approx h]]$. Since $p'' \models [[b \varepsilon g]]$, $p'' \models [[h \in g]]$, contradicting to the fact that $[[\neg h \in g]]$ is true at p', hence also at p''.

Substitutivity

Theorem

Suppose $f, g \in \mathcal{D}^{\mathcal{G}}$ and X and X' are closed formulas differing in that X' has an occurence of g in a location where X has an occurence of f. If $p \models [[f \approx g]]$, then $p \models [[X]] \leftrightarrow [[X']]$.

Proof

By induction on the complexity of the equivalent form of X, taking only $\neg$, $\wedge$, and $\forall$ as connectives and quantifiers, since if A and B are classical equivalent, then $[[A]]$ and $[[B]]$ are S4 equivalent. The atomic case is done by the lemmas. Realize that $[[A \wedge B]] \leftrightarrow ([[A]] \wedge [[B]])$ is S4 valid, the case of $\wedge$ is done. The case of $\forall$ is similar.

Now consider the case of $\neg$. Suppose that we know that whenever $[[f \approx g]]$ is true at a possible world, so is $[[X]] \leftrightarrow [[X']]$. Suppose that $p \models [[f \approx g]]$ but $p \not\models [[\neg X]] \leftrightarrow [[\neg X']]$, we derive a contradiction. Say $p \not\models [[\neg X]] \rightarrow [[\neg X']]$, thus $p \models [[\neg X]]$ and $p \not\models [[\neg X']]$, so for some p' s.t pRp', $p' \models [[f \approx g]]$, $p' \models [[X']]$ and $p' \not\models [[X]]$, which is a contradiction.

Substitutivity

This gives us the following:

Corollary

Suppose $\phi(x)$ is a formula with only x free, and with no occurence of y, The following is S4 valid in $\mathcal{M}$:

$$[[\forall x \forall y(x \approx y \to (\phi(x) \leftrightarrow \phi(y))]]$$

Extensionality

Lemma

If $f, g \in R_\alpha^{\mathcal{G}}$ and $p \models \forall x[[x \in f \leftrightarrow x \in g]]$ then $p \models [[f \approx_\alpha g]]$.

Proof

By transfinite induction on α. The 0 and limit ordinal cases are trivial. For the successor case, suppose that $f, g \in R^{\mathcal{G}}_{\alpha+1}$ and $p \models \forall x[[x \in f \leftrightarrow x \in g]]$ but $p \not\models [[f \approx_{\alpha+1} g]]$, we derive a contradiction.

Since $p \not\models [[f \approx_{\alpha+1} g]]$, say that $p \not\models [[\forall x(x \varepsilon f \rightarrow \exists y(y \varepsilon g \wedge x \approx_\alpha y))]]$, then for some a and p' s.t pRp' , $p' \models [[a \varepsilon f]]$ and $p' \models \forall y[[y \varepsilon g \rightarrow \neg a \approx_\alpha y]]$. So a has less $\mathcal{G}$-rank than f, which means that $a \in R^{\mathcal{G}}_\alpha$.

Since $p' \models [[a \varepsilon f]]$, $p' \models [[a \in f]]$, also $p \models \forall x[[x \in f \leftrightarrow x \in g]]$, so $p' \models [[a \in f \leftrightarrow a \in g]]$, so $p' \models [[a \in g]]$, so for some b, $p' \models [[a \approx b]]$ and $p' \models [[b \varepsilon g]]$, this implies that b has less $\mathcal{G}$-rank than g, which means that $b \in R^{\mathcal{G}}_\alpha$.

Obviously $p' \models [[\forall x(x \in a \leftrightarrow x \in a)]]$, also $p' \models [[a \approx b]]$, so $p' \models [[\forall x(x \in a \leftrightarrow x \in b)]]$, by induction hypothesis, $p' \models [[a \approx_\alpha b]]$. But also $p' \models \forall y[[y \varepsilon g \rightarrow \neg a \approx_\alpha y]]$, which means that $p' \models [[\neg a \approx_a lphab]]$, which is a contradiction.

Extensionality

The lemma immediately gives us the following:

Theorem

If $p \models \forall x[[x \in f \leftrightarrow x \in g]]$ then $p \models [[f \approx g]]$.

And we have the corollary, which is the translate of the axiom of extensionality:

Corollary

The following is S4 valid in $\mathcal{M}$:

$$[[\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow \forall z(x \in z \leftrightarrow y \in z))]]$$

Extensionality

Proof

If this were not true at p, then for some a, b and p' s.t pRp', $p' \models [[\forall z(z \in a \leftrightarrow z \in b)]]$, which means that $p' \models [[a \approx b]]$, also $p' \models [[\neg\forall z(a \in z \leftrightarrow b \in z))]]$, which we can get $p' \models [[\neg\forall z(b \in z \leftrightarrow b \in z))]]$, which is impossible.

We can also derive the following proposition:

Proposition

For any $p \in \mathcal{G}$ and any $f, g \in \mathcal{D}^{\mathcal{G}}$, $p \models [[f \approx g]]$ iff

$$p \models [[\forall x(x\varepsilon f \rightarrow \exists y(y\varepsilon g \wedge y \approx x)) \wedge \forall x(x\varepsilon g \rightarrow \exists y(y\varepsilon f \wedge y \approx x))]]$$

The well founded sets are present

Now we concern about the empty set axiom and the infinity axiom. We try to associate each (well founded) set in M a member of $\mathcal{D}^{\mathcal{G}}$.

Definition

To each (well founded) set x in M we associate a member $\hat{x}$ of $\mathcal{D}^{\mathcal{G}}$ as follows: $\hat{x} = \{(p, \hat{y}) | y \in x\}$.

Realize that it's an inductive definition based on the rank of x, which leads to the fact that if x has rank α then $\hat{x}$ has $\mathcal{G}$-rank α. With this observation we have the following lemma:

Lemma

If $p \models \hat{x} \approx \hat{y}$ then x and y are the same regular set.

The well founded sets are present

Proposition

Let $\phi(x_1, \ldots, x_n)$ be a Δ_0 formula with free variables among $x_1, \ldots, x_n$. Let $s_1, \ldots, s_n$ be (well founded) sets in M. Then, $\phi(s_1, \ldots, s_n)$ is true in M iff $\phi(s_1, \ldots, s_n)$ is true in M iff $[[\phi(\hat{s_1}, \ldots, \hat{s_n})]]$ is true at some world of $\mathcal{G}$ iff $[[\phi(\hat{s_1}, \ldots, \hat{s_n})]]$ is true at every world of $\mathcal{G}$.

Proof (sketch)

The first equivalence is by the absoluteness of Δ_0 formulas.
For the rest part, without loss of generality we can assume ϕ has all negations at the atomic level, and that it otherwise contains only $\wedge$, $\vee$, $\forall x \in y$ and $\exists x \in y$, since every Δ_0 formula is classically equivalent to one in this form. Then it's a routine work by induction on the complexity of ϕ.

The well founded sets are present

Corollary

Let $\phi(x_1, \ldots, x_n)$ be a Σ formula and $s_1, \ldots, s_n \in M$. If $\phi(s_1, \ldots, s_n)$ is true in M then $[[\phi(\hat{s}_1, \ldots, \hat{s}_n)]]$ is valid in $\mathcal{M}$.

Proof

Without loss of generality, we can assume that the unbounded existential quantifiers occur first, so that $\phi(x_1, \ldots, x_n) = \exists y_1 \ldots \exists y_k \psi(y_1, \ldots, y_k, x_1, \ldots, x_n)$. Then ,if $\phi(s_1, \ldots, s_n)$ is true in M, for some $t_1, \ldots, t_k$, $\psi(t_1, \ldots, t_k, s_1, \ldots, s_n)$ is true in M, by the previous proposition, $[[\psi(\hat{t}_1, \ldots, \hat{t}_k, \hat{s}_1, \ldots, \hat{s}_n)]]$ is valid in $\mathcal{M}$, so $[[\phi(\hat{s}_1, \ldots, \hat{s}_n)]]$ is valid in $\mathcal{M}$.

The well founded sets are present

Corollary

Let N be the empty set axiom and I be the axiom of infinity. Both $[[N]]$ and $[[I]]$ are true at every $p \in \mathcal{G}$.

Proof

The property x is empty is Δ_0, so the empty set axiom is Σ. Since this is true in M, its translate is valid in $\mathcal{M}$.
ω can be characterized by saying it is a limit ordinal but no member of it is a limit ordinal. The property of being a limit ordinal is Δ_0, so there is a Δ_0 characterization of ω, so case of the infinity axiom is similar to the empty set axiom.

Four more axioms

We start with the union axiom.

Theorem

For any $f \in \mathcal{D}^{\mathcal{G}}$, there is some $g \in \mathcal{D}^{\mathcal{G}}$ s.t

$$[[\forall x(x \in g \leftrightarrow x \in \cup f)]]$$

is valid in $\mathcal{M}$, where we write $x \in \cup f$ as an abbreviation for $\exists y(y \in f \wedge x \in y)$.

Four more axioms

Proof

Say $f \in R^{\mathcal{G}}_{\alpha+2}$. Define a set g as follows:
$(p, a) \in g \Leftrightarrow p \in \mathcal{G}$ and $a \in R^{\mathcal{G}}_{\alpha}$ and $p \models [[a \in \cup f]]$.
Obviously $g \in R^{\mathcal{G}}_{\alpha+1}$ and for $a \in R^{\mathcal{G}}_{\alpha}$, $p \models a\varepsilon g \Leftrightarrow p \models [[a \in \cup f]]$.
Realize that under the definition, $p \models a\varepsilon g$ implies $p \models [[a\varepsilon g]]$.
Noe we suppose that for some p, $[[\forall x(x \in g \leftrightarrow x \in \cup f)]]$ is not true at p, we derive a contradiction.

(1) Suppose that for some a, $p \not\models [[a \in g \to a \in \cup f]]$. Then there is some p' s.t pRp', $p' \models [[a \in g]]$ and $p' \models [[\neg a \in \cup f]]$. So there is some a's.t $p' \models [[a \approx a']]$ and $p' \models [[a'\varepsilon g]]$. But then, for some p'' s.t $p'Rp''$, $p'' \models a'\varepsilon g$, so $p'' \models [[a' \in \cup f]]$. By substitutivity, $p'' \models [[a \in \cup f]]$, which is impossible since $[[\neg a \in \cup f]]$ is true at p', and hence at p''.

Four more axioms

Proof

(2) Suppose that for some a, $p \not\models [[a \in \cup f \to a \in g]]$. Then for some p' s.t pRp', $p' \models [[a \in \cup f]]$ and $p' \models [[\neg a \in g]]$. Since $p' \models [[a \in \cup f]]$, $p' \models [[\exists y(y \in f \wedge a \in y)]]$, and so for some p'' s.t $p'Rp''$ and b, $p'' \models [[b \in f]]$ and $p' \models [[a \in b]]$. Then there is some b' s.t $p'' \models [[b \approx b']]$ and $p'' \models [[b' \varepsilon f]]$. By substitutivity, $p'' \models [[a \in b']]$, so there is a' s.t $p'' \models [[a \approx a']]$ and $p'' \models [[a' \varepsilon b']]$. Now we have $p'' \models [[b' \in f]] \wedge [[a' \in b']]$, so $p'' \models [[a' \in \cup f]]$ where $a' \in R^{\mathcal{G}}_{\alpha}$. By definition, $p'' \models a' \varepsilon g$, so $p'' \models [[a' \varepsilon g]]$, so $p'' \models [[a \in g]]$, which is a contradiction since $[[\neg a \in g]]$is true at p', hence also at p''.

This immediately gives us the following:

Theorem

$[[\forall f \exists g \forall x(x \in g \leftrightarrow x \in \cup f)]]$ is valid in $\mathcal{M}$.

Four more axioms

The pairing axiom is quite similar, which gives us the following:

Theorem

$[[\forall f \forall g \exists h \forall x (x \in h \leftrightarrow x \in \{f, g\})]]$ is valid in $\mathcal{M}$, where $x \in \{f, g\}$ abbreviates $x \approx f \vee x \approx g$.

Four more axioms

Now for the power set axiom. We start from a lemma.

Lemma

Suppose $f \in R^{\mathcal{G}}_{\alpha+1}$ and $[[a \subseteq f]]$ is true at p. Then for some $b \in R^{\mathcal{G}}_{\alpha+1}$, $[[a \approx b]]$ is true at p.

Proof

Suppose $f \in R^{\mathcal{G}}_{\alpha+1}$ and $p \models [[a \subseteq f]]$, define b as follows:
$(q, d) \in b \Leftrightarrow q \in \mathcal{G}$ and $d \in R^{\mathcal{G}}_{\alpha}$ and $q \models \exists w[[w \approx d \wedge w \varepsilon a]]$.
Obviously, $b \in R^{\mathcal{G}}_{\alpha+1}$ and for each q,
$q \models d \varepsilon b \Leftrightarrow d \in R^{\mathcal{G}}_{\alpha}$ and $q \models [[c \approx d]]$ and $q \models [[c \varepsilon a]]$ for some c.
In addition, if $q \models d \varepsilon b$ then $q \models [[d \varepsilon b]]$. Then it's a routine work to show that $[[a \approx b]]$ is true at p.

Four more axioms

With this lemma, it's easy to show the validity of the power set axiom, which is

Theorem

$[\![\forall f \exists g \forall x (x \in g \leftrightarrow x \subseteq f)]\!]$ is valid in $\mathcal{M}$, where $x \subseteq f$ abbreviates $\forall y (y \in x \rightarrow y \in f)$.

Four more axioms

Finally we show the validity of the axiom of well foundedness.

Lemma

If $p \models [[b \in a]]$ then for some p' s.t pRp' and c, $p' \models [[c \in a]]$ and $p' \models \forall z[[z \in a \to \neg z \in c]]$.

Proof

Assume $p \models [[b \in a]]$. Let A consists of x s.t $[[x \in a]]$ is true at some world accessible from p. Realize that pRp, so $A \neq \emptyset$. Let c be a member of A of least $\mathcal{G}$-rank. Since $c \in A$, there is some p' s.t pRp' and $p' \models [[c \in a]]$. We show c and p' satisfies the conclusion of the lemma.
First, $p' \models [[c \in a]]$ by definition.
Next, suppose $p' \not\models \forall z[[z \in a \to \neg z \in c]]$. We derive a contradiction. By the supposition, there is some p'' s.t $p'Rp''$ and d s.t $p' \models [[d \in a]]$ and $p'' \models d \in c]]$. Then, for some d', $p'' \models [[d \approx d']]$ and $p'' \models [[d' \varepsilon c]]$, which means that d' has lower $\mathcal{G}$-rank than c. But by substitutivity, $p'' \models [[d' \in a]]$, so $d' \in A$, contradicting to c being a member of A of least $\mathcal{G}$-rank.

Four more axioms

Now we have the validity of the axiom of well foundedness:

Theorem

For any $a \in \mathcal{D}^{\mathcal{G}}$,

$$[[\exists y y \in a \to \exists y(y \in a \wedge \forall z(z \in a \to \neg z \in y))]]$$

is valid in $\mathcal{M}$.

The axiom of substitution

Instead of proving the substitution axiom itself, we prove the validity of the axioms of collection:

Axioms of collection

For any relation R and set A, if for every $a \in A$ there is some x s.t $R(a, x)$, then there is a set B s.t for every $a \in A$ there is some $b \in B$ s.t $R(a, b)$.

This is equivalent to the substitution axiom provided that the universe is well founded. Using first-order formalism, we have

For each formula $\phi(x, y)$, with only x and y free,

$$\forall a(\forall x \in a \exists y \phi(x, y) \to \exists b \forall x \in a \exists y \in b \phi(x, y))$$

And our aim is to show that the translate of each of these is valid in $\mathcal{M}$.

The axiom of substitution

We know that $p \models f \varepsilon g$ doesn't imply $p \models [[f \varepsilon g]]$. Now we define g^* based on g as follows:

Definition

$g^* = \{(q, f) | (p, f) \in g \text{ for some } p \in \mathcal{G} \text{ s.t } pRq\}$

It follows easily from the definition that $p \models f \varepsilon g$ implies $p \models [[f \varepsilon g^*]]$ and one can verify that $g^* \in \mathcal{D}^{\mathcal{G}}$.

Now we proceed to the chief result:

Proposition

Suppose $p \in \mathcal{G}$, $f \in \mathcal{D}^{\mathcal{G}}$, and $p \models [[\forall x(x \in f \rightarrow \exists y \phi(x, y))]]$ for ϕ. Then for some $g^* \in \mathcal{D}^{\mathcal{G}}$, $p \models [[\forall x(x \in f \rightarrow \exists y(y \in g^* \wedge \phi(x, y)))]]$.

The axiom of substitution

Proof (sketch)

Consider the following relation between $u, v \in \mathcal{D}^{\mathcal{G}}$:
For some a, b and qRq' s.t $q' \models [[\phi(a, b)]]$, $u = (q, a)$ and $v = (q', b)$.
It can be proved that this relation is definable over M by a formula, say $psi(u, v)$, now, let
$F = \{(q, a) | q \in \mathcal{G}, q \models [[a \varepsilon f]]$ and $q \models [[\forall x(x \in f \to \exists y \phi(x, y))]]\}$.
One can verify that $F \in \mathcal{D}^{\mathcal{G}}$, so F is a set in M. It can be proved that for each $u \in F$ there is some v s.t $\psi(u, v)$ is true over M, and by applying collection on M, there is some set h in M s.t $\forall u \in F \exists v \in h \psi(u, v)$.
Let g be the subset of h s.t $v \in g$ iff $\psi(u, v)$ for some $u \in F$. Then by the defining conditions of ψ, members of g are in $\mathcal{D}^{\mathcal{G}}$, since g is a set in M, $g \in \mathcal{D}^{\mathcal{G}}$, so $g^* \in \mathcal{D}^{\mathcal{G}}$.
Then it's a routine work to verify the proposition.

Constructing classical models

Dense sets, filters and generic sets

Dense sets

A set $D \subseteq \mathcal{G}$ is dense in $(\mathcal{G}, \mathcal{R})$ if, for every $p \in \mathcal{G}$, there is some $p' \in D$ s.t pRp'.

Filters

A subset $G \subseteq \mathcal{G}$ is a filter in $(\mathcal{G}, \mathcal{R})$ if:

(1) for every $p, q \in G$ there is some $r \in G$ s.t pRr and qRr;

(2) for every $q \in G$, if pRq then $p \in G$.

Generic sets

Let M be a set and $(\mathcal{G}, \mathcal{R})$ a frame. A set $G \in \mathcal{G}$ is $(\mathcal{G}, \mathcal{R})$-generic over M if:

(1) G is A filter in $(\mathcal{G}, \mathcal{R})$;

(2) if $D \in M$ is dense in $(\mathcal{G}, \mathcal{R})$ then $D \cap G \neq \emptyset$.

When generic sets exist

Theorem

Assume M is a countable standard models in V and $(\mathcal{G}, \mathcal{R}) \in M$. Then for every $r \in \mathcal{G}$ there is a set G that is $(\mathcal{G}, \mathcal{R})$ -generic over M s.t $r \in G$.

Proof

Since M is countable, we can enumerate members of M that are dense in $(\mathcal{G}, \mathcal{R})$: $D_0, D_1, \ldots$. Now define a sequence $p_0, p_1, \ldots$ in $\mathcal{G}$ as follows:
$p_0 = r$.
Given p_n, there is some $p' \in D_n$ s.t pRp', let it be p_{n+1}.
Let $G = \{q \in \mathcal{G} | qRp_n \text{ for some } n\}$. Obviously $r \in G$ and it is $(\mathcal{G}, \mathcal{R})$ -generic over M.

When generic sets exist

Lemma

Let M be a standard model with $(\mathcal{G}, \mathcal{R}) \in M$, let G be $(\mathcal{G}, \mathcal{R})$-generic over M, and let $S \in M$ be an arbitrary subset of $\mathcal{G}$. Then either:

(1) for some $p \in G$, $p \in S$, or

(2) for some $p \in G$, p is incompatible with every member of S, that is, for each member s of S, there is no $r \in \mathcal{G}$ s.t pRr and sRr.

Proof (sketch)

Let $A = \{q \in \mathcal{G} | pRq \text{ for some } p \in S\}$, $b = \{q \in \mathcal{G} | q \text{ is incompatible with every member of } S\}$.
Then, $A \cup B$ is dense, so $G \cap (A \cup B) \neq \emptyset$, then the proof is done.

When generic sets exist

Proposition

Let M be a standard model with $(\mathcal{G}, \mathcal{R}) \in M$, let G be $(\mathcal{G}, \mathcal{R})$-generic over M, for any formula X, exactly one of:

(1) for some $p \in G$, $p \models [[X]]$, or

(2) for some $p \in G$, $p \models [[\neg X]]$.

Proof

First, (1) and (2) can't hold together since G is generic implies that every two members of G are compatible.
Now we show that at least one of these holds.
Realize that $[[X]] \leftrightarrow \Box[[X]]$ and $[[\neg X]] \leftrightarrow \Box\neg\Box[[X]]$. So it's enough to show that for each formula Y, either $p \models \Box Y$ for some $p \in G$ or $p \models \Box\neg\Box Y$ for some $p \in G$.
Let $S = \{q \in \mathcal{G} | q \models \Box Y\}$, and by the previous lemma, the proof is done.

Generic extensions

Definition

For each $f \in \mathcal{D}^{\mathcal{G}}$, and for each set $G \subseteq \mathcal{G}$, we define a set f_G, by recursion on $\mathcal{G}$-rank, as follows:
$f_G = \{g_G | p \models [[g \varepsilon f]] \text{ for some } p \in G\}$.

Now we introduce a new notion:

Dense below p

A set $D \subseteq \mathcal{G}$ is called dense below p if, for every $q \in \mathcal{G}$ s.t pRq there is some $r \in D$ s.t qRr.

With this new notion we have the following:

$g_G \in f_G$ iff $\{q | (q, g) \in f\}$ is dense below p for some $p \in G$.

Generic extensions

We recall the notion of $\hat{x}$:

To each (well founded) set x in M we associate a member $\hat{x}$ of $\mathcal{D}^{\mathcal{G}}$ as follows: $\hat{x} = \{(p, \hat{y}) | y \in x\}$.

And we have the following observation:

For each $x \in M$, $(\hat{x})_G = x$.

Now we define the model $M[G]$:

Definition

For any set $G \in V$, $M[G] = \{f_G | f \in \mathcal{D}^{\mathcal{G}}\}$.

Generic extensions

Proposition

$M \subseteq M[G]$.

Proof

If $x \in M$ then $\hat{x} \in \mathcal{D}^{\mathcal{G}}$, so $(\hat{X})_G \in M[G]$, and by the previous proposition, $x \in M[G]$.

Lemma

Suppose that G is (G, R)-generic over M, and $p \in G$. If D is dense below p then some member of D is in G.

Proof

Suppose that D and G ahs no members in common. then by the previous lemma there is some $q \in G$ that is incompatible with every member of D. Now, both p and q are in G, so there is some $rinG$, pRr and qRr. Since D is dense below p and pRr, so there is some r' s.t rRr', then qRr', which is a contradiction.

Generic extensions

Now we recall another notion:

Definition

$g^* = \{(q, f) | (p, f) \in g \text{ for some } p \in \mathcal{G} \text{ s.t } pRq\}$

The following items hold:

(1) $p \models a\varepsilon f \Rightarrow p \models a\varepsilon f^*$

(2) $p \models a\varepsilon f^* \Leftrightarrow q \models a\varepsilon f$ for some q s.t qRp

(3) $p \models a\varepsilon f^* \Rightarrow p \models [[a\varepsilon f^*]]$

Realize that for $p \in \mathcal{G}$, $\hat{p} \in \mathcal{D}^{\mathcal{G}}$, define g as follows:

$$g = \{(p, \hat{p}) | p \in G\}$$

Then $g \in \mathcal{D}^{\mathcal{G}}$ and for $p \in G$, $p \models x\varepsilon g$ iff $x = \hat{p}$. Then we have the following:

If G is $(\mathcal{G}, \mathcal{R})$-generic then $(g^*)_G = G$, so $M[G]$ has G as a member.

The truth lemma

Truth Lemma

Assume that G is $(\mathcal{G}, \mathcal{R})$-generic over M; let $\phi(x_1, \ldots, x_n)$ be a formula, and let $c_1, \ldots, c_n \in \mathcal{D}^{\mathcal{G}}$. Then, $\phi((c_1)_G, \ldots, (c_n)_G)$ is true in $M[G]$ iff $p \models [[\phi(c_1, \ldots, c_n)]]$ for some $p \in G$.

Proof (sketch)

It can be proved that $p \models [[f \approx g]]$ for some $p \in G$ iff $f_G = g_G$ in $M[G]$. Then the proof can be done by the induction on the complexity of ϕ.

Forcing via First-order intuitionistic Logic

Models

The model we use here is a quadruple $(\mathcal{G}, \mathcal{R}, \models, \mathcal{S})$ where $\mathcal{G}$ is a nonempty set, $\mathcal{R}$ is a binary relation that is transitive and reflexive, $\models$ ia a relation between elements of $\mathcal{G}$ and formulas, and $\mathcal{S}$ is a nonempty set act as the domain of the model. For $\Gamma \in \mathcal{G}$, we use Γ^* to say some $\Gamma' \in \mathcal{G}$ s.t $\Gamma R \Gamma'$, the quardruple also satisfies the following:

- if $\Gamma \models A$ then for all Γ^*, $\Gamma^* \models A$ for A atomic
- $\Gamma \models X \wedge Y$ iff $\Gamma \models X$ and $\Gamma \models Y$
- $\Gamma \models X \vee Y$ iff $\Gamma \models X$ or $\Gamma \models Y$
- $\Gamma \models \neg X$ iff for all Γ^*, $\Gamma^* \not\models X$
- $\Gamma \models X \to Y$ iff for all Γ^*, if $\Gamma^* \models X$, $\Gamma^* \models Y$
- $\Gamma \models \exists x X(x)$ iff for some $a \in \mathcal{S}$, $\Gamma \models X(a)$
- $\Gamma \models \forall x X(x)$ iff for every Γ^* and for every $a \in \mathcal{S}$, $\Gamma^* \models X(a)$

We call a model $(\mathcal{G}, \mathcal{R}, \models, \mathcal{S})$ an intuitionistic ZF model if classical equivalents of all axioms of ZF, expressed without using universal quantifier, are valid in it.

Derived models

Definition

For a model $(\mathcal{G}, \mathcal{R}, \models, \mathcal{S})$, let P be the collection of all $\mathcal{R}$-closed subsets of $\mathcal{G}$. We say f is difinable over $(\mathcal{G}, \mathcal{R}, \models, \mathcal{S})$ if $domain(f) = \mathcal{S}$, $range(f) \subseteq P$, and for some $X(x)$ with only one free variable, all constants from $\mathcal{S}$, and no universal quantifiers, for any $a \in \mathcal{S}$:

$$f(a) = \{\Gamma | \Gamma \models X(a)\}$$

Definition

For a model $(\mathcal{G}, \mathcal{R}, \models, \mathcal{S})$, let $\mathcal{S}'$ be the set of elements in $\mathcal{S}$ together with functions definable over $(\mathcal{G}, \mathcal{R}, \models, \mathcal{S})$. We define $\models'$ as follows: for $f, g \in \mathcal{S}'$,

(1) $f, g \in \mathcal{S}$, then let $\Gamma \models' f \in g$ if $\Gamma \models f \in g$

(2) $f \in \mathcal{S}$, $g \in \mathcal{S}' - \mathcal{S}$, let $\Gamma \models' f \in g$ if $\Gamma \in g(f)$

(3) $f \in \mathcal{G}' - \mathcal{G}$, let $X(x)$ be the formula that defines f, let $\Gamma \models' f \in g$ if there is $h \in \mathcal{S}$ s.t $\Gamma \models \neg\exists x \neg(x \in h \leftrightarrow X(x))$ and $\Gamma \models' h \in g$

We call the model $(\mathcal{G}, \mathcal{R}, \models', \mathcal{S}')$ the derived model of $(\mathcal{G}, \mathcal{R}, \models, \mathcal{S})$.

The intuitionistic M_α sequence

Let V be a classical model for ZF, and let $(\mathcal{G}, \mathcal{R}, \models_0, \mathcal{S}_0)$ be an intuitionistic model satisfying the following conditions:

(1) $(\mathcal{G}, \mathcal{R}, \models_0, \mathcal{S}_0) \in V$

(2) $\mathcal{S}_0$ is a collection of functions s.t if $f \in \mathcal{S}_0$, *domainf* $\subseteq \mathcal{S}_0$ and *range*$(f) \subseteq P$

(3) for $f, g \in \mathcal{S}_0$, $\Gamma \models_0 f \in g$ iff $\Gamma \in g(f)$

(4) for $f, g, h \in \mathcal{S}_0$, if $\Gamma \models_0 \neg\exists x \neg(x \in f \leftrightarrow x \in g)$ and $\Gamma \models_0 \neg f \in h$ then $\Gamma \models_0 \neg g \in h$

(5) $\mathcal{S}_0$ is well-founded with respect to the relation $x \in \mathit{domain}(y)$

Next, let $(\mathcal{G}, \mathcal{R}, \models_{\alpha+1}, \mathcal{S}_{\alpha+1})$ be the derived model of $(\mathcal{G}, \mathcal{R}, \models_\alpha, \mathcal{S}_\alpha)$. And for a limit ordinal λ, let $\mathcal{S}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{S}_\alpha$, let $\Gamma \models_\lambda f \in g$ if for some $\alpha < \lambda$, $\Gamma \models_\alpha f \in g$. Thus we have $(\mathcal{G}, \mathcal{R}, \models_\lambda, \mathcal{S}_\lambda)$.
Finally, let $\mathcal{S} = \bigcup_{\alpha \in V} \mathcal{S}_\alpha$, and let $\Gamma \models f \in g$ if for some $\alpha \in V$, $\Gamma \models_\alpha f \in g$, thus we have $(\mathcal{G}, \mathcal{R}, \models, \mathcal{S})$, which can be proved to be an intuitionistic ZF model.

Now, suppose that $(\mathcal{G}, \mathcal{R}, \models, \mathcal{S})$ is an intuitionistic ZF model s.t $\neg AC$ is valid in it, where AC is some equivalent form of axiom of choice expessed without using universal quantifiers. It follows that the axiom of choice is classical unprovable from the axioms of ZF. For otherwise, there is some finite axioms $A_1, \ldots, A_n$ s.t $\vdash_c (A_1 \wedge \cdots \wedge A_n) \rightarrow AC$, which implies that $\vdash_I (A_1 \wedge \cdots \wedge A_n) \rightarrow \neg\neg AC$, which contradicts the model $(\mathcal{G}, \mathcal{R}, \models, \mathcal{S})$.

The relation between these methods

A mapping between formulas

We define a mapping from intuitionistic formulas to QS4 formulas by:

$M(A) = \Box A$ for A atomic
$M(X \wedge Y) = M(X) \wedge M(Y)$
$M(X \vee Y) = M(X) \vee M(Y)$
$M(\neg x) = \Box \neg M(X)$
$M(X \rightarrow Y) = \Box(M(X) \rightarrow M(Y))$
$M(\exists x X) = \exists x M(X)$
$M(\forall x X) = \Box \forall x M(X)$

And we have the following:

If X is an intuitionistic formula, X is intuitionistically valid iff $M(X)$ is QS4 valid.

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