Regularity and Relative Likelihood

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Joint work with Thomas Icard and Wes Holliday

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Data on reasoning about relative likelihood

$(\varphi \succsim \psi)$	φ is at least as likely as ψ.
$(\varphi \succ \psi)$	φ is more likely than ψ.
$\triangle\varphi$	φ is more likely than not (probably φ).

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$(\varphi \gtrsim \psi)$	φ is at least as likely as ψ.
$(\varphi \succ \psi)$	φ is more likely than ψ.
$\triangle\varphi$	φ is more likely than not (probably φ).

Intuitively valid/invalid inferences:

V1	$\triangle\varphi \to \neg\triangle\neg\varphi$	V2	$\triangle(\varphi \wedge \psi) \to (\triangle\varphi \wedge \triangle\psi)$
V3	$\triangle\varphi \to \triangle(\varphi \vee \psi)$	V11	$(\psi \gtrsim \varphi) \to (\triangle\varphi \to \triangle\psi)$
V4	$(\varphi \gtrsim \bot)$	V12	$(\psi \gtrsim \varphi) \to ((\varphi \gtrsim \neg\varphi) \to (\psi \gtrsim \neg\psi))$
V5	$(\top \gtrsim \varphi)$	V13	$((\varphi \wedge \neg\psi) \succ \bot) \to ((\varphi \vee \psi) \succ \psi)$
V6	$(\Box\varphi \to \triangle\varphi)$	I1	$((\varphi \gtrsim \psi) \wedge (\varphi \gtrsim \chi)) \to (\varphi \gtrsim (\psi \vee \chi))$
V7	$(\triangle\varphi \to \Diamond\varphi)$	I2	$(\varphi \gtrsim \neg\varphi) \to (\varphi \gtrsim \psi)$
		I3	$(\triangle\varphi \to (\varphi \gtrsim \psi))$.

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- $\bot \not\succsim \top$
- $((\varphi \succsim \psi) \wedge (\psi \succsim \chi)) \rightarrow (\varphi \succsim \chi)$
- $(\varphi \succsim \psi) \leftrightarrow ((\varphi \wedge \neg\psi) \succsim (\psi \wedge \neg\varphi))$

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Three basic types of semantics:

- event-ordering semantics;
- measure-based semantics;
- world-ordering semantics.

Event-ordering semantics

Definition

An **event-ordering (EO) frame** is a tuple $\langle W, R, \succsim \rangle$ where $R \subseteq W^2$ is serial and $\succsim : W \to \wp(W)^2$ s.t. for every $w \in W$, $\succsim_w$ is a preorder on $\wp(R(w))$.

An EO model is a tuple $\langle W, R, \succsim, V \rangle$ where $\langle W, R, \succsim \rangle$ is an EO frame and $V : \text{Prop} \to \wp(W)$.

Semantics for $\succsim$: $\mathcal{M}, w \models \varphi \succsim \psi$ iff $[\![\varphi]\!]^{\mathcal{M}}_w \succsim_w [\![\psi]\!]^{\mathcal{M}}_w$.

Here $[\![\varphi]\!]^{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \models \varphi\}$, and $X_w = X \cap R(w)$.

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Event-ordering semantics is like algebraic semantics: it's too close to syntax and you get exactly the logic that correspond to the constraints you put on $\succsim$.

World-ordering semantics

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Most famously used for counter-factuals: $R(w)$ is the set of 'possible worlds' according to w, and $\succeq_w$ compares similarity/normality relative to w.

Lifting world-ordering to event-ordering

Given a preorder $\succeq$ on X, we get the following preorder $\succeq^l$ on $\wp(X)$:

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This lifting is first proposed in semantics by Lewis for counterfactuals:
$\mathcal{M}, w \models \varphi \Diamond\!\!\rightarrow \psi$ iff $(\llbracket\varphi\rrbracket \cap \llbracket\psi\rrbracket)_w \succeq^l_w (\llbracket\varphi\rrbracket \cap \llbracket\neg\psi\rrbracket)_w$, assuming that $\succeq$ is also total.

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Kratzer interpreted $\succsim$ by this lifting. Call this the *l*-**lifting semantics**:

$$\mathcal{M}, w \models \varphi \succsim \psi \iff \llbracket\varphi\rrbracket_w \succeq^l_w \llbracket\psi\rrbracket_w.$$

"For every ψ-world, there is an at least as likely φ-world."

Disjunction problem

l-lifting semantics validate the following:

$$((\varphi \gtrsim \psi) \wedge (\varphi \gtrsim \chi)) \rightarrow (\varphi \gtrsim (\psi \vee \chi)).$$

This is very bad. The problem is double-counting.

measure-based semantics

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However, we can easily require μ to be a probability measure. Then we get probabilistic frames/models.

Probability measures

Definition

A **probability measure** μ on W is a function from $\wp(W)$ to $[0, 1]$ such that:

- $\mu(\varnothing) = 0$, $\mu(W) = 1$;
- for any $A, B \subseteq X$ with $A \cap B = \varnothing$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

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For convenience, we only consider probability measures that assign numbers to **every subset** of the set of all possible worlds.

Probability models

Definition

A **sharp-probability (SP) frame** is a tuple $\langle W, R, \mu \rangle$ where $R \subseteq W^2$ is serial and for each $w \in W$, μ_w is a probability measure on $R(w)$. Models are obtained by adding a valuation. Then $\mathcal{M}, w \models \varphi \succsim \psi$ iff $\mu_w(\llbracket \varphi \rrbracket_w) \geq \mu_w(\llbracket \psi \rrbracket_w)$.

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Definition

Given a preorder $\succeq$ on W, its ***i*-lifting** $\succeq^i$ on $\wp(W)$ is defined by

$$A \succeq^i B \iff \exists f : B \to A \text{ injective and inflationary: } \forall b \in B, f(b) \succeq b.$$

Then *i*-lifting semantics defines: $\mathcal{M}, w \models \varphi \gtrsim \psi$ iff $[\![\varphi]\!]_w \succeq^i_w [\![\psi]\!]_w$.

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- in particular, it is valid on all finite models.

Thus, the *i*-lifting solution must be paired with the Noetherianess assumption. This is an apparent weak spot.

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$$(\varphi_1, \varphi_2, \cdots, \varphi_n) \equiv (\psi_1, \psi_2, \cdots, \psi_n).$$

This says that $|\{i \mid \varphi_i \text{ is true}\}| = |\{i \mid \psi_i \text{ is true}\}|$.

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- the axiom of generalized finite cancellation (GFC) ($n \in \mathbb{N}$, $k \in \mathbb{N} \setminus \{0\}$):
  $$((\varphi_1, \ldots, \varphi_n, \underbrace{\varphi', \ldots, \varphi'}_{k \text{ times}}) \equiv (\psi_1, \ldots, \psi_n, \underbrace{\psi', \ldots, \psi'}_{k \text{ times}}) \succsim \top) \to ((\bigwedge_{i=1}^{n}(\varphi_i \succsim \psi_i)) \to (\psi' \succsim \varphi'));$$

and (3) is closed under the following rules:

- the rule of necessitation for $\succsim$: if $\varphi \in$ L, then $(\varphi \succsim \top) \in$ L;
- the rule of *modus ponens*: if $\varphi, (\varphi \to \psi) \in$ L, then $\psi \in$ L.

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IP($\succsim$) is the logic of imprecise-probability frames. It is also the logic of all Noetherian WO frames.

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The logic diverge when we add $\Diamond$. This is because there can't be an injection from a non-empty set to the empty set, but a non-empty set can get probability 0.

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IP$(\succsim, \Diamond)$ and IPS$(\succsim, \Diamond)$ are defined as usual.

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IPR($\succsim, \Diamond$) is the logic of all Noetherian WO frames in $\mathcal{L}(\succsim, \Diamond)$, where

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A probability measure μ is **regular** if it assigns nonzero numbers to non-empty sets. An SP(IP)-frame(model) is regular if all the probability measures used there are regular.

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IPR($\succsim, \Diamond$) is the logic of all regular IP frames.

Extra Frame Properties

For special interpretations of R and $\succeq, \mathcal{P}$, the following properties are useful.

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- uniformity: for all $w, v \in W$, $R(w) = R(v)$ and $\succeq_w = \succeq_v$ (resp. $\mathcal{P}_w = \mathcal{P}_v$);
- universality: the conjunction of R-reflexivity and uniformity, i.e., for all $w, v \in W$, $R(w) = W$ and $\succeq_w = \succeq_v$ (resp. $\mathcal{P}_w = \mathcal{P}_v$).

Logics

Theorem

IP**X**($\succsim, \Diamond$) is the logic of all **Y** IP frames, where

- **X** has $R : \Diamond\varphi \leftrightarrow (\bot \not\succsim \varphi)$ iff **Y** has regularity;
- **X** has $Mc : (\bot \succsim \varphi) \to \neg\varphi$ iff **Y** has reflexivity;
- **X** has the introspection axioms iff **Y** has uniformity.

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Theorem

IPR$\mathbf{X}(\succsim, \Diamond)$ is the logic of all **Y** Noetherian WO frames and **Y** regular IP frames, where

- **X** has *Mc* iff **Y** has *R*-reflexivity;
- **X** has the introspective axioms iff **Y** has uniformity.

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IP$\mathbf{X}(\succsim)$ is the logic of all $\mathbf{Y}$ regular IP frames and also the logic of all $\mathbf{Y}$ Noetherian WO frames, where

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Proof idea: representation theorem, filtration, and turning regular IP models to equivalent WO models.

Summing up

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If you don't like regularity, then *i*-lifting semantics is not for you. But regularity is actually quite popular in philosophical literature.

Given regularity, for *i*-lifting semantics to generate the same logic as IP semantics does, we still need Noetherian condition, for soundness.

Noetherian assumption

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How important are infinite models for natural language semantics is debatable, but let's say we need them. Then we need to modify *i*-lifting.

Modifying i-lifting

A lifting x in general is a class function defined on all preorders s.t. that for any preorder $\succeq$ a non-empty domain W, $\succeq^x$ is a preorder on $\wp(W)$.

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Given a lifting x, the x-lifting semantics for $\succsim$ is

$$\mathcal{M}, w \models \varphi \succsim \psi \iff [\![\psi]\!]_w \succeq^x_w [\![\varphi]\!].$$

We have seen *i*-lifting semantics.

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Goal: find a lifting x s.t. the x-lifting semantics over all WO-frames generate the logic IPR($\succsim, \Diamond$).

The lifting should also be not too far away from i-lifting.

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A sufficient condition for a lifting x to work:

- x-lifting and i-lifting behaves the same on finite preorders;
- x-lifting always generate GFC orders.

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Definition

A GFC order is a preorder $\succsim$ on some $\wp(W)$ for some non-empty W s.t.

- $X \succsim \varnothing$ for all $X \subseteq W$, and $\varnothing \not\succsim W$
- for every $\langle A_1, A_2, \cdots, A_n, \underbrace{X, X, \cdots X}_{k \text{ many}} \rangle$ and $\langle B_1, B_2, \cdots, B_n, \underbrace{Y, Y, \cdots Y}_{k \text{ many}} \rangle$ that are balanced: for any $w \in W$, $|\{i \mid w \in A_i\}| + k[w \in X] = |\{i \mid w \in B_i\}| + k[w \in Y]$, if $A_i \succsim B_i$ for $i = 1 \ldots n$, then $Y \succsim X$.

Representation theorem

Theorem

Let W be a finite non-empty set and $\succsim$ a preorder on $\wp(W)$. Then, the following are equivalent:

- $\succsim$ is a GFC order;
- there is a set $\mathcal{P}$ of probability distributions on W that represents $\succsim$: for any $X, Y \subseteq W$, $X \succsim Y$ iff $\forall \mu \in \mathcal{P}, \mu(X) \geq \mu(Y)$.

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- $\succsim$ is a GFC order;
- there is a set $\mathcal{P}$ of probability distributions on W allowing hyperreals that represents $\succsim$: for any $X, Y \subseteq W$, $X \succsim Y$ iff $\forall \mu \in \mathcal{P}, \mu(X) \geq \mu(Y)$.

A minimal lifting

Definition

The *fi*-lifting of a preorder $\succeq$ on W is defined by

$$A \succeq^{fi} B \iff (B \setminus A) \text{ is finite and } \exists f : B \setminus A \to A \setminus B \text{ injective and inflationary.}$$

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Qualitative additivity is baked in, but we also have full GFC.

A minimal lifting

Theorem

For any preorder $\succeq$ on a non-empty W, $\succeq^{fi}$ is a GFC order. In fact,

- it is the smallest GFC order on $\wp(W)$ extending $\{\langle\{x\},\{y\}\rangle \mid x \succeq y\}$ and
- it is represented by

$$\{\mu \in \bigcup_{\mathcal{U}\in fUlt(\wp_{fin}(W))} \Delta(\wp(W), \Pi_{\mathcal{U}}\mathbb{R}) \mid w \succeq w' \Rightarrow \mu(w) \geq \mu(w')\}.$$

A conservative lifting

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The *ni*-lifting of a preorder $\succeq$ on W is defined by $A \succeq^{fi} B$ iff

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- $1, 1/2, 1/4, 1/8, \cdots;$

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Distinguishing *fi* and *ni* liftings:

- $1, 1/2, 1/4, 1/8, \cdots;$
- a disjoint union of $\mathbb{N}$ many totally connected $\mathbb{N}$.

Representable by completely additive measures

Definition

The *cma*-lifting of a preorder $\succeq$ on W is defined by $A \succeq^{cma} B$ iff

$$\exists f : B \cap W_{fin} \to A \cap W_{fin} \text{ injective and inflationary.}$$

Here $W_{fin} = \{w \in W \mid \{w' \in W \mid w' \succeq w\} \text{ is finite}\}$.

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Theorem

For any preorder $\succeq$ on a non-empty W, $\succeq^{cma}$ is a GFC order represented by

$$\{\mu \in \Delta^{cma}(\wp(W), [0,1]) \mid w \succeq w' \Rightarrow \mu(w) \geq \mu(w')\}.$$

Representable by finitely additive measures

Definition

The *fa*-lifting of a preorder $\succeq$ on W is defined by $A \succeq^{fa} B$ iff

$$B \setminus A \text{ finite and } \exists f : (B \setminus A) \cap W_{fin} \to (A \setminus B) \cap W_{fin} \text{ injective and inflationary.}$$

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Accepting regularity, inflationary injection based lifting semantics can be as good as numerical semantics.

I think we should be as agnostic about infinities as possible and use *fi*-lifting. I still don't understand *ni*-lifting as we lack a representation theorem.

We also don't know what's the logic of *i*-lifting over all WO frames.

Thank you!